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A FRACTIONAL MEAN VALUE THEOREM, AND A TAYLOR THEOREM, FOR STRONGLY CONTINUOUS VECTOR VALUED FUNCTIONS

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1. Introduction. The conclusion of the classical (Lagrange) mean value theorem for a, suitably restricted, real valued function $x(t)$ of a real variable t, asserts the existence of a mean value c, with $a < c < b$, such that $(x(b) - x(a))/(b - a) = x'(c)$, where x' denotes the usual first derivative. Several extensions of this classical theorem, to vector valued functions of the real variable t, have appeared recently (see, e.g. [1], [2]).

Now, again in the case of real valued $x(t)$, there is a more general mean value theorem (Cauchy) – which has been variously called the "generalized", or "fractional", mean value theorem-whose conclusion asserts the existence of an intermediate value c such that $(x(b) - x(a))/(g(b) - g(a)) = (x'(c))/(g'(c))$, where $g(t)$ is a real valued, suitably restricted, auxiliary function of the real variable t. The purpose of the present paper is to prove a mean value theorem (of "fractional type") for vector valued functions $x(t)$, which is an extension of the mean value theorem (which may be said to be "of classical type") for vector valued functions given earlier as "mean value theorem 2" of [1]. This desired fractional type theorem is theorem I of section 2. As a corollary of theorem I (not the most general corollary deducible, but merely one which is convenient to state), section 3 contains the proof of theorem II, which is a "Taylor theorem for vector valued functions".

2. A fractional theorem of the mean. This section contains the proof of the following.

Mean value theorem I. *If* (1) *the vector valued function* $x(t)$ *is defined for all real* t *such that* $a \leqq t \leqq b$, *where* $a < b$, *both* a *and* b *being finite, and its values are in a linear normed space* B (*the norm in* B *will be denoted by* $\| \quad \|$; *the scalars may be real, complex, or quaternion*); (2) *the function* $x(t)$ *is strongly continuous on* $a \leqq t \leqq b$; (3) *the real valued function* $g(t)$ *is continuous and strictly monotonic for all real* t *such that* $a \leqq t \leqq b$; then *there always exists a number* c,

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with $a < c < b$, *such that* either, *whenever both* $h > 0$ *and* $a \leqq c + h \leqq b$, *one has*

$$\left\| \frac{x(b) - x(a)}{g(b) - g(a)} \right\| \leqq \left\| \frac{x(c + h) - x(c)}{g(c + h) - g(c)} \right\|; \tag{1}$$

or, *whenever both* $h > 0$ *and* $a \leqq c - h \leqq b$, *one has*

$$\left\| \frac{x(b) - x(a)}{g(b) - g(a)} \right\| \leqq \left\| \frac{x(c) - x(c - h)}{g(c) - g(c - h)} \right\|. \tag{2}$$

Proof. Suppose, for definiteness, that the function $g(t)$ is strictly increasing, and consider the real valued function f, defined on $a \leqq t \leqq b$ by the equation

$$f(t) = \|x(t) - x(a)\| - [g(t) - g(a)] \cdot \left\| \frac{x(b) - x(a)}{g(b) - g(a)} \right\|.$$

This function f is real valued and continuous on the closed interval $a \leqq t \leqq b$; and, further, $f(a) = f(b) = 0$. Therefore, there is a number c, $a < c < b$, such that the function f has either a maximum or minimum, over the closed interval $a \leqq t \leqq b$, at the (interior) number c.

Suppose first that the function f has a maximum at c. Then, whenever both $h > 0$ and $a \leqq c - h \leqq b$, one has

$$f(c) - f(c - h) = \|x(c) - x(a)\| - \|x(c - h) - x(a)\| + \\ + [-g(c) + g(c - h)] \left\| \frac{x(b) - x(a)}{g(b) - g(a)} \right\| \geqq 0,$$

therefore

$$\|x(c) - x(c - h)\| \geqq \|x(c) - x(a)\| - \|x(c - h) - x(a)\| \geqq \\ \geqq [g(c) - g(c - h)] \left\| \frac{x(b) - x(a)}{g(b) - g(a)} \right\|.$$

Consequently, the inequality (2) follows, since $g(c) - g(c - h) > 0$.

If f has a minimum at c one obtains by a similar argument the inequality (1).

Remark 1. The conclusion of the theorem implies the weaker conclusion that, at c, either

$$\left\| \frac{x(b) - x(a)}{g(b) - g(a)} \right\| \leqq \liminf_{h \to +0} \left\| \frac{x(c + h) - x(c)}{g(c + h) - g(c)} \right\|,$$

or

$$\left\| \frac{x(b) - x(a)}{g(b) - g(a)} \right\| \leqq \liminf_{h \to +0} \left\| \frac{x(c) - x(c - h)}{g(c) - g(c - h)} \right\|,$$

where $h \to +0$ denotes, as usual, both $h > 0$ and $h \to 0$.

Remark 2. In the special case when $g(t) = t$ throughout the interval $a \leqq t \leqq b$, the present mean value theorem I, together with remark 1 above, reduce to the "mean value theorem 2" of [1].

Remark 3. If the vector valued function $x(t)$ possesses, whenever $a < t < b$, both a finite strong right hand derivative

$$x'_{+}(t) = \lim_{h \to +0} \frac{x(t+h) - x(c)}{h},$$

and a finite strong left hand derivative

$$x'_{-}(t) = \lim_{h \to +0} \frac{x(t) - x(t-h)}{h},$$

and the real valued function $g(t)$ possesses both a finite (non zero) right hand derivative $g'_{+}(t)$, and a finite (non zero) left hand derivative $g'_{-}(t)$, throughout $a < t < b$, then the present mean value theorem I implies the existence of an intermediate value c such that either

$$\left\| \frac{x(b) - x(a)}{g(b) - g(a)} \right\| \leqq \left\| \frac{x'_{+}(c)}{g'_{+}(c)} \right\|,$$

or

$$\left\| \frac{x(b) - x(a)}{g(b) - g(a)} \right\| \leqq \left\| \frac{x'_{-}(c)}{g'_{-}(c)} \right\|.$$

If we replace the assumption of the existence of one sided derivatives by the assumption of the existence of derivatives we obtain the inequality

$$\left\| \frac{x(b) - x(a)}{g(b) - g(a)} \right\| \leqq \left\| \frac{x'(c)}{g'(c)} \right\|.$$

In the special case when $g(t) = t$ throughout the interval $a \leqq t \leqq b$, then the inequality just written reduces precisely to the conclusion of the "mean value theorem of the differential calculus of vector valued functions" of [2, p. 261].

Remark 4. Suppose, in particular, that the continuous function $x(t)$ is real valued. In this special case, by considering the real valued continuous function f, defined on $a \leqq t \leqq b$ by the equation

$$f(t) = x(t) - x(a) - [g(t) - g(a)] \frac{x(b) - x(a)}{g(b) - g(a)},$$

which vanishes at a and at b, the conclusion of the theorem can be strengthened to read as follows: there is a number c, with $a < c < b$, such that, whenever both $h > 0$

and $a \leqq c - h < c < c + h \leqq b$, then either

$$\frac{x(c+h) - x(c)}{g(c+h) - g(c)} \leqq \frac{x(b) - x(a)}{g(b) - g(a)} \leqq \frac{x(c) - x(c-h)}{g(c) - g(c-h)},$$

or the inequalities just written hold with the inequality signs reversed. From these inequalities one can obtain results of the nature of those of Remarks 1 and 2. These results are related to the mean theorem of W. H. and G. C. Young concerning the Dini derivates (see [3, p. 10], in case $g(t) = t$; and also pp. 19–24 of [3], for a "general" $g(t)$).

3. Mean value theorem II (Taylor' s theorem). *If (1) the vector valued function $x(t)$ is defined for all real t such that $a \leqq t \leqq b$, where $a < b$, both a and b being finite, and its values are in a linear normed space B (the norm in B will be denoted by $\| \quad \|$; the scalars may be real, complex, or quaternion); (2) n is a non-negative integer, and the function $x(t)$ possesses n (successive) strong derivatives, $x^{(k)}(t)$, $k=0, 1, \ldots, n$, which are finite and continuous throughout the closed interval $a \leqq t \leqq b$ (these derivatives are understood to be "one sided" at a and at b; for example,*

$$x'(a) = \lim_{h \to +0} \frac{x(a+h) - x(a)}{h};$$

(3) the function $x(t)$ possesses a finite derivative of order $n+1$, $x^{(n+1)}(t)$ on the open interval $a < t < b$; (4) the real valued function $g(t)$ is defined on the closed interval $a \leqq t \leqq b$, possesses a finite, non-zero derivative on $a < t < b$, and is continuous at a and at b; then there always exists a number c, with $a < c < b$, such that

$$\left\| \frac{x(b) - \sum_{i=0}^{n} x^{(i)}(a) \frac{(b-a)^i}{i!}}{g(b) - g(a)} \right\| \leqq \left\| \frac{x^{(n+1)}(c)}{g'(c)} \right\| \frac{(b-c)^n}{n!}.$$

Proof. If $n = 0$, theorem I gives the result. Therefore, suppose that $n > 0$. Consider the auxiliary vector valued function F, defined on $a \leqq t \leqq b$ by the equation

$$F(t) = x(b) - \sum_{i=0}^{n} x^{(i)}(t) \frac{(b-t)^i}{i!}.$$

Notice that F is strongly continuous on $a \leqq t \leqq b$, and that $F(b)$ is the zero vector, while

$$F(a) = x(b) - \sum_{i=0}^{n} x^{(i)}(a) \frac{(b-a)^i}{i!}.$$

Further, on $a < t < b$:

$$F'(t) = -x^{(n+1)}(t) \frac{(b-t)^n}{n!}.$$

Now applying Remark 3 above to the vector valued function $F(t)$ it follows that there is a number c such that $a < c < b$, for which

$$\left\| \frac{F(b) - F(a)}{g(b) - g(a)} \right\| \leqq \left\| \frac{F'(c)}{g'(c)} \right\| ;$$

and this is, except for notation, precisely the desired inequality.

Remark 5. Let r be a real number satisfying $0 \leqq r \leqq n$, and choose $g(t) = (b - t)^{n+1-r}$. Then one obtains, as a special case of theorem II., that

$$\left\| x(b) - \sum_{i=0}^{n} x^{(i)}(a) \frac{(b-a)^i}{i!} \right\| \leqq \|x^{(n+1)}(c)\| \frac{(b-a)^{n+1-r}(b-c)^r}{(n+1-r)\, n!} .$$

For $r = 0$ one obtains what may be called "Taylor's theorem with Lagrange's form of the remainder"; while for $r = n$ one obtains what may be called "Taylor's theorem with Cauchy's form of the remainder".

Remark 6. If, instead of hypothesis (3) of theorem II, one assumes that the right and left hand derivatives of $x^{(n)}(t)$ are finite on $a < t < b$ (let them be denoted by $x_+^{(n+1)}(t)$ and $x_-^{(n+1)}(t)$, respectively); while, at the same time, instead of hypothesis (4), one assumes that the real valued function $g(t)$ defined on the closed interval $a \leqq t \leqq b$, possesses finite, non-zero, right and left hand derivatives, $g'_+(t)$ and $g'_-(t)$, on $a < t < b$, and is continuous at a and at b; then one can deduce the existence of a number c, such that $a < c < b$, and such that either

$$\left\| \frac{x(b) - \sum_{i=0}^{n} x^{(i)}(a) \frac{(b-a)^i}{i!}}{g(b) - g(a)} \right\| \leqq \left\| \frac{x_+^{(n+1)}(c)}{g'_+(c)} \right\| \frac{(b-c)^n}{n!} ,$$

or

$$\left\| \frac{x(b) - \sum_{i=0}^{n} x^{(i)}(a) \frac{(b-a)^i}{i!}}{g(b) - g(a)} \right\| \leqq \left\| \frac{x_-^{(n+1)}(c)}{g'_-(c)} \right\| \frac{(b-c)^n}{n!} .$$

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Резюме

ТЕОРЕМА О СРЕДНЕМ ЗНАЧЕНИИ И ТЕОРЕМА ТЕЙЛОРА ДЛЯ НЕПРЕРЫВНЫХ ВЕКТОРНЫХ ФУНКЦИЙ

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Целью работы является формулировка и элементарное доказательство теоремы о среднем значении для векторных функций, которая является в определенном смысле аналогом теоремы Коши о среднем значении. В статье также высказана и доказана теорема Тейлора для векторных функций.