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A NON-ABSOLUTELY CONVERGENT INTEGRAL IN E_m AND THE THEOREM OF GAUSS

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Let f be a function, defined on a subset of E_m. By the method developed in [2], an integral $\beta(f, A)$ which needn't be absolutely convergent is defined on a class of sets $A \subset E_m$. If A is a bounded set with finite perimeter such that the Lebesgue integral $L = \int_A f(x)\,dx$ converges, then $\beta(f, A) = L$. Further, a transformation theorem for the integral β and a theorem on the representation of a surface integral by means of the integral β are proved.

1. Throughout the paper, m denotes an integer >1, and E_m stands for the m-dimensional Euclidean space. For natural k and $A \subset E_k$ let $|A|$ denote the outer Lebesgue measure of the set A. (Of course, for $a \in E_1$ the symbol $|a|$ has its usual meaning; there is no danger of misunderstanding.) Words e.g. "measurable", "almost everywhere" are related to the Lebesgue measure on E_m. The symbols $\bar{A}$, A°, $\dot{A}$ denote the closure, the interior and the boundary of a set $A \subset E_m$, respectively.

Let $\mathfrak{V}$ denote the system of all vectors $v = [v_1, \ldots, v_m]$, where v_i are polynomials in m variables. For each bounded measurable subset A of E_m and each $v \in \mathfrak{V}$, we put $P(A, v) = \int_A \operatorname{div} v(x)\,dx$; further, we define $\|A\| = \sup P(A, v)$, with $v \in \mathfrak{V}$ fulfilling $\sum_{i=1}^{m} (v_i(x))^2 \leqq 1$ for each $x \in A$. The letter $\mathfrak{A}$ stands for the set of all $A \subset E_m$ such that $\|A\| < \infty$. For each continuous vector v on the boundary of $A \in \mathfrak{A}$, we define $P(A, v)$ according to [1], 13 and 15.

Let $\mathfrak{Z}$ denote the system of all measurable subsets of the space E_m. For $R, S \in \mathfrak{Z}$, define $RS = R \cap S$, $R + S = (R \cup S) - (R \cap S)$ respectively; then $\mathfrak{Z}$ becomes a Boolean ring (see [2], 2), with zero element equal to the empty set and E_m as unit element. For $\mathfrak{R}, \mathfrak{T} \subset \mathfrak{Z}$, let $\mathfrak{R}\mathfrak{T}$ denote the system of all $RT = R \cap T$, with $R \in \mathfrak{R}$, $T \in \mathfrak{T}$. If $\mathfrak{R}$ consists of only one element R, we write $\mathfrak{R}\mathfrak{T} = R\mathfrak{T}$.

According to [1], 35, we have

$$(1) \qquad \max(\|A \cup B\|, \|A \cap B\|, \|A - B\|) \leqq \|A\| + \|B\|$$

for arbitrary $A, B \in \mathfrak{A}$. It follows that $\mathfrak{A}$ is a subring of $\mathfrak{Z}$ (of course, there is no unit element in $\mathfrak{A}$).

Further, let $\mathfrak{P}$ denote the set of all sequences $\{A_n\}_{n=1}^{\infty}$ such that $A_n \in \mathfrak{A}$, $\sup_n \|A_n\| < \infty$, $|A_n| \to 0$. Evidently, $\{A_n\} \in \mathfrak{P}$, $A \in \mathfrak{A}$ implies $|A_n \cap A| \to 0$, $|A_n - A| \to 0$, and by (1) we have $\sup_n \|A_n \cap A\| < \infty$, $\sup_n \|A_n - A\| < \infty$; hence also $\{A_n \cap A\} \in \mathfrak{P}$, $\{A_n - A\} \in \mathfrak{P}$.

Let us define a convergence $\to$ on the ring $\mathfrak{Z}$ in the following way: the symbol $Z_n \to Z$ means that $Z_n \subset Z$ $(n = 1, 2, \ldots)$ and $\{Z - Z_n\} \in \mathfrak{P}$. From [2], 4 we infer that the assumptions of [2], 3 will be fulfilled, when we put $Z = \mathfrak{Z}$, $A = \mathfrak{A}$. The closure of a set $\mathfrak{R} \subset \mathfrak{Z}$ is defined as in [2], 1, and denoted by $\boldsymbol{u}\mathfrak{R}$. The notion of a continuous additive mapping of a set $\mathfrak{M} \subset \mathfrak{Z}$ into E_1 is defined in an evident way (see [2], 1 and 5, with $\mathfrak{G} = E_1$, of course).

The domain of definition of a map φ is denoted by Dom φ. Let $\mathscr{F}$ be the system of all realvalued functions f ($\pm\,\infty$ not excluded) such that Dom $f \subset E_m$. To each $f \in \mathscr{F}$ we attach the set $\mathfrak{M}(f)$ of all $Z \in \mathfrak{Z}$ such that finite Lebesgue integral $\lambda(f, Z) = \int_Z f(x)\,dx$ exists; instead of $\lambda(f, .)$ we shall write $\lambda(f)$. The system $\mathfrak{M}(f)$ is clearly an ideal in the ring $\mathfrak{Z}$.

2. *Let $\mathfrak{R} \subset \mathfrak{Z}$. Let $\mathfrak{T}$ denote the system of all $T \in \mathfrak{Z}$ with the following property: given any $\varepsilon > 0$, there exists $R \in \mathfrak{R}$ such that $R \subset T$, $|T - R| < \varepsilon$, $T - R \in \mathfrak{A}$. Then $\boldsymbol{u}\mathfrak{R} \subset \mathfrak{T}$.*

Proof. It is easy to see that $\mathfrak{R} \subset \mathfrak{T}$, $\boldsymbol{u}\mathfrak{T} = \mathfrak{T}$.

3. *Let $\mathfrak{R} \subset \mathfrak{Z}$, $T \in \boldsymbol{u}\mathfrak{R}$. Then there exist $R_n \in \mathfrak{R}$ such that $R_n \subset T$ $(n = 1, 2, \ldots)$, $T - R_n \in \mathfrak{A}$, $|T - \bigcup R_n| = 0$.*

(This follows immediately from 2.)

4. *Let $M \in \mathbf{u}(\mathfrak{M}(f))$. Then $f(x) \in E_1$ for almost all $x \in M$ and f is measurable on M.*

(This follows immediately from 3.)

5. *For each $f \in \mathscr{F}$, $\lambda(f)$ is a continuous additive mapping of the set $\mathfrak{M}(f)$ into E_1.*

Proof. The additivity of $\lambda(f)$ is clear. Let now $M \in \mathfrak{M}(f)$, $\{A_n\} \in \mathfrak{P}$, $A_n \subset M$. As $|A_n| \to 0$, it follows from absolute continuity of the Lebesgue integral that $\lambda(f, A_n) \to 0$. The mapping λ is therefore continuous (see [2], 6).

6. Definition. To each mapping $\lambda(f)$, there corresponds according to [2], 19 (we put $A = \mathfrak{A}$, $Z = \mathfrak{Z}$, $\mathfrak{G} = E_1$, $\mu = \lambda(f)$ there) a mapping β; we put $\beta = \beta(f)$, and for each $A \in \text{Dom}\,\beta$ we shall write $\beta(A) = \beta(f, A)$ (so that $\beta(f) = \beta(f, .)$). Of course, Dom $\beta(f) \subset \mathbf{u}(\mathfrak{M}(f))$. From [2], 22 it follows that Dom $\beta(f)$ is an ideal in $\mathfrak{A}$ and that the mapping $\beta(f)$ is continuous and additive. Using notations of [2], 24, we have, of course, $\beta(f) = \beta(\lambda(f))$. Instead of "$A \in \text{Dom}\,\beta(f)$" we shall sometimes write "$\beta(f, A)$ exists", etc. In this case $A \in \mathbf{u}(\mathfrak{M}(f))$, and from 4 we infer that $f(x) \in E_1$ for almost all $x \in A$.

Remark. Supposing $A \in \text{Dom}\, \beta(f) \cap \mathfrak{M}(f)$ we have $\beta(f, A) = \lambda(f, A)$ by [2], 20; hence, the mapping $\beta(f)$ is a kind of improper integral of the function f.

7. Theorem. *Let $f, g, h \in \mathscr{F}$; suppose that the sum $s = \beta(f, A) + \beta(g, A)$ has a meaning and $h(x) = f(x) + g(x)$ for almost all $x \in A$. Then $s = \beta(h, A)$.*

Proof. Clearly $\lambda(h, M) = \lambda(f, M) + \lambda(g, M)$ for each $M \subset A$ such that the right-hand side is meaningful. Now, we apply [2], 25.

8. Theorem. *Let $f, g \in \mathscr{F}$, $c \in E_1$; suppose that $\beta(f, A)$ exists and that $g(x) = c\, f(x)$ for almost all $x \in A$. Then $\beta(g, A) = c\, \beta(f, A)$.*

(This follows immediately from [2], 26.)

9. *Let ζ be a one-to-one regular mapping of an open set $G \subset E_m$ into E_m. Let K be compact, $K \subset G$. Then there exist $c_1, c_2 \in E_1$ such that for each measurable set $S \subset K$ the relations $|\zeta(S)| \leqq c_1(S)$, $\|\zeta(S)\| \leqq c_2 \|S\|$ hold.*

Proof. For a matrix T with elements t_{ik} we put $\nu(T) = (\sum_{i,k} t_{ik}^2)^{\frac{1}{2}}$; vectors are considered as matrices with one column. Let M be the functional matrix of the mapping ζ, $N = M^{-1} \,.\, |\det M|$. There exist $c_1, c_2 \in E_1$ such that, for each $x \in K$, $|\det M(x)| < c_1$ and $\nu(N(x)) < c_2$. Let S be a measurable subset of K. From a well-known theorem, we obtain $|\zeta(S)| = \int_S |\det M(x)|\, \mathrm{d}x \leqq c_1 |S|$. The inequality $\|\zeta(S)\| \leqq c_2 \|S\|$ is clear for $\|S\| = \infty$; suppose therefore $S \in \mathfrak{A}$ and let w be a continuous vector on the set $\zeta(G)$ such that $\nu(w(y)) \leqq 1$ for each $y \in \zeta(S)$. Putting $v(x) = N(x)\, w(\zeta(x))$, we have $\nu(v(x)) \leqq c_2$ for each $x \in S$, and, according to [1], 50, $P(\zeta(S), w) = P(S, v) \leqq c_2 \|S\|$, whence $\|\zeta(S)\| \leqq c_2 \|S\|$.

10. *Let ζ be a one-to-one regular mapping of an open set $G \subset E_m$ into E_m. Let S be bounded, $\bar{S} \subset G$, $S_n \to S$, $A \in \mathfrak{A}$, $\bar{A} \subset G$. Then $\zeta(S_n) \to \zeta(S)$, $\zeta(A) \in \mathfrak{A}$.*

(This follows immediately from 9.)

11. Theorem. *Let ζ be a one-to-one regular mapping of an open set $G \subset E_m$ into E_m, and let $f \in \mathscr{F}$. Let D be the functional determinant of the mapping ζ. Put $g(x) = f(\zeta(x))\, |D(x)|$ for each x such that the right-hand side is defined. Suppose that $\bar{A} \subset G$. Then $\beta(g, A) = \beta(f, \zeta(A))$ whenever at least one side of this equation has a meaning.*

Proof. I. Let $\beta(g, A)$ exist. First, we prove that $\zeta(A) \in \boldsymbol{u}(\mathfrak{M}(f))$. There exists a compact set $K \subset \mathfrak{A}$ such that $A \subset K \subset G$. Let $\mathfrak{F}$ denote the system of all $S \subset E_m$ such that $\zeta(S \cap K) \in \boldsymbol{u}(\mathfrak{M}(f))$. For $S_n \in \mathfrak{F}$, $S_n \to S$, we have $S_n \cap K \to S \cap K$ so that, according to 10, $\zeta(S_n \cap K) \to \zeta(S \cap K)$. Hence $S \in \mathfrak{F}$; thus, we see that $\boldsymbol{u}\mathfrak{F} = \mathfrak{F}$. Supposing $S \in \mathfrak{M}(g)$ we have $S \cap K \in \mathfrak{M}(g)$, too, and from the transformation theorem for Lebesgue integrals we get $\lambda(g, S \cap K) = \lambda(f, \zeta(S \cap K))$ so that $\zeta(S \cap K) \in \mathfrak{M}(f)$, $S \in \mathfrak{F}$; this proves that $\mathfrak{M}(g) \subset \mathfrak{F}$. Hence it follows that $A \in \boldsymbol{u}(\mathfrak{M}(g)) \subset \mathfrak{F}$, or $\zeta(A) \in \boldsymbol{u}(\mathfrak{M}(f))$.

As Dom $\beta(g)$ is an ideal in $\mathfrak{A}$, $\beta(g, S)$ exists for each S such that $S \in \mathfrak{A}$, $S \subset A$. If $T \in \zeta(A)\,\mathfrak{A}$, we have $\bar{T} \subset \zeta(\bar{A})$ and, according to 10, $\zeta^{-1}(T) \in \mathfrak{A}$. We see that on $\zeta(A)\,\mathfrak{A}$ we may define a mapping $\varphi(T) = \beta(g, \zeta^{-1}(T))$; φ is continuous. If $T \in$ $\in (\zeta(A)\,\mathfrak{A}) \cap \mathfrak{M}(f)$, then $\varphi(T) = \lambda(g, \zeta^{-1}(T)) = \lambda(f, T)$; from the definition of the mapping $\beta(f)$ we see that $\beta(f, \zeta(A)) = \varphi(\zeta(A)) = \beta(g, A)$.

II. Let $\beta(f, \zeta(A))$ exist. Let D_1 be the functional determinant of the mapping ζ^{-1}. Put $h(y) = g(\zeta^{-1}(y))\,|D_1(y)|$ for each y such that the right-hand side is meaningful. Then, for each $y \in \zeta(G) \cap \operatorname{Dom} f$, we have $h(y) = f(y).\,|D(\zeta^{-1}(y))|\,|D_1(y)| = f(y)$. According to 4, almost all $y \in \zeta(A)$ lie in $\operatorname{Dom} f$, and Theorem 8 (with $c = 1$) then gives $\beta(f, \zeta(A)) = \beta(h, \zeta(A))$. If we put in I $h, g, \zeta(A)), \zeta^{-1}$ instead of g, f, A, ζ respectively, we get $\beta(h, \zeta(A)) = \beta(g, A)$, which completes the proof.

12. *Let $A \in \mathfrak{A}$ and let v be a continuous vector on $\bar{A}$. For each $C \in A\mathfrak{A}$ put $\varphi(C) =$ $= P(C, v)$. Then the mapping φ is continuous.*

Proof. Let $\{C_n\} \in \mathfrak{P}$, $\bigcup C_n \subset A$, and $v = [v_1, \ldots, v_m]$. Choose $\varepsilon > 0$. There exist polynomials $w_1, \ldots, w_m$ such that $|w_i(x) - v_i(x)| < \varepsilon$ $(i = 1, \ldots, m)$ for each $x \in \bar{A}$. Put $w = [w_1, \ldots, w_m]$, $\sigma_1 = \sup \|C_n\|$ $(n = 1, 2, \ldots)$, $\sigma_2 = \sup |\operatorname{div} w(x)|$ $(x \in A)$. According to [1], 16, c we have $|\varphi(C_n)| \leqq |P(C_n, v - w)| + |\lambda(\operatorname{div} w, C_n)| \leqq$ $\leqq \sigma_1 \varepsilon m^{\frac{1}{2}} + \sigma_2 |C_n|$ so that $\varphi(C_n) \to 0$. According to [1], 14, remark 2, φ is additive. Hence (see [2], 6) our assertion easily follows.

13. Notation. 1) For each open set $G \subset E_m$, let $\mathfrak{R}(G)$ denote the system of all $A \in \mathfrak{A}$ such that $\bar{A} \subset G$.

2) We say that a function f is the integral divergence of a vector v on a set $M \subset E_m$, if the vector v is continuous on M and if $P(K, v) = \lambda(f, K)$ for each closed cube $K \subset M$.

14. Theorem. *Let G be an open set in E_m and let $A \in \mathbf{u}(\mathfrak{R}(G))$. Let v be a continuous vector on $\bar{A} \cup G$ and let the function f be the integral divergence of v on G. Then $P(A, v) = \beta(f, A)$.*

Proof. For each $C \in A\mathfrak{A}$ put $\varphi(C) = P(C, v)$. It follows from 12 that φ is continuous and, by [1], 24, we have $\varphi(C) = \lambda(f, C)$ for each $C \in A\mathfrak{A} \cap \mathfrak{R}(G)$. Now it follows from [2], 21 that $\beta(f, A) = \varphi(A) = P(A, v)$.

Remark 1. There are several sufficient conditions for a function f to be the integral divergence of a vector $v = [v_1, \ldots, v_m]$ on an open set $G \subset E_m$. If, for example, $\partial v_1/\partial x_1, \ldots, \partial v_m/\partial x_m$ are continuous on G or if $v_1, \ldots, v_m$ have a total differential in each point $x \in G$ and if for each closed cube $K \subset G$ the integral $\lambda\,(\operatorname{div} v, K)$ exists, we may put $f = \operatorname{div} v$. (See also [3], Theorem 4, condition 1.)

Remark 2. In theorem 23 a sufficient condition for the validity of the relation $A \in \mathbf{u}(\mathfrak{R}(G))$ will be given.

15. *Let G, G_α be open sets in E_m (α runs through an arbitrary index-set). For each α let $\mathfrak{R}(G_\alpha) \subset \mathbf{u}(\mathfrak{R}(G))$. Then also $\mathfrak{R}(\bigcup G_\alpha) \subset \mathbf{u}(\mathfrak{R}(G))$.*

Proof. Let $A \in \mathfrak{R}(\bigcup G_\alpha)$. It is easy to see that there exists $\varepsilon > 0$ such that the following holds: if K is a set with diameter $< \varepsilon$ such that $K \cap \bar{A} \neq \emptyset$, then K lies in G_α for some α. Hence there exist closed cubes $K_1, \ldots, K_n$ such that $\bar{A} \subset \bigcup_{j=1}^{n} K_j$ and, for each j, K_j lies in G_α for some α. By assumption, $A \cap K_j \in \mathbf{u}(\mathfrak{R}(G))$ for $j = 1, \ldots, n$. As the system $\mathfrak{R}(G)$ contains the union of each pair of its elements, we see from [2], 12 (where we put $P = Q = \mathfrak{R}(G)$) that the system $\mathbf{u}(\mathfrak{R}(G))$ has this property too. Hence it follows immediately that $A = \bigcup_{j=1}^{n} (A \cap K_j) \in \mathbf{u}(\mathfrak{R}(G))$.

16. Notation. Let $\mathfrak{N}$ denote the system of all $N \subset E_m$ with the following property: if G, G_1 are open sets in E_m such that $G_1 - G \subset N$, then $\mathfrak{R}(G_1) \subset \mathbf{u}(\mathfrak{R}(G))$.

17. *Let G be an open set in E_m; let $A \in \mathfrak{A}$ and $\bar{A} - G \in \mathfrak{N}$. Then $A \in \mathbf{u}(\mathfrak{R}(G))$.*

Proof. Put $F = \bar{A} - G$, $U = E_m - F$. If $T \in \mathfrak{R}(U)$, then $\overline{A \cap T} \subset \bar{A} \cap \bar{T} \subset \subset (G \cup F) \cap (E_m - F) \subset G$, whence $A\,\mathfrak{R}(U) \subset \mathfrak{R}(G)$. It follows from $E_m - U \in \mathfrak{N}$ that $\mathfrak{R}(E_m) \subset \mathbf{u}(\mathfrak{R}(U))$, and by [2], 10 we obtain $A \in A\,\mathbf{u}(\mathfrak{R}(U)) \subset \mathbf{u}(A\,\mathfrak{R}(U)) \subset \subset \mathbf{u}(\mathfrak{R}(G))$.

18. *Let N_j be closed sets, $N_j \in \mathfrak{N}$ ($j = 1, 2, \ldots$). Then $\bigcup_{j=1}^{\infty} N_j \in \mathfrak{N}$.*

Proof. Let G, G_1 be open, $G_1 - G \subset \bigcup_{j=1}^{\infty} N_j$. Let U be the maximal open set such that $\mathfrak{R}(U) \subset \mathbf{u}(\mathfrak{R}(G))$ (by 15, such U exists). Suppose that $G_1 \subset U$ is false; we shall arrive at a contradiction. Put $F = G_1 - U$. Hence, F is a non-empty G_δ – set in E_m. As $U \supset G$, we have $F \subset G_1 - G \subset \bigcup_{j=1}^{\infty} N_j$. The Baire's theorem implies that there exist an index p and an open set $V \subset G_1$ such that $\emptyset \neq V \cap F \subset F \cap N_p$; hence

$$\emptyset \neq V - U \subset N_p \in \mathfrak{N}\,. \tag{2}$$

Accordingly, $\mathfrak{R}(V) \subset \mathbf{u}(\mathfrak{R}(U)) \subset \mathbf{u}(\mathfrak{R}(G))$, and therefore $V \subset U$, a contradiction with (2). This proves that $G_1 \subset U$ so that $\mathfrak{R}(G_1) \subset \mathfrak{R}(U) \subset \mathbf{u}(\mathfrak{R}(G))$; hence $\bigcup_{j=1}^{\infty} N_j \in \mathfrak{N}$.

19. *Let $N \in \mathfrak{N}$, $M \subset E_n$. Suppose there exists a sequence $\{U_n\} \in \mathfrak{P}$ such that $M \subset \bigcap_{n=1}^{\infty} U_n^\circ$. Then $M \cup N \in \mathfrak{N}$.*

Proof. Let G, G_1 be open sets, $G_1 - G \subset N \cup M$; further, let $A \in \mathfrak{R}(G_1)$. Put $A_n = A - U_n$. Then $\bar{A}_n \subset \bar{A} - U_n^\circ \subset G_1 - M$, hence $\bar{A}_n - G \subset G_1 - G - M \subset \subset N$. According to 17 we have $A_n \in \mathbf{u}(\mathfrak{R}(G))$ for all n. As $A - A_n = U_n \cap A$, we have $A_n \to A$ so that $A \in \mathbf{u}(\mathfrak{R}(G))$. This proves this lemma.

20. Notation. For each set $M \subset E_m$, the symbol $H(M)$ will denote its outer $(m-1)$-dimensional Hausdorff measure.

21. *Let κ be the volume of the $(m-1)$-dimensional sphere with diameter* 1. *Let M be a bounded set in E_m, $H(M) < \infty$. Then there exist open sets $A_n \in \mathfrak{A}$ such that $M \subset A_n$, $\|A_n\| \leqq 2m\kappa^{-1}H(M) + n^{-1}$ $(n = 1, 2, \ldots)$, $|A_n| \to 0$.*

Proof. For each m-dimensional cube K, let $h(K)$ denote the length of its edge. It is easy to see ([1], remark to Theorem 20) that $\|K^\circ\| = \|K\| = 2m(h(K))^{m-1}$. From the definition of $H(M)$ it follows that there exist cubes K_{nk} $(n, k = 1, 2, \ldots)$ with the following properties: $K_{nk} \cap M \neq \emptyset$, $h(K_{nk}) < n^{-1}$ for all k, n, $\bigcup_k K^\circ_{nk} \supset M$, $\kappa \sum_k (h(K_{nk}))^{m-1} < H(M) + \kappa(2mn)^{-1}$ for all n. Hence

$$\sum_k \|K_{nk}\| < 2m\kappa^{-1} H(M) + n^{-1} \quad (n = 1, 2, \ldots). \tag{3}$$

Put $A_n = \bigcup_k K^\circ_{nk}$. According to [1], 37,

$$\|A_n\| \leqq \sum_k \|K_{nk}\| \quad (n = 1, 2, \ldots). \tag{4}$$

Further $|A_n| \leqq \sum_k (h(K_{nk}))^m \leqq n^{-1} \sum_k (h(K_{nk}))^{m-1} < n^{-1}(\kappa^{-1} H(M) + (2mn)^{-1}) \to$ $\to 0$. Now, our assertion follows easily from (3) and (4).

22. *Let $H(S) < \infty$, $\varepsilon > 0$. Then there exists a closed set F such that $H(F) \leqq H(S)$, $H(S - F) < \varepsilon$.*

Proof. Let κ denote the volume of the $(m-1)$-dimensional sphere with diameter 1 and let $d(R)$ be the diameter of $R \subset E_m$. There exist closed sets C_{nk} such that $\bigcup_k C_{nk} \supset$ $\supset S$, $\kappa \sum_k (d(C_{nk}))^{m-1} < H(S) + n^{-1}$ for all n and $d(C_{nk}) < n^{-1}$ for all k, n. It is possible to choose $r_1, r_2, \ldots$ such that $\sum_{n=1}^{\infty} \sum_{k=r_n+1}^{\infty} (d(C_{nk}))^{m-1} < \infty$. For $p = 1, 2, \ldots$ define $B_p = \bigcap_{n=p}^{\infty} \bigcup_{k=1}^{r_n} C_{nk}$; further, put $B = \bigcup_{p=1}^{\infty} B_p$, $Q = \bigcap_{p=1}^{\infty} \bigcup_{n=p}^{\infty} \bigcup_{k=r_n+1}^{\infty} C_{nk}$. It is easy to see that $H(B_p) \leqq H(S)$ for each p, and $H(Q) = 0$, $S \subset B \cup Q$. There exists a p such that $H(B - B_p) < \varepsilon$; write $F = B_p$. Then $H(S - F) \leqq H(S - B) + H(B - F) < \varepsilon$.

23. Theorem. *Let $A \in \mathfrak{A}$ and let G be an open set in E_m. Suppose there exist M_n with $H(M_n) < \infty$ $(n = 1, 2, \ldots)$ and $\bar{A} - G = \bigcup_{n=1}^{\infty} M_n$. Then $A \in \mathbf{u}(\mathfrak{R}(G))$.*

Proof. As it follows from 22, there exist closed sets F_n such that $H(F_n) < \infty$ $(n = 1, 2, \ldots)$ and $\sum_{n=1}^{\infty} H(M_n - F_n) < \infty$. Put $S = \bar{A} - G$ $(= \bigcup_{n=1}^{\infty} M_n)$, $T = \bigcup_{n=1}^{\infty} F_n$. Evidently, $S - T \subset \bigcup_{n=1}^{\infty} (M_n - F_n)$ so that $H(S - T) < \infty$. According to 21 and 19 (where we put $N = \emptyset$, $M = F_n$) $F_n \in \mathfrak{N}$, and from 18 we get $T \in \mathfrak{N}$. Sections 21 and 19 (we put $N = T$, $M = S - T$ there) give $S \cup T = T \cup (S - T) \in \mathfrak{N}$, and all the more $S \in \mathfrak{N}$. Now apply 17.

24. *Let A be a bounded set, $B \in \mathfrak{A}$, $\dot{A} \subset B^\circ$. Then* $\|A - B\| \leqq \|B\|$.

Proof. Put $A_1 = A - B$, $A_2 = A \cup B$. As $A_1 = \bar{A} - B$, $A_2 = \bar{A} \cup B$, both A_1, A_2 are measurable; as $A_1 \subset \bar{A} - B^\circ$, $A^\circ \cup B^\circ \subset A_2$, we have $\bar{A}_1 \subset \bar{A} - B^\circ \subset$ $\subset A^\circ \subset A^\circ \cup B^\circ \subset A_2^\circ$, and from [1], 9, d) we infer that $\|A - B\| \leqq \|A_1\| + \|A_2\| =$ $= \|A_2 - A_1\| = \|B\|$.

25. *Let A be a bounded set. Let $A_n \subset A$, $A_n \in \mathfrak{A}$, $|A - A_n| \to 0$. Then* $\|A\| \leqq$ $\leqq \liminf_{n \to \infty} \|A_n\|$.

Proof. Let $v_1, \ldots, v_m$ be polynomials such that $\sum_{i=1}^{m} (v_i(x))^2 = 1$ for each $x \in A$; put $v = [v_1, \ldots, v_m]$. From the assumption $|A - A_n| \to 0$ it follows easily that $P(A_n, v) \to P(A, v)$; as $P(A_n, v) \leqq \|A_n\|$, we have $\|A\| = \sup_v P(A, v) \leqq \liminf_{n \to \infty} \|A_n\|$.

26. Theorem. *Let A be a bounded set such that its boundary $\dot{A}$ fulfils the relation $H(\dot{A}) < \infty$. Then $A \in \mathfrak{A}$.*

Proof. According to 21 there exists a sequence $\{B_n\} \in \mathfrak{P}$ such that $\dot{A} \subset B_n^\circ$; put $A_n = A - B_n$. From 24 we have $\sup_n \|A_n\| < \infty$; as $A - A_n \subset B_n$, we conclude that $|A - A_n| \to 0$. Now we apply 25.

27. Example. Let f_1 be a continuous function on $(0, \infty)$ such that the integral $\int_0^1 f_1(t)\,dt$ converges non-absolutely. Put $A = \{[x_1, \ldots, x_m] \in E_m;\ 0 < x_i < 1$ $(i = 1, \ldots, m)\}$, $G = \{[x_1, \ldots, x_m] \in E_m;\ x_1 > 0\}$. Further, for $x = [x_1, \ldots, x_m] \in G$ put $f(x) = f_1(x_1)$ and for $x \in \bar{G}$ put $v(x) = [\int_0^{x_1} f_1(t)\,dt, 0, \ldots, 0]$. It is clear that $\lambda(f, A)$ does not exist; however, from Theorems 14 and 23 we infer that $\beta(f, A)$ exists.

Remark. Some results of this paper are contained in [4].

References

[1] *J. Mařík:* The surface integral, Czech. Math. J., *6 (81)*, 1956, 522—558.

[2] *J. Holec, J. Mařík:* Continuous additive mappings, Czech. Math. J., *15 (90)*, 1965, 237—243.

[3] *J. Mařík:* Заметка к теории поверхностного интеграла, Чех. Мат. Ж., *6 (81)*, 1956, 387—400.

[4] *J. Mařík:* Uneigentliche mehrfache Integrale, Wiss. Z. Humboldt-Univ. Berlin, Math.-nat. R. *X* (1961), 413—414.

Резюме

НЕАБСОЛЮТНО СХОДЯЩИЙСЯ ИНТЕГРАЛ В E_m И ТЕОРЕМА ГАУССА

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Пусть для каждого $A \subset E_m$ $|A|$ обозначает внешнюю меру Лебега множества A. Пусть $\mathfrak{A}$ — система всех ограниченных измеримых множеств $A \subset E_m$, периметр $\|A\|$ которых конечен. Пусть теперь f — функция, определенная в некоторой части пространства E_m. Методом, использованным в работе [2], можно для некоторого класса множеств $\mathfrak{B}(f)$ определить интеграл $\beta(f, .)$, который не должен сходится абсолютно. Если $A \in \mathfrak{A}$ и интеграл Лебега $L = \int_A f(x)\,\mathrm{d}x$ сходится, то $\beta(f, A) = L$. Система $\mathfrak{B}(f)$ является идеалом в $\mathfrak{A}$. Если $A \in \mathfrak{B}(f)$, $A_n \subset A$, $|A_n| \to 0$, $\sup \|A_n\| < \infty$ $(n = 1, 2, \ldots)$, то $\beta(f, A_n) \to 0$. Отображение $\beta(f, A)$ является аддитивным относительно A и линейным относительно f. Для взаимно однозначных регулярных отображений интеграл β преобразуется соответственно известной формуле. Пусть, далее, для $M \subset E_m$ $H(M)$ — мера Хаусдорфа размерности $m - 1$. Пусть A — ограниченная часть E_m, $\dot{A}$ — её граница, и G — открытое множество в E_m. Пусть, далее, $H(\dot{A}) < \infty$ и пусть существуют такие множества M_n, что $H(M_n) < \infty$ $(n = 1, 2, \ldots)$, и $\bar{A} - G = \bigcup_{n=1}^{\infty} M_n$. Предположим, наконец, что v — непрерывный вектор, определенный на $G \cup \bar{A}$, имеющий на G непрерывные частные производные первого порядка. Тогда существует $\beta(\operatorname{div} v, A)$ и равняется поверхностному интегралу вектора v через границу множества A.