Czechoslovak Mathematical Journal

Václav Doležal

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Czechoslovak Mathematical Journal, Vol. 15 (1965), No. 3, 436–453

Persistent URL: http://dml.cz/dmlcz/100685

Чехословацкий математический журнал, т. 15 (90) 1965, Прага

AN EXTENSION OF POPOV'S METHOD FOR VECTOR-VALUED NONLINEARITIES

VÁCLAV DOLEŽAL, Praha

(Received December 3, 1964)

The article presents some sufficient conditions for stability in large of certain types of nonlinear vector integro-differential equations.

1. Consider the equations

$$\sigma(t) = z(t) + \int_0^t k(t - \tau) f(\sigma(\tau)) \, d\tau - \gamma \xi(t) , \tag{1.1}$$

$$\xi'(t) = f(\sigma(t)) , \quad t \geqq 0 \tag{1.2}$$

with a given initial condition $\xi(0)$, where $\sigma(t)$, $z(t)$, $\xi(t)$, $f(\sigma)$ are real n-vectors, $k(t)$ and γ real $n \times n$ matrices.

We shall assume that $\|z(t)\|$ and $\|k(t)\|$ are bounded on every finite interval $\langle 0, T \rangle$ and that there is a $\mu > 0$ such that $\|f(\sigma_1) - f(\sigma_2)\| \leqq \mu \|\sigma_1 - \sigma_2\|$ for any pair of n-vectors σ_1, σ_2. Then obviously (1.1), (1.2) possess a uniquely determined solution $\sigma(t)$, $\xi(t)$ for given $z(t)$, $\xi(0)$.

Theorem 1.1. *Let $k(t)$ possess a derivative almost everywhere for $t \geqq 0$ such that*

$$\|k(t)\|, \ \|k'(t)\| \leqq C \exp(-\alpha t) \tag{1.3}$$

with $\alpha > 0$, and let γ be a symmetric positive definite matrix; let $z(t)$ possess a second derivative almost everywhere for $t \geqq 0$ such that

$$\|z(t)\|, \ \|z'(t)\|, \ \|z''(t)\| \leqq Z \exp(-\beta t) \tag{1.4}$$

with some Z and a fixed $\beta > 0$. Moreover, let $f(\sigma)$ satisfy the conditions:

a) *There is a real scalar function $U(\sigma)$ possessing continuous first partial derivatives everywhere such that $f(\sigma) = \operatorname{grad} U(\sigma)$, i.e. for the i-th component of $f(\sigma)$,*

$$f_i(\sigma) = \frac{\partial U(\sigma)}{\partial \sigma_i} , \quad i = 1, 2, \ldots, n . \tag{1.5}$$

b) *There are numbers* $h_1 > 0$ *and* h_2 *such that*

$$h_1 \|\sigma\|^2 \leqq f^{`}(\sigma)\,\sigma\,, \quad \|f(\sigma)\| \leqq h_2 \|\sigma\| \tag{1.6}$$

for every σ.

Let $h > h_2^2 h_1^{-1}$ *and* $\tilde{k}(\omega) = \int_0^\infty k(t) \exp(-i\omega t)\,\mathrm{d}t$, $-\infty < \omega < \infty$; *if there is a* $q > 0$ *such that the matrix*

$$A(\omega) = (1 + i\omega q)\,\tilde{k}(\omega) - (h^{-1}I + q\gamma) \tag{1.7}$$

fulfills the condition $\operatorname{Re} \bar{\eta}^{`} A(\omega)\,\eta \leqq 0$ *for every real* ω *and every constant n-vector* η, *then there are functions* $S(x, y)$ *and* $K(x, y)$ *continuous everywhere and vanishing at the origin* $x = y = 0$ *such that*

$$\|\sigma(t)\| \leqq S(Z, \|\xi(0)\|)\,, \quad \|\xi(t)\| \leqq K(Z, \|\xi(0)\|)\,, \quad t \geqq 0\,, \tag{1.8}$$

where $\sigma(t)$, $\xi(t)$ *is the solution of* (1.1), (1.2) *corresponding to* $z(t)$, $\xi(0)$. *Moreover, we have* $\sigma(t) \to 0$, $\xi(t) \to 0$ *as* $t \to \infty$.

Proof. First of all, from (1.6) we get easily $h_1\|\sigma\| \leqq \|f(\sigma)\| \leqq h_2\|\sigma\|$, $h_1\|\sigma\|^2 \leqq \leqq f^{`}(\sigma)\,\sigma \leqq h_2\|\sigma\|^2$ and

$$f^{`}(\sigma)\,\sigma - h^{-1}\|f(\sigma)\|^2 \geqq h_3\|\sigma\|^2 \tag{1.9}$$

with $h_3 = h^{-1}(hh_1 - h_2^2) > 0$.

Next, according to assumption a), the curvilinear integral $\int_{\sigma_1}^{\sigma_2} f^{`}(s)\,\mathrm{d}s$ is independent of the path joining points σ_1 and σ_2 and equals $U(\sigma_2) - U(\sigma_1)$. Thus, for a given σ take a line-segment joining the origin with σ as the integration path; then $s = \lambda\sigma$, $0 \leqq \lambda \leqq 1$, $\mathrm{d}s = \sigma\,\mathrm{d}\lambda$, and consequently,

$$U(\sigma) - U(0) = \int_0^1 f^{`}(\lambda\sigma)\,(\lambda\sigma)\,\lambda^{-1}\,\mathrm{d}\lambda\,.$$

Making use of (1.6), we get immediately

$$\tfrac{1}{2}h_1\|\sigma\|^2 \leqq U(\sigma) - U(0) \leqq \tfrac{1}{2}h_2\|\sigma\|^2\,. \tag{1.10}$$

Now, let $\sigma(t)$, $\xi(t)$ be the unique solution of (1.1), (1.2) corresponding to $z(t)$, $\xi(0)$; obviously, both $\sigma(t)$ and $\xi(t)$ have a continuous derivative for $t \geqq 0$. Choosing a $T > 0$ define the vector functions

$$\begin{aligned} f_T(t) &= f(\sigma(t)) \quad \text{for} \quad 0 \leqq t \leqq T\,, \\ &= 0 \qquad\qquad \text{elsewhere}\,, \end{aligned} \tag{1.11}$$

$$\begin{aligned} w_T(t) &= \sigma(t) + \gamma\xi(t) - z(t) && \text{for} \quad 0 \leqq t < T\,, \\ &= \int_0^T k(t - \tau)\,f(\sigma(\tau))\,\mathrm{d}\tau && \text{for} \quad t > T\,, \\ &= 0 && \text{for} \quad t < 0\,. \end{aligned} \tag{1.12}$$

It can be easily verified that $w'_T(t)$ exists everywhere in $(0, \infty)$ except at $t = T$ and that due to (1.3) both $w_T(t)$ and $w'_T(t)$ possess Fourier transforms $\tilde{w}_T(\omega)$, $\widetilde{(w'_T(t))}$, respectively. At the same time, we have $\widetilde{(w'_T(t))} = i\omega\, \tilde{w}_T(\omega)$, since $w_T(0) = 0$ by (1.12). Moreover, defining $k(t) = 0$ for $t < 0$, we have from (1.11), (1.12),

$$w_T(t) = \int_{-\infty}^{\infty} k(t - \tau) f_T(\tau)\, d\tau \tag{1.13}$$

for every t. Hence, by the convolution theorem [5], (1.13) yields

$$\tilde{w}_T(\omega) = \tilde{k}(\omega) \tilde{f}_T(\omega)\,, \tag{1.14}$$

where $\tilde{f}_T(\omega)$ is the Fourier transform of $f_T(t)$.

Next define the number $\varrho(T)$ by

$$\varrho(T) = \int_0^T f_T^{\backprime}\{w_T - h^{-1} f_T + q(w'_T - \gamma f_T)\}\, dt\,. \tag{1.15}$$

Since $f_T(t)$ vanishes outside $\langle 0, T\rangle$, bounds $-\infty$, ∞ may be written in the latter integral; thus we have by Parseval's equality [5],

$$\varrho(T) = \int_{-\infty}^{\infty} f_T^{\backprime}\{\ldots\}\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}_T^{\backprime}\widetilde{\{\ldots\}}\, d\omega\,.$$

Furthermore, since $\varrho(T)$ is real, we have by (1.14),

$$\begin{aligned} \varrho(T) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{Re} \tilde{f}_T^{\backprime}\{\tilde{w}_T - h^{-1}\tilde{f}_T + q(i\omega\tilde{w}_T - \gamma\tilde{f}_T)\}\, d\omega = \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{Re} \tilde{f}_T^{\backprime}\{(1 + i\omega q)\, \tilde{k}(\omega) - (h^{-1} I + \gamma q)\}\, \tilde{f}_T\, d\omega\,. \end{aligned} \tag{1.16}$$

Hence, by the assumption of the theorem, $\varrho(T) \leqq 0$. Invoking (1.11), (1.12), we have from (1.15),

$$\int_0^T f^{\backprime}(\sigma)\{\sigma - \gamma\xi - z - h^{-1} f(\sigma) + q(\sigma' - \gamma\xi' - z' - \gamma f(\sigma))\}\, dt \leqq 0\,, \tag{1.17}$$

and consequently, by (1.12),

$$\begin{aligned} &\int_0^T f^{\backprime}(\sigma)\{\sigma - h^{-1} f(\sigma)\}\, dt + \int_0^T \xi^{\backprime\prime}\gamma\xi\, dt + q\int_0^T f^{\backprime}(\sigma)\, \sigma'\, dt \leqq \\ &\leqq \int_0^T \xi^{\backprime\prime}(z + qz')\, dt = \\ &= \xi'(T)\,(z(T) + qz'(T)) - \xi(0)\,(z(0) + qz'(0)) - \int_0^T \xi^{\backprime}(t)\,(z' + qz'')\, dt \leqq \\ &\leqq \|\xi(T)\|\,(1 + q)\, Z + \|\xi(0)\|\,(1 + q)\, Z + \beta^{-1}(1 + q)\, Z \sup_{t\in\langle 0,T\rangle} \|\xi(t)\|\,. \end{aligned} \tag{1.18}$$

On the other hand, by (1.9),

$$\int_0^T f`(\sigma)\{\sigma - h^{-1} f(\sigma)\} \, dt \geqq h_3 \int_0^T \|\sigma(t)\|^2 \, dt \geqq 0 .$$

Moreover, since γ is symmetric,

$$\int_0^T \xi`'\gamma\xi \, dt = \tfrac{1}{2}\xi`(T)\,\gamma\xi(T) - \tfrac{1}{2}\xi`(0)\,\gamma\xi(0) .$$

At the same time, since γ is positive definite, there is a $\mu > 0$ such that

$$\xi`(T)\,\gamma\xi(T) \geqq \mu\|\xi(T)\|^2 .$$

Finally, we have

$$\int_0^T f`(\sigma(t))\,\sigma'(t) \, dt = \int_{\sigma(0)}^{\sigma(T)} f`(s) \, ds ,$$

where the latter integral is taken along the trajectory $\sigma(t)$ between points $\sigma(0)$ and $\sigma(T)$.

Introducing these relationships into (1.18) and making use of (1.10) we get

$$\begin{aligned} (1.19) \qquad & h_3 \int_0^T \|\sigma(t)\|^2 \, dt + \tfrac{1}{2}\mu\|\xi(T)\|^2 + \tfrac{1}{2}q h_1 \|\sigma(T)\|^2 \leqq \\ & \leqq \|\xi(0)\|\,(1 + q)\,Z + \tfrac{1}{2}\|\gamma\| \,.\, \|\xi(0)\|^2 + \tfrac{1}{2}q h_2 \|\sigma(0)\|^2 + \\ & + (1 + \beta^{-1})(1 + q)\,Z \sup_{t\in\langle 0,T\rangle} \|\xi(t)\| \leqq M_0 + M_1 \sup_{t\in\langle 0,T\rangle} \|\xi(t)\| \end{aligned}$$

with

$$\begin{aligned} M_0 &= \|\xi(0)\|\,(1 + q)\,Z + \tfrac{1}{2}\|\gamma\| \,.\, \|\xi(0)\|^2 + \tfrac{1}{2}q h_2 (Z + \|\gamma\| \,.\, \|\xi(0)\|)^2 , \\ M_1 &= (1 + \beta^{-1})(1 + q)\,Z . \end{aligned}$$

However, (1.19) implies that $\frac{1}{2}\mu\|\xi(T)\|^2 \leqq M_0 + M_1 \sup_{t\in\langle 0,T\rangle} \|\xi(t)\|$, and as this inequality is true for any $T > 0$, $\|\xi(t)\|$ must be bounded in $\langle 0, \infty)$. Putting $M_2 = \sup_{t\geqq 0} \|\xi(t)\|$, we have $\frac{1}{2}\mu M_2^2 \leqq M_0 + M_1 M_2$, and consequently,

$$(1.20) \qquad M_2 \leqq \mu^{-1}(M_1 + (M_1^2 + 2\mu M_0)^{1/2}) = K(Z, \|\xi(0)\|) ,$$

where $K(x, y)$ has the properties stated in the theorem.

On the other hand, (1.19) implies that

$$\tfrac{1}{2}q h_1 \|\sigma(T)\|^2 \leqq M_0 + M_1 K(Z, \|\xi(0)\|) ,$$

so that

$$\|\sigma(t)\| \leqq S(Z, \|\xi(0)\|) \tag{1.21}$$

for every $t \geqq 0$, where $S(x, y)$ has again the properties given in the theorem.

Furthermore, by (1.1), (1.2),

$$\sigma'(t) = z'(t) + \int_0^t k'(t - \tau) f(\sigma(\tau)) \, d\tau + k(0) f(\sigma(t)) - \gamma f(\sigma(t)) . \tag{1.22}$$

From this it follows by the above estimates that

$$\|\sigma'(t)\| \leqq M_3 , \quad t \geqq 0 . \tag{1.23}$$

Finally, again from (1.19), $h_3 \int_0^T \|\sigma(t)\|^2 \, dt \leqq M_0 + M_1 K$, i.e. $\int_0^\infty \sigma_i^2(t) \, dt \leqq \tilde{M}$ for i = 1, 2, ..., n, $\sigma_i(t)$ being the i-th component of $\sigma(t)$. From this and (1.23), however, we have $\sigma_i(t) \to 0$ as $t \to \infty$. Actually, assuming conversely that $\sigma_i(t)$ does not converge, we can find a $\delta > 0$ and a sequence $t_1 < t_2 < t_3 < \ldots$, $t_i \to \infty$ such that $|\sigma_i(t_k)| > \delta$ for $k = 1, 2, \ldots$; then by (1.23) there is an interval I_k with length δ / M_3 containing t_k such that $|\sigma_i(t)| > \delta/2$ for every $t \in I_k$, $k = 1, 2, \ldots$ Then, of course, we have for any integer $N > 0$, $\int_0^\infty \sigma_i^2(t) \, dt \geqq \delta^3 N / 4 M_3$, which is a contradiction. Hence, $\sigma(t) \to 0$ as $t \to \infty$.

Moreover, due to assumption (1.3) we have

$$\left\| \int_0^t k(t - \tau) f(\sigma(\tau)) \, d\tau \right\| \leqq C h_2 \int_0^t e^{-\alpha(t-\tau)} \|\sigma(\tau)\| \, d\tau \to 0$$

as $t \to \infty$; consequently, from (1.1), $\gamma \xi(t) \to 0$, and since γ is a regular matrix, $\xi(t) \to 0$ as $t \to \infty$. Hence, Th. 1.1 is proven.

The assumptions of Th. 1.1 may be modified as follows:

Theorem 1.2. *Let* (1.3) *and* (1.4) *be satisfied, and let γ be symmetric positive definite. Moreover, let $f(\sigma)$ fulfill condition* a) *of Th.* 1.1 *and the condition*

$$f^{`}(\sigma) \, \sigma > 0 \tag{1.24}$$

for every $\sigma \neq 0$. Let $\tilde{k}(\omega)$ have the same meaning as in Th. 1.1. *If there is a $q \geqq 0$ such that the matrix*

$$\tilde{A}(\omega) = (1 + i\omega q) \, \tilde{k}(\omega) - q\gamma \tag{1.25}$$

fulfills the condition $\operatorname{Re} \bar{\eta}^{`} \tilde{A}(\omega) \, \eta \leqq 0$ *for every real ω and every constant vector η, then the assertion of Th.* 1.1 *is true.*

Proof: From the continuity of $f(\sigma)$ we have $f(0) = 0$. Furthermore, $\|f(\sigma)\| > 0$ for $\sigma \neq 0$. Analogously as in the proof of Th. 1.1 we get

$$U(\sigma) - U(0) > 0 \quad \text{for} \quad \sigma \neq 0 . \tag{1.26}$$

Define vector functions $f_T(t)$, $w_T(t)$ again by (1.11), (1.12). Then (1.13), (1.14) are true. Instead of (1.15) put

$$\varrho(T) = \int_0^T f_T^{\prime}\{w_T + q(w_T' - \gamma f_T)\}\, \mathrm{d}t\,. \tag{1.27}$$

Using the same procedure as before we conclude that $\varrho(T) \leqq 0$, and from this the inequality

$$\begin{aligned}\int_0^T f^{\prime}(\sigma)\,\sigma\,\mathrm{d}t + \tfrac{1}{2}\xi^{\prime}(T)\,\gamma\xi(T) + q(U(\sigma(T)) - U(0)) \leqq \\ \leqq \overline{M}_0 + \overline{M}_1 \sup_{t\in\langle 0,T\rangle} \|\xi(t)\|\,,\end{aligned} \tag{1.28}$$

where $\overline{M}_0$, $\overline{M}_1$ are again continuous functions of Z and $\|\xi(0)\|$, which vanish for $Z = \|\xi(0)\| = 0$ and are independent of T.

From (1.28) we conclude that $\|\xi(t)\| \leqq \overline{K}(Z, \|\xi(0)\|)$ for every $t \geqq 0$, where $\overline{K}$ has the required properties.

On the other hand, substituting (1.2) into (1.1) and integrating by parts, we obtain

$$\begin{aligned}\sigma(t) &= z(t) + \int_0^t k(t-\tau)\,\xi'(\tau)\,\mathrm{d}\tau - \gamma\xi(t) = \\ &= z(t) + (k(0) - \gamma)\,\xi(t) + k(t)\,\xi(0) + \int_0^t k'(t-\tau)\,\xi(\tau)\,\mathrm{d}\tau\,.\end{aligned} \tag{1.29}$$

Hence,

$$\|\sigma(t)\| \leqq Z + (C + \|\gamma\|)\,\overline{K} + C\|\xi(0)\| + \alpha^{-1}C\overline{K} = \bar{S}(Z, \|\xi(0)\|)\,. \tag{1.30}$$

Furthermore, let $H = \sup_{\|\sigma\| \leqq \bar{S}} \|f(\sigma)\|$; then we have by (1.22),

$$\|\sigma'(t)\| \leqq Z + \alpha^{-1}CH + CH + \|\gamma\|\,H = \overline{M}_2\,. \tag{1.31}$$

Finally, from (1.28) we get $\int_0^T f^{\prime}(\sigma)\,\sigma\,\mathrm{d}t \leqq \overline{M}_0 + \overline{M}_1 K$, i.e.,

$$\int_0^\infty f^{\prime}(\sigma)\,\sigma\,\mathrm{d}t \leqq \overline{M}_3 \tag{1.32}$$

which together with (1.30), (1.31) implies that $\sigma(t) \to 0$ as $t \to \infty$. As a matter of fact, assume conversely that there is a $\delta > 0$ and a sequence $t_1 < t_2 < t_3 < \ldots$, $t_i \to \infty$ such that $\|\sigma(t_i)\| > \delta$. Since $\|\sigma\|^2 = \sigma^{\prime}\sigma$, we have $\|\sigma\| \,.\, \|\sigma\|' = \sigma^{\prime}\sigma'$, and consequently, $|\|\sigma\|'| \leqq \|\sigma'\|$ for $\sigma \neq 0$. Consider a point t_i; then for $|t - t_i| < \delta/2\overline{M}_2$ we have by the mean-value theorem, $\|\sigma(t)\| = \|\sigma(t_i)\| + \|\sigma(\xi)\|' (t - t_i)$ with ξ lying between t_i and t, and at the same time, $|\|\sigma(\xi)\|' (t - t_i)| \leqq \delta/2$. Consequently, $\|\sigma(t)\| > \delta/2$ on an interval I_i with length $\delta/\overline{M}_2$ which contains t_i. Putting now

$\eta = \inf_{\|\sigma\| > \delta/2} f'(\sigma)\,\sigma$, we have by (1.24), $\eta > 0$. Hence, for any integer $N > 0$, $\int_0^\infty f'(\sigma)\,\sigma\,dt \geqq N\eta\delta/\overline{M}_2$, which contradicts (1.32), Q.E.D.

From this we conclude in the same manner as in the proof of Th. 1.1 that $\xi(t) \to 0$ as $t \to \infty$. Thus, Th. 1.2 is proven.

The proofs of the previous theorems suggest that the requirements on $f(\sigma)$ may be relaxed as follows:

Theorem 1.3. *Let $k(t)$ fulfill condition (1.3) and let γ be a symmetric positive definite matrix; further, let $z(t)$ satisfy the inequalities*

$$\|z(t)\|,\ \|z'(t)\| \leqq Z \exp(-\beta t) \tag{1.33}$$

with a fixed $\beta > 0$. Let $f(\sigma)$ be a continuous vector function which satisfies the in equalities (1.6) with some $h_1 > 0$, h_2. If there is a $h > h_2^2 h_1^{-1}$ such that the matrix

$$A^*(\omega) = \tilde{k}(\omega) - h^{-1} I \tag{1.34}$$

(with $\tilde{k}(\omega)$ having the meaning given in Th. 1.1) fulfills the condition $\operatorname{Re} \bar{\eta}' A^*(\omega)\,\eta \leqq 0$ *for any real ω and η, then the assertion of Th. 1.1 is true.*

Proof: Here again (1.9) is true; defining $f_T(t)$, $w_T(t)$ by (1.11), (1.12), the equality (1.13) holds. Putting

$$\varrho(T) = \int_0^T f_T'\{w_T - h^{-1} f_T\}\,dt\,,$$

we obtain by Parseval's equality that $\varrho(T) \leqq 0$, and consequently,

$$\int_0^T f'(\sigma)\{\sigma - h^{-1} f(\sigma)\}\,dt + \int_0^T \xi'\gamma\xi\,dt \leqq \int_0^T \xi' z\,dt\,.$$

Arranging this inequality as in the proof of Th. 1.1, we get

$$h_3 \int_0^T \|\sigma(t)\|^2\,dt + \tfrac{1}{2}\mu\|\xi(T)\|^2 \leqq M_0^* + M_1^* \sup_{t\in\langle 0,T\rangle} \|\xi(t)\|\,, \tag{1.35}$$

where M_0^*, M_1^* depend continuously on Z, $\|\xi(0)\|$, vanish at the origin and are independent of T. From (1.35) we get immediately $\|\xi(t)\| \leqq K^*(Z, \|\xi(0)\|)$. From (1.30) it follows then that $\|\sigma(t)\| \leqq S^*(Z, \|\xi(0)\|)$.

The remaining part of the proof follows from inequalities (1.31) and $\int_0^\infty \|\sigma(t)\|^2\,dt \leqq M_2^*$.

The assumptions of Th. 1.3 on the behavior of $f(\sigma)$ may also be modified as in Th. 1.2. We have

Theorem 1.4. *Let $k(t)$, $z(t)$, γ fulfill the conditions in Th. 1.3. Let $f(\sigma)$ be a continuous vector function satisfying the inequality $f'(\sigma)\,\sigma > 0$ for $\sigma \neq 0$. If the matrix $\tilde{k}(\omega)$ satisfies the condition* $\operatorname{Re} \bar{\eta}' \tilde{k}(\omega)\,\eta \leqq 0$ *for every real ω and every η, then the assertion of Th. 1.1 is true.*

The proof follows the same pattern as that of Th. 1.3 and Th. 1.2 and therefore is omitted.

Note 1. Theorems 1.3 and 1.4 do not require $f(\sigma)$ to be a gradient, thus imposing obviously the weakest restrictions on $f(\sigma)$. On the other hand, condition a) in Th. 1.1 is satisfied, if the components $f_i(\sigma)$ of $f(\sigma)$ fulfill the condition $f_i(\sigma) = \varphi_i(\sigma_i)$, φ_i being a continuous scalar function, $i = 1, 2, \ldots, n$. Also, (1.6) are evidently true, if $h_1\sigma_i^2 \leqq \leqq \varphi_i(\sigma_i)\,\sigma_i \leqq h_2\sigma_i^2$ with some $h_1 > 0$, h_2, $i = 1, 2, \ldots, n$. Similarly, $\varphi_i(\sigma_i)\,\sigma_i > 0$ for $\sigma_i \neq 0$, $i = 1, 2, \ldots, n$ imply (1.24). (See also [2]).

Note 2. It can be easily verified that (1.1), (1.2) describe the behavior of any physical system, which consists of a linear subsystem with constant elements governed by the time-domain transfer-matrix $\gamma - k(t)$, and a non linear one, governed by the equation $\eta = f(\sigma)$. If, particularly, conditions of Th. 1.4 are satisfied, then both the linear and nonlinear system are dissipative, i.e. unable to produce energy.

2. In this part the system

$$\sigma(t) = z(t) + \int_0^t k(t - \tau)\,f(\sigma(\tau))\,\mathrm{d}\tau - a\,\xi(t) - b\,\eta(t)\,, \tag{2.1}$$

$$\xi'(t) = \eta(t)\,, \quad \eta'(t) = f(\sigma(t)) \tag{2.2}$$

with initial conditions $\xi(0)$, $\eta(0)$, where $\sigma(t)$, $\xi(t)$, $\eta(t)$, $z(t)$, $f(\sigma)$ are real-valued n-vectors and $k(t)$, a, b real $n \times n$ matrices, will be considered. As in part 1. we shall assume that conditions guaranteeing the existence and uniqueness of a solution $\sigma(t)$, $\xi(t)$, $\eta(t)$ are satisfied.

Theorem 2.1. *Let $k(t)$, $z(t)$ fulfill the condition*(1.3) *and* (1.4), *respectively, given in Th.* 1.1 *and let a be a symmetric positive definite matrix; furthermore, let $f(\sigma)$ satisfy conditions a), b) in Th.* 1.1, *and let $\tilde{k}(\omega)$ have the usual meaning. If there is a positive definite matrix ε (not necessarily symmetric and with $\|\varepsilon\|$ however small) such that the matrix*

$$A(\omega) = i\omega\tilde{k}(\omega) - b + \varepsilon \tag{2.3}$$

satisfies the condition $\operatorname{Re} \bar{v}' A(\omega)\, v \leqq 0$ *for every real ω ane every v, then there are functions $S(x, y, v)$, $K(x, y, v)$, $E(x, y, v)$ continuous everywhere and vanishing at the origin such that for a solution $\sigma(t)$, $\xi(t)$, $\eta(t)$ of* (2.1), (2.2) *corresponding to initial conditions $\xi(0)$, $\eta(0)$ we have*

$$\begin{gathered}\|\sigma(t)\| \leqq S(Z, \|\xi(0)\|, \|\eta(0)\|)\,, \quad \|\xi(t)\| \leqq K(Z, \|\xi(0)\|, \|\eta(0)\|)\,, \\ \|\eta(t)\| \leqq E(Z, \|\xi(0)\|, \|\eta(0)\|)\end{gathered} \tag{2.4}$$

for every $t \geqq 0$. Moreover, we have $\sigma(t) \to 0$, $\xi(t) \to 0$, $\eta(t) \to 0$ as $t \to \infty$.

Proof: First note that estimates (1.10) are true. Since $\sigma(t)$ has a derivative, we have from (2.1), (2.2),

$$\sigma'(t) = z'(t) + \int_0^t k'(t - \tau) f(\sigma(\tau))\,\mathrm{d}\tau + (k(0) - b) f(\sigma(t)) - a\,\eta(t)\,. \tag{2.5}$$

Choosing a $T > 0$ put

$$\begin{aligned} f_T(t) &= f(\sigma(t)) \quad \text{for} \quad 0 \leqq t \leqq T\,, \\ &= 0 \qquad\qquad \text{elsewhere}\,, \end{aligned} \tag{2.6}$$

$$\begin{aligned} w_T(t) &= \sigma'(t) - z'(t) - (k(0) - b) f(\sigma(t)) + a\,\eta(t) && \text{for} \quad 0 \leqq t \leqq T\,, \\ &= \int_0^T k'(t - \tau) f(\sigma(\tau))\,\mathrm{d}\tau && \text{for} \quad t > T\,, \\ &= 0 && \text{for} \quad t < 0\,. \end{aligned}$$

Obviously, both $f_T(t)$ and $w_T(t)$ possess Fourier transforms; moreover, it can be easily verified that

$$w_T(t) = \int_{-\infty}^{\infty} k'(t - \tau) f_T(\tau)\,\mathrm{d}\tau\,, \tag{2.7}$$

where we define $k'(t) = 0$ for $t < 0$. Thus, by the convolution theorem, $\tilde{w}_T = \widetilde{(k')}\tilde{f}_T$. But $\widetilde{(k')} = i\omega\tilde{k} - k(0)$ so that we have

$$\tilde{w}_T(\omega) = (i\omega\,\tilde{k}(\omega) - k(0))\,\tilde{f}_T(\omega)\,. \tag{2.8}$$

Next, put

$$\varrho(T) = \int_0^T f^{\backprime}(\sigma)\,(\sigma' + a\eta - z')\,\mathrm{d}t\,. \tag{2.9}$$

Using (2.6), Parseval's equality and (2.8) we get

(2.10)

$$\begin{aligned} \varrho(T) &= \int_{-\infty}^{\infty} f^{\backprime}_T\{w_T + (k(0) - b) f_T\}\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{Re} \bar{\tilde{f}}^{\backprime}_T\,\{\tilde{w}_T + (k(0)) - b)\,\tilde{f}_T\}\,\mathrm{d}\omega = \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{Re} \bar{\tilde{f}}^{\backprime}_T (A(\omega) - \varepsilon)\,\tilde{f}_T\,\mathrm{d}\omega \leqq -\frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{Re} \bar{\tilde{f}}^{\backprime}_T \varepsilon f_T\,\mathrm{d}\omega < 0\,. \end{aligned}$$

Since ε is positive definite, there is a number $\kappa > 0$ such that $\operatorname{Re} \bar{\tilde{f}}^{\backprime}_T \varepsilon \tilde{f}_T \geqq \kappa \bar{\tilde{f}}^{\backprime}_T \tilde{f}_T$. Hence,

$$\varrho(T) \leqq -\frac{\kappa}{2\pi}\int_{-\infty}^{\infty} \bar{\tilde{f}}^{\backprime}_T \tilde{f}_T\,\mathrm{d}\omega < 0\,. \tag{2.11}$$

Thus, from (2.9) we have by (2.2),

$$\int_{\sigma(0)}^{\sigma(T)} f^{\backprime}(s)\,\mathrm{d}s + \int_0^T \eta^{\backprime\prime} a\eta\,\mathrm{d}t < \int_0^T \eta^{\backprime\prime} z'\,\mathrm{d}t\,, \tag{2.12}$$

the trajectory $\sigma(t)$ being the integration path in the first integral.

Now, expressing the first integral in (2.12) by $U(\sigma)$ and estimating by (1.10), the second one by the primitive function $\frac{1}{2}\eta^{\backprime}(t)\,a\,\eta(t) \geqq v\|\eta(t)\|^2$ with $v > 0$, and integrating by parts the third one, we get the inequality

$$\begin{aligned} \tfrac{1}{2}h_1\|\sigma(T)\|^2 + v\|\eta(T)\|^2 &< \tfrac{1}{2}h_2\|\sigma(0)\|^2 + \tfrac{1}{2}\|a\|\,.\,\|\eta(0)\|^2 + \\ &+ Z\|\eta(0)\| + Z(1+\beta^{-1}) \sup_{t\in\langle 0,T\rangle} \|\eta(t)\|\,. \end{aligned} \tag{2.13}$$

Taking into account that $\|\sigma(0)\| \leqq Z + \|a\|\,.\,\|\xi(0)\| + \|b\|\,.\,\|\eta(0)\|$ by (2.1), we immediately conclude from (2.13) that

$$\|\eta(t)\| \leqq E(Z, \|\xi(0)\|, \|\eta(0)\|)\,, \quad \|\sigma(t)\| \leqq S(Z, \|\xi(0)\|, \|\eta(0)\|) \tag{2.14}$$

for every $t \geqq 0$, where the functions E, S possess the properties given in the theorem. Since a is a regular matrix it follows from (2.1) by (2.14) that

$$\|\xi(t)\| \leqq K(Z, \|\xi(0)\|, \|\eta(0)\|\,, \quad t \geqq 0\,. \tag{2.15}$$

Moreover, by (2.2), $\|\xi'(t)\| \leqq E$, and $\|\sigma'(t)\| \leqq M_1$ from (2.5).

Next, referring back to (2.9), we have

$$\begin{aligned} -\varrho(T) &= -\int_0^T f^{\backprime}\sigma'\,\mathrm{d}t - \int_0^T f^{\backprime} a\eta\,\mathrm{d}t + \int_0^T f^{\backprime} z'\,\mathrm{d}t = \\ &= -U(\sigma(T)) + U(\sigma(0)) + \tfrac{1}{2}(\eta^{\backprime}(T)\,a\,\eta(T) - \eta^{\backprime}(0)\,a\,\eta(0)) + \\ &+ \eta^{\backprime}(T)\,z'(T) - \eta^{\backprime}(0)\,z'(0) - \int_0^T \eta^{\backprime}\,z''\,\mathrm{d}t \leqq \\ &\leqq \frac{h_2}{2}(\|\sigma(\tau)\|^2 + \|\sigma(0)\|^2) + \tfrac{1}{2}\|a\|\,(\|\eta(T)\|^2 + \|\eta(0)\|^2) + Z(\|\eta(T)\| + \|\eta(0)\|) + \\ &+ \beta^{-1}Z \sup_{t\in\langle 0,T\rangle} \|\eta(t)\| \leqq h_2S^2 + \|a\|\,E^2 + 2ZE + \beta^{-1}ZE = M_2\,, \end{aligned} \tag{2.16}$$

where M_2 is independent of T. Thus, by (2.11),

$$M_2 \geqq -\varrho(T) \geqq \frac{\kappa}{2\pi}\int_{-\infty}^{\infty} \tilde{\bar{f}}_T^{\backprime}\tilde{f}_T\,\mathrm{d}\omega > 0 \tag{2.17}$$

for any $T > 0$. Consequently, by Parseval's equality,

$$0 < \kappa \int_{-\infty}^{\infty} f_T' f_T \, dt \leqq M_2 \,, \quad \text{i.e., by (2.6)} \,,$$

$$\kappa \int_0^T \|f(\sigma(t))\|^2 \, dt \leqq M_2 \tag{2.18}$$

for any $T > 0$. Since $\|f(\sigma)\| \geqq h_1 \|\sigma\|$, we have $\int_0^\infty \|\sigma(t)\|^2 \, dt < M_3$ so that in view of (2.14) and the inequality $\|\sigma'(t)\| \leqq M_1$ we easily conclude as before that $\sigma(t) \to 0$ as $t \to \infty$.

Thus, summarizing we have $\|\xi(t)\| \leqq K$, $\|\xi'(t)\| \leqq E$, $\|\xi''(t)\| \leqq M_3$ and $\xi''(t) \to 0$ as $t \to \infty$. (The last two relationships follow from (2.2)). From this, however, we get $\eta(t) = \xi'(t) \to 0$ as $t \to \infty$. As a matter of fact, assume conversely that $\xi'(t) \to 0$ is not true, i.e. that there is a component $\xi_k'(t)$ of $\xi'(t)$, a $\delta > 0$ and a sequence $t_1 < t_2 < t_3 < \ldots, t_i \to \infty$ such that $|\xi_k'(t_i)| > \delta$ for $i = 1, 2, \ldots$ Choose an index i and consider first the case that $\xi_k'(t_i) > \delta$; then there is a t_i^* such that $\xi_k'(t) > \delta/2$ for $t_i \leqq t < t_i^*$ and $\xi_k'(t_i^*) = \delta/2$. Such a t_i^* really exists as in the opposite case we would have $\xi_k'(t) > \delta/2$ for every $t \geqq t_i$, and consequently,

$$\xi_k(t) - \xi_k(t_i) = \int_{t_i}^t \xi_k'(\tau) \, d\tau > \frac{\delta}{2}(t - t_i) \,,$$

which would contradict the assumption $|\xi_k(t)| \leqq K$. Thus, we have

$$\xi_k(t_i^*) - \xi_k(t_i) > \frac{\delta}{2}(t_i^* - t_i) \,,$$

and since the left hand side does not exceed $2K$, we get

$$t_i^* - t_i < 4K/\delta \,. \tag{2.19}$$

Furthermore, by the mean-value theorem there is a τ_i with $t_i < \tau_i < t_i^*$ such that

$$\xi_k''(\tau_i) = \frac{\xi_k'(t_i) - \xi_k'(t_i^*)}{t_i - t_i^*} \,,$$

so that $|\xi_k''(\tau_i)| > \frac{\delta}{2} |t_i^* - t_i|^{-1}$, i.e. by (2.19),

$$|\xi_k''(\tau_i)| > \delta^2/8K \,. \tag{2.20}$$

Repeating the whole consideration for the case that $-\xi_k'(t_i) > \delta$, we conclude that (2.20) is again true. Hence, we have found a sequence $\tau_1 < \tau_2 < \tau_3 < \ldots, \tau_i \to \infty$ such that (2.20) is true for $i = 1, 2, 3, \ldots$, which contradicts the fact that $\xi''(t) \to 0$. Thus, $\xi'(t) \to 0$ as $t \to \infty$.

Finally, starting from (2.1) and using the fact that a is a nonsingular matrix we obtain $\xi(t) \to 0$ as $t \to \infty$; hence, Th. 2.1 is proven.

3. In this part the vector equation

$$\sigma(t) = z(t) + \int_0^t k(t-\tau) f(\sigma(\tau))\, d\tau\,, \tag{3.1}$$

where $\sigma(t)$, $f(\sigma)$, $z(t)$ are real n-vectors, $k(t)$ a real $n \times n$ matrix will be considered.

Assuming again that conditions guaranteeing the existence and uniqueness of a solution of (3.1) are satisfied, we have the following assertion:

Theorem 3.1. *Let $k(t)$ fulfill condition (1.3), $z(t)$ condition (1.33); furthermore, let $f(\sigma)$ satisfy conditions a), b) in Th. 1.1 and let $h > h_2^2 h_1^{-1}$. If there is a $q > 0$ such that the matrix*

$$A(\omega) = (1 + i\omega q)\, \tilde{k}(\omega) - h^{-1} I \tag{3.2}$$

with $\tilde{k}(\omega)$ having the usual meaning fulfills the inequality $\operatorname{Re} \bar{\eta}' A(\omega)\, \eta \leqq 0$ for every real ω and every η, then there is a continuous function $S(x)$ vanishing at $x = 0$ such that

$$\|\sigma(t)\| \leqq S(Z)\,, \quad t \geqq 0\,, \tag{3.3}$$

where $\sigma(t)$ is a solution of (3.1). Moreover, we have $\sigma(t) \to 0$ as $t \to \infty$.

Proof: Choosing a $T > 0$ put

$$\begin{aligned}
f_T(t) &= f(\sigma(t)) && \text{for } 0 \leqq t \leqq T, \\
&= 0 && \text{elsewhere}, \\
\sigma_T(t) &= \sigma(t) && \text{for } 0 \leqq t \leqq T, \\
&= 0 && \text{elsewhere}, \\
z_T(t) &= z(t) && \text{for } 0 \leqq t \leqq T, \\
&= -\int_0^T k(t-\tau) f(\sigma(\tau))\, d\tau && \text{for } t > T, \\
&= 0 && \text{elsewhere};
\end{aligned}$$

then we have $\sigma_T(t) - z_T(t) = \int_{-\infty}^{\infty} k(t-\tau) f_T(\tau)\, d\tau$, and consequently, $\tilde{\sigma}_T - \tilde{z}_T = \tilde{k}\tilde{f}_T$. Defining $\varrho(T)$ by

$$\varrho(T) = \int_0^T f'(\sigma) \{\sigma - z - h^{-1} f(\sigma) + q(\sigma' - z')\}\, dt\,, \tag{3.4}$$

we easily obtain by Parseval's equality that $\varrho(T) \leqq 0$. Following the pattern of previous proofs we conclude from (3.4) that

$$\begin{aligned}
\tfrac{1}{2} q h_1 \|\sigma(T)\|^2 + h_3 \int_0^T \|\sigma(t)\|^2\, dt \leqq \\
\leqq \tfrac{1}{2} q h_2 \|\sigma(0)\|^2 + \beta^{-1} Z(1 + q)\, h_2 \sup_{t \in \langle 0, T\rangle} \|\sigma(t)\|\,.
\end{aligned} \tag{3.5}$$

From this, however, we have immediately (3.3). Taking then the first derivative of (3.1) and making use of (3.3), we get $\|\sigma'(t)\| \leqq M$. From $\int_0^T \|\sigma(t)\|^2 \, dt \leqq M'$ we conclude that $\sigma(t) \to 0$ as $t \to \infty$ which finishes the proof.

Note 3. The assertion of Th. 3.1 remains true if condition b) is replaced by $f'(\sigma)\,\sigma > 0$ for $\sigma \neq 0$ and $\|f(\sigma)\| \leqq \kappa\|\sigma\|$, and the matrix (3.2) by $A'(\omega) = (1 + i\omega q)\,\tilde{k}(\omega)$. The proof of this follows from Th. 3.1 applied on an equation obtained from (3.1) by substitution $f(\sigma) = g(\sigma) - \varepsilon\sigma$, where $\varepsilon > 0$ is sufficiently small; its idea is the same as that used in [6], pp. 56.

4. In this part we shall consider certain systems related to (1.1), (1.2) and (2.1), (2.2) and (3.1) on the one hand, and a relationship of the above results to the Liapunov's theory on the other.

Let the system

$$x' = Ax + Bf(\sigma)\,, \quad \xi' = f(\sigma)\,, \quad \sigma = Cx - \gamma\xi\,, \tag{4.1}$$

where $\sigma(t)$, $\xi(t)$, $f(\sigma)$ are n-vectors, $x(t)$ an m-vector, A a real constant $m \times m$ matrix B a real constant $m \times n$ matrix, C a real constant $n \times m$ matrix, γ a real constant $n \times n$ matrix, with initial conditions $x(0)$, $\xi(0)$ be given.

From the first equation (4.1) we have

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}\, Bf(\sigma(\tau))\, d\tau\,;$$

substituting this into the second one, we get

$$\sigma(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}\, Bf(\sigma(\tau))\, d\tau - \gamma\, \xi(t)\,. \tag{4.2}$$

Hence, system (4.1) is equivalent to (4.2) with $\xi' = f(\sigma)$, i.e. reduces to the system (1.1), (1.2).

Thus, if every eigenvalue of A has a negative real part, the vector $z(t) = Ce^{At}x(0)$ satisfies condition (1.4) and the matrix $k(t) = Ce^{At}B$ satisfies (1.3). Moreover, if $f(\sigma)$ fulfills the conditions stated either in Th. 1.1 or Th. 1.2, we have $f(0) = 0$, and consequently, (4.1) possesses the trivial solution $x = 0$, $\xi = \sigma = 0$. Thus, we have the assertion:

If the assumptions of any one of the theorems 1.1 *to* 1.4 *are satisfied by* $k(t)$ *defined above,* $f(\sigma)$ *and* γ, *then the trivial solution of* (4.1) *is stable in large, i.e. it is stable and asymptotically stable, the corresponding stability regions being the entire space.*

Next, consider the system

$$x' = Ax + Bf(\sigma)\,, \quad \xi' = \eta\,, \quad \eta' = f(\sigma)\,, \quad \sigma = Cx - a\xi - b\eta \tag{4.3}$$

with initial conditions $x(0)$, $\xi(0)$, $\eta(0)$, where $\xi(t)$, $\eta(t)$, $\sigma(t)$, $f(\sigma)$ are n-vectors, $x(t)$ an m-vector, and A, B, C, a, b constant matrices of corresponding types. Expressing again $x(t)$ from the first equation (4.3) and substituting it into the last one, we get

$$(4.4) \qquad \sigma(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bf(\sigma(\tau))\,d\tau - a\,\xi(t) - b\,\eta(t)\,,$$

i.e. the equation (2.1). Thus, we have the assertion:

If all the eigenvalues of A have negative real parts and the matrices a, b, $k(t) = Ce^{At}B$ together with $f(\sigma)$ fulfill the requirements given in Th. 2.1, then the trivial solution of (4.3) is stable in large.

Finally, it is obvious that the system

$$(4.5a) \qquad x' = Ax + Bf(\sigma)\,, \quad \sigma = Cx$$

reduces to (3.1); hence, Th. 3.1 may be applied.

Now, let us pay our attention to a relationship with the Liapunov's theory.

Obviously, (4.1) is equivalent to the system

$$(4.5) \qquad x' = Ax + Bf(\sigma)\,, \quad \sigma' = CAx + (CB - \gamma)f(\sigma)\,.$$

In the following we shall assume that all eigenvalues of A have negative real parts and γ is symmetric. For the sake of brevity introduce the notation:

$$(4.6) \qquad v(t) = -Ce^{At}B\,, \quad N(i\omega) = \int_0^\infty v(t)\exp(-i\omega t)\,dt\,,$$

$$G(i\omega) = N(i\omega) + (i\omega)^{-1}\gamma\,.$$

Furthermore, let $\mathfrak{A}$ be the set of all real continuous n-vector functions $f(\sigma)$, which are gradients and satisfy the condition

$$f^{\backprime}(\sigma)\,\sigma > 0 \quad \text{for} \quad \sigma \neq 0\,.$$

Then we have:

Theorem 4.1. *If there is a positive definite matrix H and a $\beta > 0$ such that for any $f(\sigma) \in \mathfrak{A}$ the function*

$$(4.7) \qquad V = x^{\backprime}Hx + 2\beta(U(\sigma) - U(0))$$

with $f(\sigma) = \operatorname{grad} U(\sigma)$ is a Liapunov function for (4.5), i.e. V' along the trajectory is negative for $x \neq 0$ or $\sigma \neq 0$, then there is number $q > 0$ such that

$$(4.8) \qquad \operatorname{Re} \bar{\eta}^{\backprime}(1 + i\omega q)\,G(i\omega)\,\eta \geqq 0$$

for every real ω and every η.

Observe that condition (4.8) coincides with the condition imposed on the matrix $\tilde{A}(\omega)$ in Th. 1.2, provided $\tilde{k}(\omega) = -N(i\omega)$. (Compare also with 4.6 and 4.2).

Proof of Th. 4.1. Choose a symmetric positive definite $n \times n$ matrix h and specify $f(\sigma) = h\sigma$. Then obviously $f(\sigma) \in \mathfrak{A}$. Actually, we have $f`(\sigma)\,\sigma = \sigma`h\sigma > 0$ for every $\sigma \neq 0$. Moreover, putting $U(\sigma) = \frac{1}{2}\sigma`h\sigma$, we get

$$\frac{\partial U(\sigma)}{\partial \sigma_i} = \left(\frac{\partial \sigma}{\partial \sigma_i}\right)` h\sigma\,, \quad i = 1, 2, \ldots, n\,,$$

and consequently, grad $U(\sigma) = h\sigma = f(\sigma)$.

Thus, we have

$$V = x`Hx + \beta\sigma`h\sigma\,, \tag{4.9}$$

and by (4.5) with $f(\sigma) = h\sigma$,

$$\begin{aligned} V' &= x`(H + H`)\,x' + 2\beta\sigma`h\sigma' = \\ &= x`(H + H`)\,Ax + x`\{(H + H`)\,Bh + 2\beta A`C`h\}\,\sigma + \\ &\quad + 2\beta\sigma`h(CB - \gamma)\,h\sigma\,. \end{aligned} \tag{4.10}$$

It can be easily verified that with

$$Q = \left[\begin{array}{c|c} (H + H`)\,A & (H + H`)\,Bh \\ \hline 2\beta hCA & 2\beta h(CB - \gamma)\,h \end{array}\right], \quad w = \left[\begin{array}{c} x \\ \hline \sigma \end{array}\right] \tag{4.11}$$

we have $V' = w`Qw$. Since by assumption Q is negative definite, we obtain for any complex vector $w \neq 0$ that

$$\operatorname{Re} \bar{w}`Qw < 0\,. \tag{4.12}$$

Consequently, expanding (4.12),

$$\begin{aligned} &\operatorname{Re}\,\{\bar{x}`(H + H`)\,Ax + \bar{x}`(H + H`)\,Bh\sigma + \\ &+ \bar{\sigma}`2\beta hCAx + \bar{\sigma}`\,2\beta h(CB - \gamma)\,h\sigma\} < 0 \end{aligned} \tag{4.13}$$

for $x \neq 0$ or $\sigma \neq 0$.

Define the $m \times n$ matrix $M(i\omega)$ by

$$M(i\omega) = \int_0^\infty e^{-i\omega t} e^{At} B\, dt\,. \tag{4.14}$$

Then obviously

$$i\omega\, M(i\omega) = B + AM(i\omega)\,. \tag{4.15}$$

Next, choose a constant n-vector $\eta \neq 0$, an ω and put $x = M(i\omega)\,\eta$, $\sigma = h^{-1}\eta$. Since $\sigma \neq 0$, we have from (4.13),

$$\text{(4.16)} \qquad \operatorname{Re} \bar{\eta}'\{\bar{M}'(H + H')\,AM + \bar{M}'(H + H')\,B + 2\beta CAM + \\ + 2\beta(CB - \gamma)\}\,\eta < 0\,.$$

Invoking (4.15) it follows that

$$\text{(4.17)} \qquad \operatorname{Re} \bar{\eta}'\,\{i\omega\bar{M}'(H + H')\,M + i\omega\,2\beta CM - 2\beta\gamma\}\,\eta < 0\,.$$

Since $\bar{\eta}'\bar{M}'(H + H')\,M\eta$ is real and $\beta > 0$, (4.17) yields

$$\text{(4.18)} \qquad \operatorname{Re} \bar{\eta}'\{i\omega CM - \gamma\}\,\eta < 0\,.$$

On the other hand, from (4.6), (4.14) we have $N(i\omega) = -\,CM(i\omega)$; consequently,

$$\text{(4.19)} \qquad \operatorname{Re} \bar{\eta}'\{i\omega\,N(i\omega) + \gamma\}\,\eta = \operatorname{Re} \bar{\eta}'\{i\omega\,G(i\omega)\}\,\eta > 0$$

for every ω and $\eta \neq 0$.

Now we are going to show that there are constants $K_1 > 0$, $K_2 > 0$ such that

$$\text{(4.20)} \qquad \operatorname{Re} \bar{\eta}'\{i\omega\,G(i\omega)\}\,\eta \geqq K_1\|\eta\|^2\,,$$

$$\text{(4.21)} \qquad |\operatorname{Re} \bar{\eta}'\,G(i\omega)\,\eta| \leqq K_2\|\eta\|^2$$

for every ω and η.

As a matter of fact, since $\|v(t)\| \leqq K_3 \exp(-\alpha t)$ with $\alpha > 0$, we have from (4.6), $\|N(i\omega)\| \leqq K_3\alpha^{-1}$ for every ω. At the same time,

$$|\operatorname{Re} \bar{\eta}'\,G(i\omega)\,\eta| = |\operatorname{Re}\{\bar{\eta}'\,N(i\omega)\,\eta + (i\omega)^{-1}\,\bar{\eta}'\gamma\eta\}| = \\ = |\operatorname{Re} \bar{\eta}'\,N(i\omega)\,\eta| \leqq \|\eta\|^2\,\|N(i\omega)\| \leqq \alpha^{-1}K_3\|\eta\|^2\,.$$

Thus, (4.21) is established.

Next, the integration by parts yields

$$\int_0^\infty e^{-i\omega t}\,v''(t)\,dt = -v'(0) - i\omega\,v(0) + (i\omega)^2\,N(i\omega)\,.$$

Since $v''(t) = -CA^2e^{At}B$, we have $\|\int_0^\infty e^{-i\omega t}\,v''(t)\,dt\| \leqq K_4$ for every ω. Consequently,

$$\text{(4.22)} \qquad i\omega\,N(i\omega) - v(0) \to 0 \quad \text{as} \quad |\omega| \to \infty\,.$$

On the other hand, from (4.13) we have for $x = 0$, $\sigma = h^{-1}\eta \neq 0$,

$$\text{(4.23)} \qquad \operatorname{Re} \bar{\eta}'(CB - \gamma)\,\eta < 0\,.$$

Since $v(0) = -CB$ by (4.6), we have by (4.22),

$$i\omega\, G(i\omega) = i\omega\, N(i\omega) + \gamma \to \gamma - CB = F\,, \tag{4.24}$$

which is a positive definite matrix in view of (4.23). Thus, there is a $K_5 > 0$ such that $\operatorname{Re} \bar{\eta}' F \eta \geqq K_5$ for any η with $\|\eta\| = 1$.

On the other hand, by (4.24) there is a $P > 0$ such that $\|i\omega\, G(i\omega) - F\| < K_5/2$ for every $|\omega| \geqq P$. Thus, for every $|\omega| \geqq P$ and η with $\|\eta\| = 1$ we have

$$\operatorname{Re} \bar{\eta}'\, i\omega\, G(i\omega)\, \eta = \operatorname{Re} \bar{\eta}' F \eta + \operatorname{Re} \bar{\eta}'(i\omega\, G(i\omega) - F)\, \eta$$

and

$$|\operatorname{Re} \bar{\eta}'(i\omega\, G(i\omega) - F)\, \eta| \leqq \|i\omega\, G(i\omega) - F\| < K_5/2\,.$$

Hence,

$$\operatorname{Re} \bar{\eta}'\, i\omega\, G(i\omega)\, \eta > K_5/2 \tag{4.25}$$

for any $|\omega| \geqq P$ and $\|\eta\| = 1$.

Next, choose an $\omega_0 \in \langle -P, P \rangle$; in view of (4.19) there is a $K_{\omega 0} > 0$ such that

$$\operatorname{Re} \bar{\eta}' \{i\omega_0\, G(i\omega_0)\}\, \eta \geqq K_{\omega 0} \tag{4.26}$$

for any η with $\|\eta\| = 1$. Since the matrix $i\omega\, G(i\omega)$ is continuous for every ω, there is a neighborhood $I_{\omega 0}$ of ω_0 such that $\|i\omega\, G(i\omega) - i\omega_0\, G(i\omega_0)\| < K_{\omega 0}/2$ for any $\omega \in I_{\omega 0}$. But as

$$\operatorname{Re} \bar{\eta}'\, i\omega\, G(i\omega)\, \eta = \operatorname{Re} \bar{\eta}'\, i\omega_0\, G(i\omega_0)\, \eta + \operatorname{Re} \bar{\eta}' \{i\omega\, G(i\omega) - i\omega_0\, G(i\omega_0)\}\, \eta$$

and

$$|\operatorname{Re} \bar{\eta}' \{i\omega\, G(i\omega) - i\omega_0\, G(i\omega_0)\}\, \eta| < K_{\omega 0}/2$$

for any $\omega \in I_{\omega 0}$ and $\|\eta\| = 1$, we have

$$\operatorname{Re} \bar{\eta}'\, i\omega\, G(i\omega)\, \eta > \tfrac{1}{2} K_{\omega 0}\,, \quad \omega \in I_{\omega 0}\,, \quad \|\eta\| = 1\,. \tag{4.27}$$

The system of all intervals $I_{\omega 0}$, $\omega_0 \in \langle -P, P \rangle$, however, covers $\langle -P, P \rangle$; hence, by Borel's theorem, there is a finite subsystem with the same property. Combining this result with (4.25) we conclude that there is a $K_1 > 0$ such that

$$\operatorname{Re} \bar{\eta}' \{i\omega\, G(i\omega)\}\, \eta \geqq K_1$$

for any ω and η with $\|\eta\| = 1$, i.e. that (4.20) is true. But from (4.20), (4.21) it follows that there is a $q > 0$ such that (4.8) is true for every ω and η. Hence, Th. 4.1 is proven.

Observe that Th. 4.1 is true for $\gamma = 0$ and that (4.5a) is equivalent to

$$x' = Ax + Bf(\sigma)\,, \quad \sigma' = CAx + CBf(\sigma)\,, \tag{4.28}$$

i.e. to (4.5) with $\gamma = 0$; hence, Th. 4.1 appears also as a counterpart of Th. 3.1 considered for (4.28).

References

[1] *Попов В. М.*: Об одном критическом случае абсолютной устойчивости, Автом. и телемех., Т. *XXIII.*, № 1, 1962.

[2] *Popov V. N.:* Nouveaux critériums de stabilité pour les systèmes automatiques non-linéaires, Rev. d'électrotechn. et d'énergetique, Acad. R. P. R., t. *V.*, No 1, 1960.

[3] *Halanay A.:* Teoria calitativă a ecuaţiilor diferenţiale, Editura Academiei Rep. Pop. Romine, 1963.

[4] *Халанай А.*: Абсолютная устойчивость некоторых нелинейных регулируемых систем с запаздыванием, Автом. и телемех., Т. *XXV*, № 3, 1964.

[5] *S. Bochner, K. Chandrasekharan:* Fourier Transforms, Princeton Univ. Press, 1949.

[6] *М. А. Айзерман, Ф. Р. Гантмахер:* Абсолютная устойчивость регулируемых систем, Изд. А. Н. СССР, 1963.

Резюме

РАСПРОСТРАНЕНИЕ МЕТОДА ПОПОВА НА ВЕКТОРНЫЕ НЕЛИНЕЙНОСТИ

ВАЦЛАВ ДОЛЕЖАЛ (Václav Doležal), Прага

В статье выведены некоторые достаточные условия для абсолютной устойчивости решения определенных типов векторных интегродифференциальных уравнений Вольтерра. Оказывается, что устойчивость решения гарантирована выполнением определенного алгебраического условия, налагаемого на образ ядра по Фурье.

В первой части работы исследуется система уравнений (1.1), (1.2), во второй части система (2.1), (2.2) и в третьей части уравнение (3.1).

В четвертой части показаны, во-первых, приложения результатов к системам уравнений (4.1), (4.3), (4.5a) и, во-вторых, связь с теорией устойчивости Ляпунова. Доказана теорема, утверждающая, что если (4.7) является функцией Ляпунова для всех систем (4.5), то выполнено условие (4.8), равносильное достаточному условию устойчивости, высказанному в теореме 1.2.