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CHARACTERISTICS OF MODULAR FINITE-LENGTH LATTICES

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Two integer-valued characteristics of modular lattices of finite length are described, and their relation with the defect (as introduced in [1]) exhibited.

This paper is closely connected with [1], and the notation, terminology and definitions of [1] are assumed (the results of [1] will be referred to directly). In particular, m. l. f. l. means modular lattice of finite length.

We shall introduce, in definitions 1 and 2, two integer-valued characteristics of m. l. f. l., with apparently intuitive meaning.

Definition 1. Let L be a lattice not **1**. The discriminator $d(L)$ of L is the least cardinal n such that for any $x \neq y$ in L there is a lattice M with $l_M \leqq n$ and a homomorphism $h : L \to M$ such that $hx \neq hy$.

We have immediately the following elementary properties: $d(L)$ is not changed if we also require h to be onto. Also $d(L) \leqq l_L$, so that $d(L)$ is finite if L has finite length. If L_1 is a sublattice or a factor lattice of L (i.e. $L_1 = L/\theta$ for some congruence relation θ on L) then $d(L_1) \leqq d(L)$. Finally,

$$(1) \qquad L \textit{ simple implies } d(L) = l_L .$$

Lemma 1. *If L is a m. l. f. l. then $d(L)$ in definition 1 is not changed if we require further that M be simple.*

Proof. Let $h : L \to M$ be a homomorphism onto; then $M \approx L/\theta$ for the congruence relation θ defined by $x \equiv y\,(\theta)$ iff $h(x) = h(y)$. Since the congruence lattice Θ_L of a m. l. f. l. is a Boolean algebra, θ is the intersection of all dual atoms $\beta \geqq \theta$. Thus, if $hx \neq hy$, then $x \not\equiv y\,(\theta)$ and hence $x \not\equiv y\,(\beta)$ for some dual atom β. But then $S = L/\beta$ is simple, the natural homomorphism $L \to S$ does not identify x, y, and $l(L/\beta) \leqq l(L/\theta) = l_M$. This completes the proof.

For m. l. f. l. L, let B be the set of all dual atoms β in Θ_L (i. e. the set of all congruence relations θ such that L/θ is simple nontrivial). Then we have from lemma 1

$$(2) \qquad d(L) = \max_B l(L/\beta)$$

and thus, since $\delta_M = l_M - 2$ for simple m. l. f. l.,

$$d(L) = 2 + \max_B \delta(L/\beta) . \tag{3}$$

From (1) and (2) we then conclude

$$d(L) = \max_B d(L/\beta) . \tag{4}$$

Lemma 2. *Let L be a m. l. f. l. If $L \leqq \mathbf{P}L_a$ then*

$$d(L) = \max d(L_a) . \tag{5}$$

L is simple if and only if $d(L) = l_L$.

Proof. Let $L_a = L/\theta_a$. Then $d(L_a) = d(L/\theta_a) \leqq d(L)$, so that

$$\max d(L_a) \leqq d(L) . \tag{6}$$

Since the decomposition is subdirect, we have $\bigwedge\theta_a = O$, so that each dual atom $\beta \geqq \theta_a$ for some θ_a, and then

$$l(L/\beta) \leqq \max \{l(L/\beta) : \theta_a \leqq \beta \in B\} = d(L_a)$$

on applying (2). Thus $l(L/\beta) \leqq \max d(L_a)$, and (2) again yields $d(L) \leqq \max d(L_a)$. With (6), this proves (5). As for the second statement of lemma 2, we already have (1). If L is not simple, then it is an exact subdirect product of more than one simple nontrivial m. l. f. l. (cf. [1], lemmas 9, 13, and (12)): $L \leqq \mathbf{P}M_j$ with

$$l_L - 1 = \sum(l(M_j) - 1) , \quad l(M_j) \geqq 2 ,$$

and hence (5 and 1)

$$d(L) = \max d(M_j) = \max l(M_j) = l(M_k) < l_L ,$$

so that $d(L) = l_L$ is excluded. This completes the proof of lemma 2.

Definition 2. The trivialiser $t(L)$ of a finite-length lattice $L\,(\neq \mathbf{1})$ is the maximal integer n such that every homomorphism $h : L \to M$ with $l_M \leqq n$ is constant.

Obviously $1 \leqq t(L) < l(L)$; also

$$L \textit{ simple implies } t(L) = l(L) - 1 . \tag{7}$$

Lemma 3. *In definition 2, if L is a m. l. f. l. then we may add the requirement that M be simple.*

Proof. Definition 2 may also be formulated thus (h denotes a homomorphism): $t(L) \geqq n$ iff $h : L \to M$ is constant whenever $l_M \leqq n$. Now define: $t_1(L) \geqq n$ iff $h : L \to M$ is constant whenever $l_M \leqq n$ and M is simple.

Then obviously $t(L) \leqq t_1(L)$. Now assume $t(L) + 1 \leqq t_1(L)$, and aim at a contradiction. There is a nonconstant $h : L \to M$ with $l_M \leqq t(L) + 1$; necessarily though,

$l_M = t(L) + 1$. Since h is nonconstant and $l_M \leqq t_1(L)$, M cannot be simple. Then, as in the proof of lemma 2,

$$M \leqq \mathbf{P}M_j\,, \quad M_j \text{ simple not } \mathbf{1}\,,$$

$$l(M_j) - 1 \leqq l_M - 1 = t(L) < t_1(L)\,.$$

Thus $l(M_j) \leqq t_1(L)$; by definition, every $\eta_j h : L \to M_j$ is constant (η_j is the natural projection $M \to M_j$). But then h itself is constant; this contradiction proves $t(L) \geqq \geqq t_1(L)$ and our lemma.

Using lemma 3, we may conclude immediately that

$$t(L) = -1 + \min_B l(L/\beta) \tag{8}$$

where B is the set of all dual atoms β of Θ_L. Using (7),

$$t(L) = \min_B t(L/\beta) = 1 + \min_B \delta(L/\beta)\,. \tag{9}$$

From (3) and (9) there follows

$$t(L) < d(L)\,.$$

Lemma 4. *If L is a m. l. f. l. and $L \leqq \mathbf{P}L_a$ then $t(L) = \min t(L_a)$.*

Proof. Essentially, the proof is similar to that of lemma 2. Let $L_a = L/\theta_a$, $\bigwedge\theta_a = O$. By (8), $t(L) = \min_{\beta\in B} (-1 + l(L/\beta)) = \min_a \min_{\beta \geqq \theta_a} (-1 + l(L/\beta)) = \min_a t(L_a)\,.$

Lemma 5. *Let L be a m. l. f. l. Then $d(L) = t(L) + 1$ if and only if L is a subdirect product of simple lattices all with length $d(L)$ (i.e. defect $d(L) - 2$).*

The proof follows from lemmas 2, 4 and formulae (2), (8).

The representation theorems of [1], theorems 4 and 5, may now be completed by the following (D and M have the same meaning as in theorem 4, l.c.).

Theorem. *Let L be a m. l. f. l. Then there is a unique subdirect decomposition $L \leqq \mathbf{P}L_j$ $(0 \leqq j \leqq l_L - 1)$ such that, for each j, either $L_j = \mathbf{1}$ or*

$$d(L_j) - 2 = t(L_j) - 1 = j\,. \tag{10}$$

Furthermore the decomposition is then exact, $L_0 = D$ is finite distributive, every L_j is exactly decomposable into say n_j simple lattices with defects j ($n_j = 0$ iff $L_j = \mathbf{1}$);

$$\delta(L_j) = n_j \,.\, j\,, \quad l(L_j) = 1 + n_j(j + 1)\,, \quad \lambda(L_j) = 1 + n_j\,;$$

$$\delta_L = \sum n_j \,.\, j\,,\ l_L = 1 + \sum n_j(j + 1)\,, \quad \lambda_L = 1 + \sum n_j\,;$$

$$l_M = 1 + \sum_{j\geqq 1} n_j(j + 1)\,, \quad \lambda_M = 1 + \sum_{j \geqq 1} n_j\,.$$

Proof. First prove existence. If B is the set of all dual atoms of Θ_L, set

$$B_j = \{\beta \in B : \delta(L/\beta) = j\}\,, \quad \theta_j = \bigwedge\{\beta : \beta \in B_j\}\,, \quad L_j = L/\theta_j\,.$$

Then either B_j is empty and $L_j = \mathbf{1}$, or (10) holds (lemma 5: L_j is an exact subdirect product of simple L/β with $\beta \in B_j$). Obviously $B = \bigcup B_j$ is a disjoint decomposition, so that L decomposes subdirectly into the L_j and this decomposition is exact.

As mentioned, L_j is an exact subdirect product of simple lattices, say M_{ij}, with the same defect j. By exactness, then, $\delta(L_j) = \sum_i \delta(M_{ij}) = n_j \, . \, j$ with n_j integral (and $n_j = 0$ iff $L_j = \mathbf{1}$). Then n_j is the number of simple exact factors of L_j; thus (cf. [1], lemma 9) $\lambda_L - 1 = \sum n_j$. Also $l_L - 1 = \delta_L + \lambda_L - 1 = \sum n_j \, . \, j + \sum n_j$ by the previous result. This proves the formulae for λ_L, l_L; those for λ_M, l_M are similar.

Finally, consider unicity. Thus, let $L \leqq \mathbf{P} L/\tau_j$ with $\bigwedge \tau_j = O$, and either $\tau_j = I$ or

$$d(L/\tau_j) - 2 = t(L/\tau_j) - 1 = j \, .$$

Take any $\tau_j \neq I$; let $\beta \geqq \tau_j$ be any dual atom. Then by (2),

$$l(L/\beta) \leqq \max_{\beta \geqq \tau_j} l(L/\beta) = d(L/\tau_j) = j + 2$$

and by (8),

$$-1 + l(L/\beta) \geqq \min_{\beta \geqq \tau_j} (-1 + l(L/\beta)) = t(L/\tau_j) = j + 1 \, .$$

Thus $l(L/\beta) = j + 2$, and since L/β is simple, $\delta(L/\beta) = j$. Summarising, if $\beta \leqq \tau_j$ for a dual atom β, then $\delta(L/\beta) = j$, and hence, by construction, $\beta \in B_j$, $\beta \geqq \theta_j$. This proves that $\tau_j \geqq \theta_j$ for each j; from $\bigwedge \tau_j = O$ it then follows that $\tau_j = \theta_j$ for all j. This completes the proof of the theorem.

The two following corollaries may be proved as in [1], corollary to theorem 4, corollary 2 to theorem 5.

Corollary 1. *In the theorem, if L is complemented, then the subdirect decompositions (of L and of L_j) are direct.*

Corollary 2. *Let L, L_a be m. l. f. l. Let L_j and L_{aj} be the factors of the decompositions described in the theorem, of L and L_a, respectively. Then if L is a subdirect (exact, direct) product of L_{aj}'s, then L_j is a subdirect (exact, direct) product of the L_{aj}'s.*

Next, consider some elementary consequences of the formula $l_L - 1 = \sum_0^{l_L - 1} n_j (j + 1)$. (Write l in place of l_L.) Since all summands are nonnegative, we have immediately that

$$n_{l-1} = 0 \, , \quad \textit{i.e.} \quad L_{l-1} = \mathbf{1} \, .$$

Now, assume that some L_j is non-simple (L is not "square-free"), i.e. that some $n_j \geqq 2$. Then $l - 1 \geqq 2(j + 1)$, $j \leqq \frac{1}{2}(l - 1) - 1$, and we conclude

$$\textit{for } j > \tfrac{1}{2}(l - 1) - 1, \; L_j \textit{ is simple} \, .$$

If (and only if) $n_{l-2} \geqq 1$ then necessarily all other $n_j = 0$ and $n_{l-2} = 1$, i.e.

$$L_{l-2} \neq \mathbf{1} \ \textit{implies} \ L = L_{l-2} \ \textit{is simple}.$$

Now assume that this is not the case. Then $n_{l-2} = 0$, $l \geqq 3$, and if $n_{l-3} \neq 0$ then we have, principially, several cases:

(i) $n_{l-3} \geqq 3$ implies $l - 1 \geqq 3(l - 2)$ and $l \leqq 2$, a contradiction;

(ii) $n_{l-3} = 2$ implies $l - 1 \geqq 2(l - 2)$ and $l \leqq 3$, thus $l = 3$; since L was assumed non-simple, the only possibilities are $L = \mathbf{3}$ or $L = \mathbf{2}^2$;

(iii) $n_{l-3} = 1$; then $l - 1 = \sum_0^{l-1} n_j(j + 1) = l - 2 + \sum_0^{l-4} n_j(j + 1)$ and we conclude $n_0 = 1$ and all other $n_j = 0$ (for $0 \neq j \neq l - 3$); thus the only nontrivial factors L_j are

$$L_0 = \mathbf{2}, \text{ simple } L_{l-3}.$$

Thus we have that $n_j = 0$ for $j \geqq l - 3$ with the following exceptions: L is simple or a subdirect product of a simple lattice and $\mathbf{2}$.

References

[1] *Hájek O.*: Representation of finite-length modular lattices, Czech. Math. J. *15* (*90*) (1965), 503—520.

Резюме

ХАРАКТЕРИСТИКИ ДЕДЕКИНДОВЫХ СТРУКТУР КОНЕЧНОЙ ДЛИНЫ

ОТОМАР ГАЕК (Otomar Hájek) Прага

В настоящем продолжении работы [1] определяются две численные характеристики дедекиндовых структур L конечной длины (д. с. к. д.) – дискриминатор $d(L)$ и тривиализатор $t(L)$. Доказана следующая теорема:

Для всякой д. с. к. д. L существует единственное полупрямое разложение $L \leqq \mathbf{P}_{0 \leqq j < l_L} L_j$ такое, что или $L = \mathbf{1}$ или $d(L_j) - 2 = t(L_j) - 1 = j$.