Czechoslovak Mathematical Journal

Karl Heinrich Hofmann; Paul S. Mostert

Connected extensions of simple semigroups

Czechoslovak Mathematical Journal, Vol. 15 (1965), No. 2, 295–298

Persistent URL: http://dml.cz/dmlcz/100671

Чехословацкий математический журнал, т. 15 (90) 1965, Прага

CONNECTED EXTENSIONS OF SIMPLE SEMIGROUPS

K. H. HOFMANN and P. S. MOSTERT[1]), New Orleans

(Received February 27, 1964)

In the study of compact topological semigroups, one learns early that the minimal ideal is a Rees product of a compact group G and a pair of spaces X and Y [2]. Under many conditions, this reduces to a direct product, and indeed no examples seem to be in the literature of compact connected semigroups with identity in which the minimal ideal failed to be a direct product. CLIFFORD [1] gave general construction techniques to illustrate all algebraic extensions of a given Rees product (or completely simple semigroup). We have followed his technique to develop methods of finding many examples of compact connected semigroups with non-trivial Rees products (i.e. not direct products) as minimal ideal. In fact we have the following result:

Every compact connected semigroup S with identity can be embedded in a compact connected semigroup T with identity whose minimal ideal N is a non-trivial Rees product in such a way that $T \setminus S \subset N$.

Insofar as examples are concerned, it is possible in many ways to construct from any given semigroup, another with a non-trivial Rees product for its minimal ideal and such that the Rees quotients of the two semigroups are isomorphic. One will also observe that our techniques are applicable to far wider classes of topological semigroups than compact connected ones.

1. Clifford's extension conditions. We first recapitulate the necessary and sufficient conditions given by Clifford for constructing extensions of completely simple semigroups. Following this technique, we then have associativity of multiplication, and so it is then only necessary to check the continuity of multiplication once we have found the right topological space on which the semigroup is to live.

Let $M = [X, Y \mid G]$ be a Rees product, i.e. a semigroup defined in the following fashion: The set M is the cartesian product $X \times Y \times G$ of a set X, a set Y and a group G. Multiplication is defined as follows: $(x, y \mid g)(u, v \mid h) = (x, v \mid g\, \varphi(u, y)\, h)$ where φ is any fixed function $\varphi : X \times Y \to G$. We may assume without loss of generality

[1]) This work was partially supported by NSF Grant GP 1877.

that φ is normed. That is, $\varphi(a \times Y) = \varphi(X \times b) = e =$ identity of G for some $(a, b) \in X \times Y$. In any Rees product, such a normed φ can be found. Then M is isomorphic to a product if and only if $\varphi(x, y) = e$ for all $(x, y) \in X \times Y$.

Let T be a semigroup with zero 0 and let $T_0 = T \backslash \{0\}$. The following problem was considered by Clifford: Find a semigroup S with a minimal ideal isomorphic to M such that the Rees quotient modulo this minimal ideal is isomorphic to T (on a purely algebraic basis). His result is as follows:

Theorem: *Suppose that there exist mappings*

$$(t, x) \to t \,.\, x : T_0 \times X \to X\,, \quad (y, t) \to y \,.\, t : Y \times T_0 \to Y$$
$$\mu\colon \{(s, t) \in T_0 : st = 0\} \to X\,, \quad \nu\colon \{(s, t) \in T_0 : st = 0\} \to Y\,, \quad \gamma\colon T_0 \to G$$

such that the following conditions are satisfied:

a) $$s \,.\, (t \,.\, x) = \begin{cases} st \,.\, x & \text{if } st \neq 0 \\ \mu(s, t) & \text{if } st = 0 \end{cases} \qquad (y \,.\, s) \,.\, t = \begin{cases} y \,.\, st & \text{if } st \neq 0 \\ \nu(s, t) & \text{if } st = 0 \end{cases}$$

b) $$\varphi(s \,.\, a, y)\, \gamma(s)\, \varphi(x, y \,.\, s) = \varphi(s \,.\, x, y)\, \gamma(s)\, \varphi(x, b \,.\, s)$$

c) $$\gamma(st) = \gamma(s)\, \varphi(t \,.\, a, b \,.\, t)\, \gamma(t) \quad \text{if} \quad st \neq 0\,.$$

Let S be the disjoint union of T_0 and M and define the multiplication as follows

$$s \circ t = \begin{cases} st & \text{if } s, t \in T_0,\ st \neq 0 \\ (\mu(s, t), \nu(s, t) \mid \gamma(s)\, \varphi(t \,.\, a, b \,.\, s)\, \gamma(t)) & \text{if } s, t \in T_0,\ st = 0 \end{cases}$$
$$s \circ (x, y \mid g) = (s \,.\, x, y \mid \gamma(s)\, \varphi(x, b \,.\, s)\, g) \quad \text{if} \quad s \in T_0$$
$$(x, y \mid g) \circ s = (x, y \,.\, s \mid g\, \varphi(s \,.\, a, y)\, \gamma(s)) \quad \text{if} \quad s \in T_0$$
$$(x, y \mid g) \circ (u, v \mid h) = (x, y \mid g)\,(u, v \mid h) = (x, v \mid g\, \varphi(u, y)\, h)$$

Then S is a semigroup with minimal ideal M such that the Rees quotient modulo M is isomorphic to T.

Conversely every semigroup S with minimal ideal isomorphic to M and Rees quotient isomorphic to T can be obtained in this fashion with appropriate functions.

2. General constructions. Here we illustrate a general construction technique which enables us to embed any compact semigroup S in a compact semigroup T with minimal ideal N which is a non-trivial Rees product in such a way that $T \backslash S \subset N$, where we identify S with its embedded image in T.

Let $s \to s'\colon S \to S'$ be an epimorphism onto a compact semigroup with zero $0'$. For $s \in S$, $x \in S'$ we define actions $S' \times S \to S'$ and $S \times S' \to S'$ of S acting on the right and left, respectively, by

(1) $$x \,.\, s = xs'\,, \quad s \,.\, x = s'x\,.$$

Let M be the minimal ideal of S and choose a fixed idempotent $e \in M$, and let $G = eMe$. Then G is maximal subgroup relative to the idempotent e. Define $\gamma: S \to G$ by

$$\gamma(s) = ese. \tag{2}$$

If M is isomorphic to a direct product of G and its set of idempotents, then γ is a homomorphism of M onto G. Hence, for $s, t \in S$

$$\gamma(st) = este = e(es)\, e \,.\, e(te)\, e = \gamma(s)\, \gamma(t)$$

so that γ is also a homomorphism of S onto M. Let now G^* be a compact group which contains an isomorphic copy of G (which we identify with G) such that the centralizer Z of $\gamma(S \backslash M)$ in G^* has non-trivial component. (If G already has this property, we may take $G^* = G$. If not, we let $G^* = \mathbf{T} \times G$ where $\mathbf{T}$ is the circle group.) Now take any non-trivial map $\varrho : S' \to G^*$ such that $\varrho(0') = e$ and $\varrho(S') \subset Z$. In the space $S \times S' \times S' \times G^*$, let T denote the compact subspace

$$T = \{(s, 0', 0', \gamma(s)) : s \in S\backslash M\} \cup M \times S' \times S' \times G^* .$$

We define multiplication by

$$(s, x, u, g)\,(t, y, v, h) = \begin{cases} (st, 0', 0', \gamma(st)) & \text{if } s \notin M,\ t \notin M \\ (st, s\,.\,y, v, \gamma(s)\, h) & \text{if } s \notin M,\ t \in M \\ (st, x, u\,.\,t, g\, \gamma(t)) & \text{if } s \in M,\ t \notin M \\ (st, x, v, g\, \varrho(yu)\, h) & \text{if } s \in M,\ t \in M\,. \end{cases}$$

It is a simple task to check the continuity of multiplication in this definition, observing that $\varrho(yu) = e$ whenever either u or y is $0'$. The proof of the associativity is purely algebraic and we can rely on the results of Clifford quoted above. Algebraically, we can consider S as the union of $S\backslash M$ and $M \times S' \times S' \times G^* = N$. In the latter semigroup, M is a direct factor, and we can consider $S\backslash M$ as the Rees quotient $S\backslash M$ deprived of its zero. We observe that the multiplication on $S' \times S' \times G^*$ agrees with that defined using Clifford's theorem where $\mu(s, t) = 0' = \nu(s, t)$, when $st \in M$ and $t\,.\,x$ and $y\,.\,t$ as introduced above, γ as given, and $\varrho(x, y) = \varrho(yx)$. The points a and b we take as $0'$. Then $\varphi(s\,.\,a, y) = \varrho(ys0') = \varrho(0') = e = \varphi(x, b\,.\,s)$. Comparison of this multiplication with that defined above shows complete coincidence apart from the first component which appears in our definition. We know, however, that M, being a direct factor of N and the minimal ideal of S, does not disturb the associativity. Thus, we need only check a)–c) in Clifford's theorem. Since S acts on S' on both right and left, and $\mu(s, t) = \nu(s, t) = 0'$ if $st = 0'$, a) is satisfied. The identity b) reduces in our case to $\gamma(s)\, \varrho((y\,.\,s)\, x) = \varrho(y(s\,.\,x))\, \gamma(s)$ which is indeed satisfied because of the associativity in S' and the choice of the range of ϱ in the centralizer of γ. Since $\varphi(t\,.\,a, b\,.\,s) = \varrho(0') = e$, and γ is a homomorphism on S, c) is also satisfied. This shows that T is a compact semigroup. Clearly T is connected if S is connected. If 1 is the identity for S, then $(1, 0', 0', e)$ is the identity for T. Hence, we have the following result:

Proposition. *Let S be a compact semigroup with minimal ideal M. If M is not a non-trivial Rees product, then there is a compact semigroup T with minimal ideal N which is a non-trivial Rees product and an isomorphism $\pi\colon S \to T$ such that*

(i) $\pi(S \backslash M) = T \backslash N$,

(ii) *T is connected if S is connected,*

(iii) *T has an identity if S has an identity.*

Remark. Instead of choosing a morphism for γ in Clifford's theorem it is possible to choose γ quite arbitrary subject to the condition $\gamma(1) = e$, and then determine φ from the relation c). Details of this and other constructions will be given in our forthcoming book.

References

[1] *A. H. Clifford:* Extensions of semigroups, Trans. Amer. Math. Soc., *68* (1950), 165—173.

[2] *A. D. Wallace:* The Rees-Suschewitsch structure theorem for compact simple semigroups, Proc. Amer. Math. Soc., *42* (1956), 430—432.

Резюме

СВЯЗНЫЕ РАСШИРЕНИЯ ПРОСТЫХ ПОЛУГРУПП

К. Г. ГОФМАН, П. С. МОСТЕРТ, (K. H. Hofmann, P. S. Mostert) Нью Орлеанс

В статье доказывается следующая теорема. Всякую компактную связную полгруппу S с единицей можно погрузить в компактную связную полугруппу T с единицей так, что минимальный двусторонний идеал N полугруппы T — нетривиальное Рисово произведение и $T \backslash S \subset N$. При этом нетривиальным Рисовым произведением называется произведение, которое нередуцируется на обычное прямое произведение.