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A CONDITION OF THE EXISTENCE OF AN UNIVERSAL BEST ε-STABILIZING CONTROL

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This note is a direct continuation of the paper [1]; therefore, we shall use the concepts and notations introduced in [1] without defining them again.

In [1] the concept of the best ε-stabilizing control and of the ε_0-universal best ε-stabilizing control under a given class of perturbations was introduced. It was shown that if the best ε-stabilizing control u_ε in the sense of the Euclidean norm in the system (s_ε) exists for $\varepsilon \in (0, \varepsilon_0\rangle$ and if for every $\varepsilon \leqq \eta \leqq \varepsilon_0$ the relation

$$y_\varepsilon \in G_\eta \tag{1}$$

is valid, then also the ε_0-universal best ε-stabilizing control in the sense of the Euclidean norm exists. It is easy to see that the validity of (1) for sufficiently small $\eta \leqq 0$ and $\varepsilon \leqq \eta$ is also necessary for the existence of an universal best ε-stabilizing control in (s_ε) in the sense of the Euclidean norm. Further it was shown by an example, that the condition (1) is not always satisfied. The objective of this note is to investigate the question of the validity of condition (1).

Let us denote

$$k = \min \{p^2 : (-1, p^2) \in P\} .$$

The result may be stated as a

Theorem. *If $\beta + k > 0$, then for sufficiently small $\eta > 0$ and every $\varepsilon \leqq \eta$ the relation (1) is valid. If $\beta + k < 0$, then for sufficiently small $\eta > 0$ and $\varepsilon \leqq \eta$ sufficiently close to η the relation*

$$y_\varepsilon \notin G_\eta \tag{2}$$

is valid.

Lemma. *Let us denote*

$$k_\varepsilon = \min \{\bar{p}^2 : (\bar{p}^1, \bar{p}^2) \in P,\ (a, \bar{p}) = \max_{p \in P} (a, p)\,,\ a^2 \geqq -\varepsilon,\ a^1 \leqq 0,\ \|a\| = 1\}\ ^{1)} .$$

[1]) If a, b are vectors, we shall denote their scalar product by (a, b): although the vector with the components a, b is denoted in the same way provided a, b are numbers, the meaning of the symbol will be always evident from the context. The minimum in the formula is taken over all a, satisfying the given conditions.

Then $$\liminf_{\varepsilon\to 0} k_\varepsilon \geqq k\,.$$

Proof. Let us suppose the contrary. Then an $\kappa > 0$ and sequences of vectors $\{a_n\}$, $\{p_n\}$ exist such that $a_n^2 > -n^{-1}$, $a_n^1 \leqq 0$, $\|a_n\| = 1$, $p_n \in P$, $(a_n, p_n) = \max_{p\in P}(a_n, p)$ $p_n^2 < k - \kappa$. Further, a sequence $\{n_\nu\}$ of positive integers exists such that the sequences $\{a_{n_\nu}\}$, $\{p_{n_\nu}\}$ are convergent. Denote a_0, p_0 their limits. Then, we have evidently $a_0^2 \geqq 0$, $a_0^1 \leqq 0$, $\|a_0\| = 1$, $p_0 \in P$, $p_0^2 \leqq k - \kappa$ and

$$(3)\qquad (a_0, p_0) = \max_{p\in P}(a_0, p)\,.$$

Hence, as $(-1, k) \in P$,

$$(a_0, p_0) \geqq (a_0, (-1, k))\,,\quad a_0^1 p_0^1 + a_0^2 p_0^2 \geqq -a_0^1 + a_0^2 k\,,\quad a_0^1(p_0^1 + 1) - a_0^2\kappa \geqq 0\,.$$

This is possible only if $p_0^1 \leqq -1$, which contradicts (3) and the definition of k.

Let us now denote

$$p_\varepsilon^2(y) = \min_{p\in P_m(y)} p^2\,,\quad p_\varepsilon^1(y) = \min_{p\in P_m(y)\,,\, p^2 = p_\varepsilon{}^2(y)} p^1\,,$$

where $P_m(y)$ is defined in the proof of Lemma 4.3 of [1]. With the aid of non-essential changes in the proof of Lemma 4.3 of [1] it may be easily shown, that $p_\varepsilon(y)$ is measurable. Further, we denote for brevity $p_\varepsilon(y^2) = p_\varepsilon(\gamma_\varepsilon^+(y^2), y^2)$. We have

$$(4)\qquad \frac{y^2 + p_\varepsilon^1(y^2)}{\alpha\gamma_\varepsilon^+(y^2) + \beta y^2 + 1 + p_\varepsilon^2(y^2)} = \min_{p\in\varepsilon P} \frac{y^2 + p^1}{\alpha\gamma_\varepsilon^+(y^2) + \beta y^2 + 1 + p^2}$$

and, hence,

$$(5)\qquad \begin{aligned} &-p_\varepsilon^1(y^2)\,(\alpha\gamma_\varepsilon^+(y^2) + \beta y^2 + 1 + p_\varepsilon^2(y^2)) + p_\varepsilon^2(y^2)\,(y^2 + p_\varepsilon^1(y^2)) = \\ &= \max_{p\in\varepsilon P}\{-p^1(\alpha\gamma_\varepsilon^+(y^2) + \beta y^2 + 1 + p_\varepsilon^2(y^2)) + p^2(y^2 + p_\varepsilon^1(y^2))\} \end{aligned}$$

Denote further $a_\varepsilon(y^2)$ the unit vector, parallel to the vector $(-\alpha\gamma_\varepsilon^+(y^2) - \beta y^2 - 1 - p_\varepsilon^2(y^2), y^2 + p_\varepsilon^1(y^2))$. From (5) it follows that

$$(p_\varepsilon(y^2), a_\varepsilon(y^2)) = \max_{p\in\varepsilon P}(p, a_\varepsilon(y^2))\,,\quad (\varepsilon^{-1} p_\varepsilon(y^2), a_\varepsilon(y^2)) = \max_{p\in P}(p, a_\varepsilon(y^2))\,.$$

For $\varepsilon > 0$ sufficiently small we have

$$|\alpha\gamma_\varepsilon^+(y^2) + \beta y^2 + 1 + p_\varepsilon^2(y^2)| \geqq \tfrac{1}{2}$$

and, therefore,

$$(6)\quad \begin{aligned} a_\varepsilon^2(y^2) &= [(y^2 + p_\varepsilon^1(y^2))^2 + (\alpha\gamma_\varepsilon^+(y^2) + \beta y^2 + 1 + p_\varepsilon^2(y^2))^2]^{-\frac{1}{2}}\,(y^2 + p_\varepsilon^1(y^2)) \geqq \\ &\geqq 2(y^2 + p_\varepsilon^1(y^2)) \geqq 2(-2\varepsilon) = -4\varepsilon\,. \end{aligned}$$

From (5) it follows further

$$\frac{y^2 + p_\varepsilon^1(y^2)}{\alpha\gamma_\varepsilon^+(y^2) + \beta y^2 + 1 + p_\varepsilon^2(y^2)} \leqq \frac{y^2 - \varepsilon}{\alpha\gamma_\varepsilon^+(y^2) + \beta y^2 + 1 + k\varepsilon} \leqq 0$$

from where we obtain

$$y^2 + p_\varepsilon^1(y^2) \leqq 0$$

(7) $$\varepsilon^{-1}p_\varepsilon^1(y^2) + 1 \leqq \frac{y^2 + p_\varepsilon^1(y^2)}{\alpha\gamma_\varepsilon^+(y^2) + \beta y^2 + 1 + p_\varepsilon^2(y^2)}(k - \varepsilon^{-1}p_\varepsilon^2(y^2))$$

and, therefore, with regard to (6), $k_{4\varepsilon} \leqq \varepsilon^{-1}p_\varepsilon^2(y^2) \leqq k$.

Applying the Lemma, we obtain

Corollary 1. $\lim_{\varepsilon\to 0} \varepsilon^{-1}p_\varepsilon^2(y^2) = k$ uniformly with respect to y^2, $|y^2| \leqq \varepsilon$. [2])

From (7), Corollary 1 and the relation (7.5) of [1] we get

Corollary 2. $\varepsilon^{-1}p_\varepsilon^1(y^2) = -1 + o(\varepsilon)$ uniformly with respect to y^2, $|y^2| \leqq \varepsilon$.

Proof of the theorem. Let $\varepsilon < \eta$. The relation (1) will be obviously satisfied if, and only if the solution of the equation (mR_ε^+) starting at the point y_1 of intersection of the curve Γ_η^+ with the line $y^2 = -\varepsilon$, intersects the line $y^2 = \varepsilon$ at a point, which is not closer to the origin than y_1, i.e.

(8) $$\gamma_\varepsilon^+(\varepsilon, y_1^1) \leqq -y_1^1,$$

where

(9) $$y_1^1 = \gamma_\eta^+(-\varepsilon).$$

(8) and (9) may be written in the following way

(10) $$\gamma_\varepsilon^+(\varepsilon, \gamma_\eta^+(-\varepsilon)) + \gamma_\eta^+(-\varepsilon) < 0$$

and, hence, we have to investigate the sign of the expression $\gamma_\varepsilon^+(\varepsilon, \gamma_\eta^+(-\varepsilon)) + \gamma_\eta^+(-\varepsilon)$ for sufficiently small $\eta > 0$ and $\varepsilon \leqq \eta$ sufficiently close to η.

We have

(11) $$\gamma_\varepsilon^+(\varepsilon, \gamma_\eta^+(-\varepsilon)) + \gamma_\eta^+(-\varepsilon) = 2\gamma_\eta^+(-\varepsilon) + \int_{-\varepsilon}^{+\varepsilon} m_\varepsilon^+(\gamma_\varepsilon^+(y^2, y_1^1), y^2)\,dy^2 =$$

$$= 2y_\eta^1 + 2\int_{-\eta}^{-\varepsilon} m_\eta^+(\gamma_\eta^+(y^2), y^2)\,dy^2 + \int_{-\varepsilon}^{+\varepsilon} m_\varepsilon^+(\gamma_\varepsilon^+(y^2, y_1^1), y^2)\,dy^2 =$$

[2]) In a similar way it may be proven that $\lim_{\varepsilon\to 0} k_\varepsilon = k$.

$$= -\int_{-\eta}^{+\eta} m_\eta^+(\gamma_\eta^+(y^2), y^2)\,dy^2 + \int_{-\eta}^{-\varepsilon} m_\eta^+(\gamma_\eta^+(y^2), y^2)\,dy^2 + \int_{-\varepsilon}^{+\varepsilon} m_\varepsilon^+(\gamma_\varepsilon^+(y^2, y_1^1), y^2)\,dy^2 =$$

$$= \int_{-\eta}^{+\varepsilon} m_\eta^+(\gamma_\eta^+(y^2), y^2)\,dy^2 + \int_{-\varepsilon}^{+\varepsilon} [m_\varepsilon^+(\gamma_\varepsilon^+(y^2, y_1^1), y^2) - m_\eta^+(\gamma_\eta^+(y^2), y^2)]\,dy^2 -$$

$$- \int_{\varepsilon}^{\eta} m_\eta^+(\gamma_\eta^+(y^2), y^2)\,dy^2 .$$

Denote the integrals of the last expression of (11) by I_1, I_2, I_3. Then we have

$$(12)\quad |I_1| = \left|\int_{\varepsilon}^{\eta} m_\eta^+(\gamma_\eta^+(y^2), y^2)\,dy^2\right| = \left|\int_{\varepsilon}^{\eta} \frac{y^2 + p_\eta^1(y^2)}{\alpha\gamma_\eta^+(y^2) + \beta y^2 + 1 + p_\eta^2(y^2)}\,dy^2\right| \leqq$$

$$\leqq \int_{\varepsilon}^{\eta} \frac{\eta - \varepsilon}{1 - |\alpha|\, y_\eta^1 - (|\beta| + 1)\,\eta}\,dy^2 = O(\eta)\,(\eta - \varepsilon)^2$$

$$I_3 = \int_{-\eta}^{-\varepsilon} m_\eta^+(\gamma_\eta^+(y^2), y^2)\,dy^2 = \int_{-\eta}^{-\varepsilon} \frac{y^2 + p_\eta^1(y^2)}{\alpha\gamma_\eta^+(y^2) + \beta y^2 + 1 + p_\eta^2(y^2)}\,dy^2 =$$

$$= \int_{-\eta}^{-\varepsilon} \frac{-2\eta}{\alpha\gamma_\eta^+(y^2) + 1 + (k - \beta)\,\eta}\,dy^2 + \int_{-\eta}^{-\varepsilon} \left[\frac{y^2 + p_\eta^1(y^2)}{\alpha\gamma_\eta^+(y^2) + \beta y^2 + 1 + p_\eta^2(y^2)} + \right.$$

$$\left. + \frac{2\eta}{\alpha\gamma_\eta^+(y^2) + 1 + (k - \beta)\,\eta}\right] dy^2 .$$

From Lemma 7.1 of [1] we get $\gamma_\eta^+(y^2) = o(\eta)$ uniformly with respect to $|y^2| \leqq \varepsilon$ and from Corollary 2 we get $p_\eta^1(y^2) + \eta = o(\eta^2)$. Hence,

$$(13)\quad \int_{-\eta}^{-\varepsilon} \left[\frac{y^2 + p_\eta^1(y^2)}{\alpha\gamma_\eta^+(y^2) + \beta y^2 + 1 + p_\eta^2(y^2)} + \frac{2\eta}{\alpha\gamma_\eta^+(y^2) - \beta\eta + 1 + k\eta}\right] dy^2 =$$

$$= \int_{-\eta}^{-\varepsilon} \frac{(y^2 + p_\eta^1)(\alpha\gamma_\eta^+(y^2) - \beta\eta + 1 + k\eta) + 2\eta(\alpha\gamma_\eta^+(y^2) + \beta y^2 + 1 + p_\eta^2(y^2))}{1 + O(\eta)}\,dy^2 =$$

$$= (1 + O(\eta)) \int_{-\eta}^{-\varepsilon} [(y^2 + \eta)(\alpha\gamma_\eta^+(y^2) + 1) + (p_\eta^1(y^2) + \eta)(\alpha\gamma_\eta^+(y^2) + 1) -$$

$$- y^2\beta\eta + y^2 k\eta - p_\eta^1(y^2)\,\beta\eta + p_\eta^1(y^2)\,k\eta + 2\eta\beta y^2 + 2\eta\, p_\eta^2(y^2)]\,dy^2 =$$

$$= (1 + O(\eta)) \cdot \int_{-\eta}^{-\varepsilon} [(y^2 + \eta)(\alpha\gamma_\eta^+(y^2) + 1) + (p_\eta^1(y^2) + \eta)(\alpha\gamma_\eta^+(y^2) + 1) +$$

$$+ \beta\eta(-p_\eta^1(y^2) - \eta) + \beta\eta(-p_\eta^1(y^2) - \eta) + \beta\eta(y^2 + \eta) + k\eta(y^2 + \eta) +$$

$$+ \eta(-k\eta + p_\eta^2(y^2)) + k\eta(p_\eta^1(y^2) + \eta) + \eta(p_\eta^2(y^2) - k\eta)]\,dy^2 =$$

$$= (1 + O(\eta)) \int_{-\eta}^{-\varepsilon} [O(\eta - \varepsilon) + o(\eta^2)]\,dy^2 = O(\eta - \varepsilon)^2 + o(\eta^2)\,(\eta - \varepsilon) .$$

For $\eta > 0$ sufficiently small we have $y_\eta^1 < 1$,

$$\tfrac{1}{2} < \alpha y^1 + \beta y^2 + 1 + p_\eta^2(y^2) < 2 \quad \text{for} \quad |y^1| \leqq y_\eta^1,\ |y^2| \leqq \eta\,;$$

hence,

$$\begin{aligned}
|m_\varepsilon^+(y) - m_\eta^+(y)| &= m_\varepsilon^+(y) - m_\eta^+(y) = f^+(y, p_\varepsilon(y)) - f^+(y, p_\eta(y)) \leqq \\
&\leqq f^+(y, \varepsilon\eta^{-1}p_\eta(y)) - f^+(y, p_\eta(y)) = \\
&= \frac{y^1 + \varepsilon\eta^{-1}p_\eta^1(y)}{\alpha y^1 + \beta y^2 + 1 + \varepsilon\eta^{-1}p_\eta^2(y)} - \frac{y^1 + p_\eta^1(y)}{\alpha y^1 + \beta y^2 + 1 + p_\eta^2(y)} = \\
&= (\alpha y^1 + \beta y^2 + 1 + \varepsilon\eta^{-1}p_\eta^2(y))^{-1}(\alpha y^1 + \beta y^2 + 1 + p_\eta^2(y^2))^{-1} \,. \\
&\quad .\, [(y^1 + \varepsilon\eta^{-1}\, p_\eta^1(y))(\alpha y^1 + \beta y^2 + 1 + p_\eta^2(y)) - (y^1 + p_\eta^1(y))\,. \\
&\quad .\, (\alpha y^1 + \beta y^2 + 1 + \varepsilon\eta^{-1}p_\eta^2(y))] \leqq \\
&\leqq 4[|y^1 p_\eta^2(y)(1 - \varepsilon\eta^{-1})| + 2|p_\eta^1(y)(\varepsilon\eta^{-1} - 1)|] \leqq \\
&\leqq 4[l(\eta - \varepsilon) + 2(\eta - \varepsilon)] = L_1(\eta - \varepsilon)
\end{aligned}$$

for sufficiently small η. From this, Lemma 4.3 of [1] and Theorem 2.1, Chapter I of [2] we conclude that for sufficiently small η and $\varepsilon \leqq \eta$,

$$\gamma_\varepsilon^+(y^2, y_1^1) - \gamma_\eta^+(y^2) \leqq L^{-1}L_1(\eta - \varepsilon)(e^{L(y^2+\varepsilon)} - 1) = O(\eta)(\eta - \varepsilon)$$

for $y^2 \in \langle -\varepsilon, \varepsilon\rangle$. Hence, we have

(14)

$$\begin{aligned}
I_2 &= \int_{-\varepsilon}^{+\varepsilon} [m_\varepsilon^+(\gamma_\varepsilon^+(y^2, y_1^1), y^2) - m_\eta^+(\gamma_\eta^+(y^2), y^2)]\, dy^2 \leqq \\
&\leqq \int_{-\varepsilon}^{+\varepsilon} [f^+(\gamma_\varepsilon^+(y^2, y_1^1), y^2, \varepsilon\eta^{-1}\, p_\eta(y^2)) - f^+(\gamma_\eta^+(y^2), y^2, p_\eta(y^2))]\, dy^2 = \\
&= \int_{-\varepsilon}^{+\varepsilon} \left[\frac{y^2 + \varepsilon\eta^{-1}\, p_\eta(y^2)}{\alpha\gamma_\varepsilon^+(y^2, y_1^1) + \beta y^2 + 1 + \varepsilon\eta^{-1}\, p_\eta^2(y^2)} - \frac{y^2 + p_\eta^1(y^2)}{\alpha\gamma_\eta^+(y^2) + \beta y^2 + 1 + p_\eta^2(y^2)}\right] dy^2 = \\
&= \int_{-\varepsilon}^{+\varepsilon} (\alpha\gamma_\varepsilon^+(y^2, y_1^1) + \beta y^2 + 1 + \varepsilon\eta^{-1}\, p_\eta^2(y^2))^{-1}(\alpha\gamma_\eta^+(y^2) + \beta y^2 + 1 + p_\eta^2(y^2))^{-1} \,. \\
&\quad .\, [(y^2 + \varepsilon\eta^{-1}\, p_\eta^1(y^2))(\alpha\gamma_\eta^+(y^2) + \beta y^2 + 1 + p_\eta^2(y^2)) - (y^2 + p_\eta^1(y^2))\,. \\
&\quad .\, (\alpha\gamma_\varepsilon^+(y^2, y_1^1) + \beta y^2 + 1 + \varepsilon\eta^{-1}\, p_\eta^2(y^2))\, dy^2 = \\
&= \int_{-\varepsilon}^{+\varepsilon} [1 + 2\beta y^2 + (\eta + \varepsilon)\eta^{-1}\, p_\eta^2(y^2) + o(\eta)]^{-1} [\alpha y^2(\gamma_\eta^+(y^2) - \gamma_\varepsilon^+(y^2, y_1^1)) + \\
&\quad + y^2\eta^{-1}\, p_\eta^2(y^2)(\eta - \varepsilon) + \alpha\, p_\eta^1(y^2)(\gamma_\eta^+(y^2) - \gamma_\varepsilon^+(y^2, y_1^1)) + \\
&\quad + \eta^{-1}\alpha\, p_\eta^1(y^2)\,\gamma_\eta^+(y^2)(\varepsilon - \eta) + \beta\eta^{-1}y^2\, p_\eta^1(y^2)(\varepsilon - \eta) + p_\eta^1(y^2)\,\eta^{-1}(\varepsilon - \eta) = \\
&= (\eta - \varepsilon)\int_{-\varepsilon}^{+\varepsilon} \frac{1 + (k + \beta)\, y^2}{1 + 2\beta y^2 + (\eta + \varepsilon)\, k + o(\eta)}\, dy^2 + o(\eta^2)(\eta - \varepsilon)\,.
\end{aligned}$$

For the sake of brevity denote $p_{\varepsilon\eta}(y^2) = p_\varepsilon(\gamma_\varepsilon^+(y^2), \gamma_\eta^+(-\varepsilon)), y^2)$.

In a similar way as the Corollaries 1, 2 of the Lemma we conclude that

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} p_{\varepsilon\eta}^2(y^2) = k\,, \quad \varepsilon^{-1} p_{\varepsilon\eta}^1(y^2) = -1 + o(\eta)$$

uniformly with respect to $|y^2| \leqq \varepsilon$ and $\varepsilon \leqq \eta$. Hence,

(15)

$$I_2 \geqq \int_{-\varepsilon}^{+\varepsilon}\left[\frac{y^2 + p_{\varepsilon\eta}^1(y^2)}{\alpha\gamma_\varepsilon^+(y^2, y_1^1) + \beta y^2 + 1 + p_{\varepsilon\eta}^2(y^2)} - \frac{y^2 + \varepsilon^{-1}\eta\, p_{\varepsilon\eta}^1(y^2)}{\alpha\gamma_\eta^+(y^2)\,\beta y^2 + 1 + \varepsilon^{-1}\eta\, p_{\varepsilon\eta}^2(y^2)}\right] \mathrm{d}y^2 =$$

$$= \int_{-\varepsilon}^{+\varepsilon}(\alpha\gamma_\varepsilon^+(y^2, y_1^1) + \beta y^2 + 1 + p_{\varepsilon\eta}^2(y^2))^{-1}\,(\alpha\gamma_\eta^+(y^2) + \beta y^2 + 1 + \varepsilon^{-1}\eta\, p_{\varepsilon\eta}^2(y^2))^{-1}\,.$$

$$.\,[(y^2 + p_{\varepsilon\eta}^1(y^2))\,(\alpha\gamma_\eta^+(y^2) + \beta y^2 + 1 + \varepsilon^{-1}\eta\, p_{\varepsilon\eta}^2(y^2)) - (y^2 + \varepsilon^{-1}\eta\, p_{\varepsilon\eta}^1(y^2))\,.$$

$$.\,(\alpha\gamma_\varepsilon^+(y^2, y_1^1) + \beta y^2 + 1 + p_{\varepsilon\eta}^2(y^2))]\,\mathrm{d}y^2 =$$

$$= \int_{-\varepsilon}^{+\varepsilon}[1 + 2\beta y^2 + (\eta + \varepsilon)\,\varepsilon^{-1}\,p_{\varepsilon\eta}^2(y^2) + o(\eta)]^{-1}\,[\alpha y^2(\gamma_\eta^+(y^2) - \gamma_\varepsilon^+(y^2, y_1^1)) +$$

$$+ y^2\varepsilon^{-1}\,p_{\varepsilon\eta}^2(y^2)\,(\eta - \varepsilon) + \alpha\, p_{\varepsilon\eta}^1(y^2)\,(\gamma_\eta^+(y^2) - \gamma_\varepsilon^+(y^2, y_1^1)) +$$

$$+ \alpha\varepsilon^{-1}\,p_{\varepsilon\eta}^1(y^2)\,\gamma_\varepsilon^+(y^2, y_1^1)\,(\varepsilon - \eta) +$$

$$+ \beta y^2\varepsilon^{-1}\,p_{\varepsilon\eta}^1(y^2)\,(\varepsilon - \eta) + \varepsilon^{-1}\,p_{\varepsilon\eta}^1(y^2)\,(\varepsilon - \eta)]\,\mathrm{d}y^2 =$$

$$= (\eta - \varepsilon)\int_{-\varepsilon}^{+\varepsilon}\frac{1 + (k + \beta)\,y^2}{1 + 2\beta y^2 + (\eta + \varepsilon)\,k + o(\eta)}\,\mathrm{d}y^2 + o(\eta^2)\,(\eta - \varepsilon)\,.$$

From (14) and (15) we conclude,

$$(16) \qquad I_2 = (\eta - \varepsilon)\int_{-\varepsilon}^{+\varepsilon}\frac{1 + (k + \beta)\,y^2}{1 + 2\beta y^2 + (\eta + \varepsilon)\,k + o(\eta)}\,\mathrm{d}y^2 + o(\eta^2)\,(\eta - \varepsilon)\,.$$

Suppose $\beta \neq 0$. Then, as $\ln(1 + x) = x - \frac{1}{2}x^2 + o(x^2)$,

$$\int_{-\varepsilon}^{+\varepsilon}\frac{1 + (k + \beta)\,y^2}{1 + 2\beta y^2 + (\eta + \varepsilon)\,k + o(\eta)}\,\mathrm{d}y^2 =$$

$$= \frac{1}{2\beta}\left\{(k + \beta)\,2\varepsilon + \frac{2\beta - (\beta + k)\,[1 + (\eta + \varepsilon)\,k + o(\eta)]}{2\beta}\,.\right.$$

$$\left..\,\ln\left[1 + \frac{4\beta\varepsilon}{1 + (\eta + \varepsilon)\,k - 2\beta\varepsilon + o(\eta)}\right]\right\} =$$

$$= \frac{1}{2\beta}\left\{(k + \beta)\,2\varepsilon + \left[1 - \frac{(\beta + k)\,[1 + (\eta + \varepsilon)\,k + o(\eta)]}{2\beta}\right].\right.$$

$$\left..\left[\frac{4\beta\varepsilon}{1 + (\eta + \varepsilon)\,k - 2\beta\varepsilon + o(\eta)} - \frac{16\beta^2\varepsilon^2}{2[1 + (\eta + \varepsilon)\,k - 2\beta\varepsilon + o(\eta)]^2}\right]\right\} + o(\eta^2)\,.$$

Hence,

$$I_2 = \frac{\eta - \varepsilon}{2\beta}\left\{(k + \beta)\, 2\varepsilon + \left[1 - \frac{(\beta + k)\left[1 + (\eta + \varepsilon)\, k + o(\eta)\right]}{2\beta}\right] \cdot\right.$$
$$\left.\cdot \left[\frac{4\beta\varepsilon}{1 + (\eta + \varepsilon)\, k - 2\beta\varepsilon + o(\eta)} - \frac{8\beta^2\varepsilon^2}{\left[1 + (\eta + \varepsilon)\, k - 2\beta\varepsilon + o(\eta)\right]^2} + o(\eta^2)\right]\right\}.$$

From (12), (13) and (16) we obtain

$$\gamma_\varepsilon^+(\varepsilon, \gamma_\eta^+(-\varepsilon)) + \gamma_\eta^+(-\varepsilon) = I_1 + I_2 + I_3 = \frac{2(\varepsilon - \eta)\,\eta}{1 + (k - \beta)\,\eta + o(\eta)} +$$
$$+ \frac{\eta - \varepsilon}{2\beta}\left\{(k + \beta)\, 2\varepsilon + \left[1 - \frac{(\beta + k)\left[1 + (\eta + \varepsilon)\, k + o(\eta)\right]}{2\beta}\right] \cdot\right.$$
$$\left.\cdot \frac{4\beta\varepsilon}{1 + (\eta + \varepsilon)\, k - 2\beta\varepsilon + o(\eta)} - \frac{8\beta^2\varepsilon^2}{\left[1 + (\eta + \varepsilon)\, k - 2\beta\varepsilon + o(\eta)\right]^2}\right\} +$$
$$+ O(\eta - \varepsilon)^2 + o(\eta^2)\,(\eta - \varepsilon) = \frac{2(\varepsilon - \eta)\,\eta}{1 + (k - \beta)\,\eta + o(\eta)} + \frac{(\eta - \varepsilon)\,(k + \beta)\, 2\varepsilon}{2\beta} +$$
$$+ \frac{(\eta - \varepsilon)\, 4\beta\varepsilon}{2\beta\left[1 + (\eta + \varepsilon)\, k - 2\beta\varepsilon + o(\eta)\right]^2} - \frac{(\beta + k)\left[1 + (\eta + \varepsilon)\, k + o(\eta)\right] 4\beta\varepsilon(\eta - \varepsilon)}{4\beta^2\left[1 + (\eta + \varepsilon)\, k - 2\beta\varepsilon + o(\eta)\right]^2} -$$
$$- \frac{8\beta^2\varepsilon^2(\eta - \varepsilon)}{2\beta\left[1 + (\eta + \varepsilon)\, k - 2\beta\varepsilon + o(\eta)\right]^2} + \frac{(\beta + k)\left[1 + (\eta + \varepsilon)\, k + o(\eta)\right] 8\beta^2\varepsilon^2(\eta - \varepsilon)}{4\beta^2\left[1 + (\eta + \varepsilon)\, k - 2\beta\varepsilon + o(\eta)\right]^2} +$$
$$+ O(\eta - \varepsilon)^2 + o(\eta^2)\,(\eta - \varepsilon)\,.$$

Adding the first to the third, the second to the fourth and the fifth to the sixth term, we obtain

$$\gamma_\varepsilon^+(\varepsilon, \gamma_\eta^+(-\varepsilon)) + \gamma_\eta^+(-\varepsilon) = (\eta - \varepsilon)\left\{\frac{2(\varepsilon - \eta) + 2\beta\varepsilon\eta - 2k\eta^2}{1 + O(\eta)} - \frac{2\varepsilon^2(k + \beta)}{1 + O(\eta)} +\right.$$
$$\left.+ \frac{-4\beta\varepsilon^2 + (\beta + k)\, 2\varepsilon^2 + o(\eta^2)}{1 + O(\eta)} + O(\eta - \varepsilon) + o(\eta^2)\right\} =$$
$$= (\eta - \varepsilon)\,\{2\beta\varepsilon\eta - 2k\eta^2 - 2(k + \beta)\,\varepsilon^2 - 4\beta\varepsilon^2 + (\beta + k)\, 2\varepsilon^2 +$$
$$+ O(\eta - \varepsilon) + o(\eta^2)\} = (\eta - \varepsilon)\,\{(2\beta - 2k - 2k - 2\beta - 4\beta + 2\beta + 2k)\,\eta^2 +$$
$$+ O(\eta - \varepsilon) + o(\eta^2)\} = (\eta - \varepsilon)\,\{-2(\beta + k)\,\eta^2 + O(\eta - \varepsilon) + o(\eta^2)\}\,.$$

Letting $\eta > 0$ and then $\eta - \varepsilon > 0$ be small enough we see that the inequality (10) is satisfied, if $\beta + k > 0$, and not satisfied, if $\beta + k < 0$.

Now let $\beta = 0$. Then

$$I_2 = (\eta - \varepsilon)\int_{-\varepsilon}^{+\varepsilon} \frac{1 + ky^2}{1 + k(\eta + \varepsilon) + o(\eta)}\, \mathrm{d}y^2 + o(\eta^2)\,(\eta - \varepsilon) =$$
$$= \frac{(\eta - \varepsilon)\, 2\varepsilon}{1 + (\eta + \varepsilon)\, k + o(\eta)} + o(\eta^2)\,(\eta - \varepsilon)$$

and we have

$$\gamma_\varepsilon^+(\varepsilon, \gamma_\eta^+(-\varepsilon)) + \gamma_\eta^+(-\varepsilon) = \frac{2(\varepsilon - \eta)\,\eta}{1 + k\eta + o(\eta)} + \frac{(\eta - \varepsilon)\,2\varepsilon}{1 + k(\varepsilon + \eta) + o(\eta)} +$$

$$+ O(\eta - \varepsilon)^2 + o(\eta^2)(\eta - \varepsilon) =$$

$$= \frac{2(\varepsilon - \eta)\,\eta[1 + (\varepsilon + \eta)\,k + o(\eta)] + (\eta - \varepsilon)\,2\varepsilon[1 + k\eta + o(\eta)]}{1 + O(\eta)} +$$

$$+ O(\eta - \varepsilon)^2 + o(\eta^2)(\eta - \varepsilon) =$$

$$= \frac{\eta - \varepsilon}{1 + O(\eta)}\,[-2\eta + 2\varepsilon - 2k\varepsilon\eta - 2k\eta^2 + 2k\varepsilon\eta + o(\eta^2)] +$$

$$+ O(\eta - \varepsilon)^2 + o(\eta^2)(\eta - \varepsilon) = (\eta - \varepsilon)\,\{-k\eta^2 + O(\eta - \varepsilon) + o(\eta^2)\}\,;$$

from this we see that the inequality (10) is satisfied for sufficiently small $\eta > 0$ and $\varepsilon \leqq \eta$ sufficiently close to η if $k > 0$, and not satisfied, if $k < 0$. This completes the proof.

References

[1] *P. Brunovský:* On the best stabilizing control under a given class of perturbations. Czech. math. journal *15*, (*90*), (1965), 329–369.

[2] *E.A. Coddington, N. Levinson:* Theory of ordinary differential equations. New York, 1955.

Резюме

УСЛОВИЯ СУЩЕСТВОВАНИЯ УНИВЕРСАЛЬНОГО НАИЛУЧШЕГО ε-СТАБИЛИЗИРУЮЩЕГО УПРАВЛЕНИЯ

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Статья является прямым продолжением статьи [1]. Поэтому без определений используются ее понятия и определения.

Рассматривается вопрос о выполнении условия (1) для достаточно малых $0 \leqq \varepsilon \leqq \eta$, являющегося в случае существования наилучшего ε-стабилизирующего управления при данном классе возмущений в смысле евклидовой нормы необходимым и достаточным условием существования ε_0 — универсального наилучшего ε — стабилизирующего управления в смысле евклидовой нормы.

Пусть

$$k = \min\{p^2 : (-1, p^2) \in \dot{P}\}\,.$$

Доказывается

Теорема. Если $\beta + k > 0$, то для достаточно малых $\eta > 0$ и $\varepsilon \leqq \eta$ условие (1) выполняется. Если $\beta + k < 0$, то для достаточно малых $\eta > 0$ и $\varepsilon \leqq \eta$ достаточно близких к η выполняется соотношение (2).