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THE INTERVAL TOPOLOGY OF A CERTAIN L-GROUP

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To ALEXANDER DONIPHAN WALLACE
on the occasion of his 60th birthday.

It is the purpose of this note to present a counterexample to a long-standing conjecture about lattice-ordered groups (l-groups). It will be shown that there exists an l-group which is not totally ordered, but which is a topological group and a topological lattice in its interval topology.

An l-group is a partially ordered group which is a lattice in its partial ordering. If G is an l-group, the *interval topology* of G is that topology obtained by taking as a subbasis for the closed sets, the sets (called *cones*) of the form $\{f \in G \mid f \geqq f_0\}$ and $\{g \in G \mid g \leqq g_0\}$. It is easily verified that any totally ordered l-group is a topological group and a topological lattice in its interval topology. In [1], BIRKHOFF raised the question whether *every* l-group is a topological group and a topological lattice in its interval topology. NORTHAM [6] showed that there is an l-group which is not a Hausdorff space in its interval topology, and hence is not a topological group. CHOE [2], WOLK [7], CONRAD [3], and JAKUBÍK [5] found larger and larger classes of l-groups such that if an l-group belongs to the class and is a Hausdorff space in its interval topology, then it is totally ordered. In particular, the class of all l-groups which are subdirect sums of totally ordered groups is such a class, and hence so are the commutative l-groups [5]. In all of this work, no example of a non-totally ordered l-group which is a Hausdorff space in its interval topology was presented. In fact, the nature of the theorems proved, especially in [7], [3], and [5], would lead one to conjecture that no such l-group exists. Nevertheless, we now give an example of a non-totally ordered l-group which is not only a Hausdorff space, but even a topological group and a topological lattice in its interval topology.

Let G denote the set of all order-preserving permutations f of the ordered set R of real numbers, which satisfy the condition

$$(*) \qquad (x - 1)f = xf - 1 \quad \text{for all} \quad x \in R\,.$$

It was observed in [4] that G is an l-group under the operation of composition of

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functions, and the order: $f \leqq g$ if and only if $xf \leqq xg$ for all $x \in R$. Clearly, G is not totally ordered.

Lemma. *If $a, b, a', b' \in R$, if $a < b < a + 1$, and if $a' < b' < a' + 1$, then there exists $f \in G$ such that $af = a'$ and $bf = b'$.*

Proof. Obviously, there is an order-preserving permutation f' of R such that $af' = a'$, $(a + 1)f' = a' + 1$, and $bf' = b'$. Let f'' be the restriction of f' to the interval $[a, a + 1]$. Then f'' can be uniquely extended, via the equation (*), to an element f of G.

As it is somewhat more natural to deal with open sets, we take the equivalent definition of the interval topology as that topology whose open sets are subgenerated by sets of the form $\{f \in G \mid f \nleqq f_0\}$ and $\{g \in G \mid g \ngeqq g_0\}$. It is easy to see that if U is a subbasic open set, then so are U^{-1}, Uf, and fU for any $f \in G$. It is also obvious that the interval topology is always T_1. Thus, in order to show that G is a topological group, it suffices to show that if U is a subbasic open set containing the identity e of G, then there exists an open set V containing e such that $V^2 \subseteq U$. This we proceed to do.

Let $U = \{g \in G \mid g \nleqq g_0\}$ be a subbasic open set containing $e \in G$. (The case of a U of the other form is omitted, but can be handled similarly.) As $e \in U$, $e \nleqq g_0$, so there is some $\gamma \in R$ such that $\gamma g_0 < \gamma = \gamma e$. Choose real numbers u, v, w, x, y, and z, such that

$$\gamma g_0 < u < v < w < x < y < z < \gamma\,, \quad x < w + 1\,, \quad v < u + 1\,,$$
$$\gamma < z + 1 \quad \text{and} \quad y < x + 1\,.$$

Then since $w < x < w + 1$ and $v < u + 1 < v + 1$, by the lemma there is an $f_1 \in G$ such that $wf_1 = v$ and $xf_1 = u + 1$. Also, since $z < \gamma < z + 1$ and $y < x + 1 < y + 1$, there is an $f_2 \in G$ such that $zf_2 = y$ and $\gamma f_2 = x + 1$.

Let $V = \{f \in G \mid f \nleqq f_1 \text{ and } f \nleqq f_2\}$. Then V is an open set (being the intersection of two subbasic open sets). Also, $e \in V$, because $zf_2 = y < z = ze$ and $wf_1 = v < w = we$.

Suppose $g, g' \in V$. Then since $g' \nleqq f_1$, for some $\alpha \in R$, $\alpha f_1 < \alpha g'$, and since g' and f_1 satisfy (*), we may assume $x - 1 < \alpha < x$. Likewise, for some $\beta \in R$, $\gamma - 1 < \beta < \gamma$, $\beta f_2 < \beta g$. Therefore,

$$\begin{aligned}\gamma g_0 < u = xf_1 - 1 = (x - 1) f_1 < \alpha f_1 < \alpha g' < xg' = \\ = (\gamma f_2 - 1)\, g' = (\gamma - 1) f_2 g' < \beta f_2 g' < \beta g g' < \gamma g g'\,.\end{aligned}$$

Thus, $gg' \nleqq g_0$, and so $gg' \in U$, or $V^2 \subseteq U$. Hence G is a topological group in its interval topology.

It follows, of course, that G is a Hausdorff space. However, we prefer to show this directly, since, in fact, G is a Hausdorff space in a very strong sense. It was observed in [6] that a lattice L is a Hausdorff space in its interval topology if and only if for each two elements g and g' of L, there is a covering of L by a finite number of cones such that no cone contains both g and g'. We shall show that for each $g, g' \in G$, $g \neq g'$ there are *two* cones whose union contains G and such that neither cone contains both g and g'.

Without loss of generality, assume $g' = e$ and $g \nleqq e$. Then for some $\gamma \in R$, $\gamma < \gamma g$. Choose $u, v, w, x, y, z \in R$ such that

$$\gamma < u < v < w < x < y < z < \gamma g\,, \quad v < u + 1\,, \quad z < y + 1\,,$$
$$w < v + 1 \quad \text{and} \quad y < x + 1\,.$$

Then since $u < v < u + 1$ and $z < y + 1 < z + 1$, by the lemma, there exists $f_1 \in G$ such that $uf_1 = z$ and $vf_1 = y + 1$. Similarly, there exists $f_2 \in G$ such that $wf_2 = x$ and $(v + 1)f_2 = y$. Let $U = \{f \in G \mid f \leqq f_1\}$ and $V = \{f \in G \mid f \geqq f_2\}$. Then since $uf_1 = z < \gamma g < ug$, $g \notin U$. Likewise, since $wf_2 = x > w = we$, $e \notin V$, and hence neither cone contains both e and g.

Let $f \in G$. If $f \notin V$ then for some $\alpha \in R$, $v \leqq \alpha \leqq v + 1$, $\alpha f < \alpha f_2$. Let $\beta \in R$, $v \leqq \beta \leqq v + 1$. Then

$$\beta f \leqq (v + 1) f = vf + 1 \leqq \alpha f + 1 < \alpha f_2 + 1 \leqq (v + 1) f_2 + 1 =$$
$$= y + 1 = vf_1 \leqq \beta f_1\,.$$

Hence $f < f_1$ and $f \in U$. Therefore the union of U and V contains G.

Finally, we wish to show that G is a topological lattice. It will suffice to show that if $f_1, f_2 \in G$ such that $f_1 \cap f_2 = e$ ($\cap$ and $\cup$ indicate the lattice operations in G), and if W is a subbasic open set containing e, then there exist open sets U and V such that $f_1 \in U$, $f_2 \in V$, and for every $f \in U$, $g \in V$, $f \cap g \in W$. The continuity of $\cap$ everywhere follows by homogeneity, and the continuity of $\cup$ follows by duality.

As $\cap$ does not play the same role with respect to the two different kinds of subbasic open sets, we must consider two cases.

Case 1. $W = \{f \in G \mid f \nleqq g_0\}$. Since $e \in W$, $e \nleqq g_0$, and there exists $\gamma \in R$ with $\gamma g_0 < \gamma$. Choose $x, y \in R$ such that

$$\gamma g_0 < x < y < \gamma < y + 1 \quad \text{and} \quad x < \gamma g_0 + 1\,.$$

Then by the lemma, there exists $h \in G$ such that $yh = x$ and $\gamma h = \gamma g_0 + 1$.

Let $U = V = \{f \in G \mid f \nleqq h\}$. Since $yh = x < y = ye$, $e \nleqq h$, and therefore $f_1 \nleqq h$ and $f_2 \nleqq h$. Hence $f_1 \in U$ and $f_2 \in V$. Now if $f \in U$, then for some $\alpha \in R$, $\gamma - 1 < \alpha < \gamma$, $\alpha f > \alpha h$. Hence

$$\gamma g_0 = \gamma h - 1 = (\gamma - 1)\, h < \alpha h < \alpha f < \gamma f\,.$$

Thus if $f \in U$ and $g \in V = U$, $\gamma(f \cap g) = \min(\gamma f, \gamma g) > \gamma g_0$, so $f \cap g \nleqq g_0$ and $f \cap g \in W$.

Case 2. $W = \{f \in G \mid f \ngeqq g_0\}$. By duality with case 1, we can choose $\gamma < x < < y < \gamma g_0$ and $h \in G$ such that $xh = y$ and $(\gamma + 1) h = \gamma g_0$. Let $U = \{f \in G \mid f \ngeqq h\}$. By duality, if $f \in U$, $\gamma f < \gamma g_0$. Now, as $\min(xf_1, xf_2) = x(f_1 \cap f_2) = xe = x < y = = xh$, then either $xf_1 < xh$ or $xf_2 < xh$. That is, either $f_1 \in U$ or $f_2 \in U$. Say $f_1 \in U$. Let $V = G$. Then $f_2 \in V$. Now if $f \in U$ and $g \in V$, $\gamma(f \cap g) \leqq \gamma f < \gamma g_0$, so $f \cap g \ngeqq g_0$ and $f \cap g \in W$.

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Резюме

ИНТЕРВАЛЬНАЯ ТОПОЛОГИЯ В НЕКОТОРОЙ *l*-ГРУППЕ

ЧАРЛЕС ГОЛАНД, (Charles Holland), Медисон, Висконсин

В статье строится пример структурно упорядоченной группы, которая не является линейно упорядоченной, но является топологической группой и топологической структурой относительно интервальной топологии. Этим решается вопрос, поставленный в работе [3].