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Jaroslav Holec; Jan Mařík

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CONTINUOUS ADDITIVE MAPPINGS

JAROSLAV HOLEC and JAN MAŘÍK, Praha

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Let Z be a Boolean ring and $\mathfrak{G}$ an Abelian group. Suppose that a convergence on Z and a convergence on $\mathfrak{G}$ with certain properties are given and let μ be a continuous additive mapping of a suitable set $M \subset Z$ into $\mathfrak{G}$. We construct a set B, contained in the closure of M, and a continuous additive mapping β of B into $\mathfrak{G}$ that coincides with μ on $B \cap M$. The results enable us in a further paper to extend the mapping μ.

1. Let M, N be non-empty sets. A mapping of N into M will sometimes be denoted by the symbol $\{x_n\}_{n \in N}$ or simply $\{x_n\}$, where x_n is the image of n in the mapping under study. Let $\mathfrak{B}$ be the set of all mappings of N into M and let a subset $\mathfrak{K}$ of the Cartesian product $\mathfrak{B} \times M$ be given. Instead of $[\{x_n\}, x] \in \mathfrak{K}$ we usually write $x_n \to x$; the set $\mathfrak{K}$ is called a convergence (with support N). In the sequel, we often define directly the meaning of the symbol $x_n \to x$; the corresponding set $\mathfrak{K}$ is then, of course, the set of all pairs $[\{x_n\}, x]$ such that $x_n \to x$.

A set $F \subset M$ is called closed (with respect to the given convergence), if the implication $(x_n \in F,\ x_n \to x) \Rightarrow (x \in F)$ is valid. It is easy to see that the intersection of an arbitrary class of closed sets is closed and that the set M is closed. For each $P \subset M$ there exists, therefore, the smallest closed set, containing P; this set will be denoted by $\boldsymbol{u}P$. Evidently, a set Q is closed if and only if $Q = \boldsymbol{u}Q$.

Let R be a further non-empty set and let $\mathfrak{K}^*$ be a convergence on R with support N. For $[\{r_n\}_{n \in N}, r] \in \mathfrak{K}^*$ we shall write $r_n \to r$ again; there is no danger of misunderstanding. If φ is a mapping of a set $P \subset M$ into R such that the relations $x_n \in P$ $(n \in N)$, $x \in P$, $x_n \to x$ imply $\varphi(x_n) \to \varphi(x)$, we say that φ is continuous (with respect to the given convergences).

2. An algebraical ring Y is called a Boolean ring, if $yy = y$ for each $y \in Y$. (We don't suppose that Y has a unit.) The zero of Y will be denoted by 0.

Let Y be a Boolean ring. If $x, y \in Y$, we have $x + y = (x + y)(x + y) = x + xy + yx + y$ so that $xy + yx = 0$; if we put $y = x$, we get $x + x = 0$. At the same time we see that $xy = yx$; the ring Y is therefore commutative.

For $x, y \in Y$ we put $x \vee y = x + y + xy$. If $P, Q \subset Y$, we denote by $P + Q$ the set of all $x + y$, where $x \in P$, $y \in Q$; in a similar way we define PQ, $P \vee Q$. If P consists of only one element x, we write $P + Q = x + Q$ etc.

The union, the intersection and the difference of sets S, V will be denoted by $S \cup V$, $S \cap V$ and $S - V$ respectively. If $P, Q, R \subset Y$, we write $PQ \cap R = (PQ) \cap R$.

Remark. Let X be a ring of sets (i.e. a non-empty class of sets that contains with every pair of its elements their union and difference). If we put $x + y = (x - y) \cup \cup (y - x)$, $xy = x \cap y\,(= x - (x - y))$ for $x, y \in X$, we see easily that X is a Boolean ring. Clearly $x \cup y = x \vee y$, $x - y = x + xy$ and we have $x \subset y$ if and only if $xy = x$.

3. In the whole paper, Z is a Boolean ring, A is its subring and a convergence on Z with support N is defined such that the following conditions are fulfilled:

1) If $x_n \to x$, then $xx_n = x_n$, $x + x_n \in A$ $(n \in N)$.

2) If $x_n \to x$, $a \in A$, $z \in Z$, $xz = 0$, then $ax_n \to ax$, $x_n + ax_n \to x + ax$, $x_n + z \to \to x + z$.

The next assertion shows how such a convergence can be defined.

4. *Let Y be a Boolean ring and let N be a non-empty set. Let $B \subset Y$ and let $\mathfrak{P}$ be a set whose elements are mappings of N into B. Suppose that $\{bb_n\} \in \mathfrak{P}$, $\{b_n + bb_n\} \in \in \mathfrak{P}$ for each $\{b_n\} \in \mathfrak{P}$ and each $b \in B$. Define a convergence on Y in the following way: The relation $x_n \to x$ means that*

$$(1) \qquad xx_n = x_n\ (n \in N)\,, \quad \{x + x_n\} \in \mathfrak{P}\,.$$

Then $by_n \to by$, $y_n + by_n \to y + by$, $y_n + z \to y + z$, whenever

$$(2) \qquad y_n \to y\,, \quad b \in B\,, \quad z \in Y\,, \quad yz = 0\,.$$

Proof. Let (2) hold. Plainly $(y + by)(y_n + by_n) = y_n + by_n$; since $y + by + + y_n + by_n = y + y_n + b(y + y_n)$, we have $\{y + by + y_n + by_n\} \in \mathfrak{P}$ so that $y_n + by_n \to y + by$. The relations $by_n \to by$, $y_n + z \to y + z$ can be proved similarly.

5. Throughout the paper, $\mathfrak{G}$ is an Abelian group (its zero will be denoted by 0 again) and a convergence on $\mathfrak{G}$ with support N is defined such that the following implications hold:

3) $(\alpha_n \to \alpha,\ \beta_n \to \beta) \Rightarrow (\alpha_n - \beta_n \to \alpha - \beta)$;

4) $(\alpha_n \to \alpha,\ \alpha_n = 0\ (n \in N)) \Rightarrow \alpha = 0$.

If $\alpha_n \to \alpha$, $\beta_n \to \beta$, then $\alpha_n + \beta_n = \alpha_n - ((\beta_n - \beta_n) - \beta_n) \to \alpha - (0 - \beta) = \alpha + \beta$. If φ, ψ are continuous mappings of a set $Q \subset Z$ into $\mathfrak{G}$, then the mappings $\varphi + \psi$, $\varphi - \psi$ are continuous as well.

A mapping φ of a set $Q \subset Z$ into $\mathfrak{G}$ fulfilling the relation

$$(3) \qquad (x \in Q,\ y \in Q,\ x + y \in Q,\ xy = 0) \Rightarrow (\varphi(x + y) = \varphi(x) + \varphi(y))$$

is called additive.

6. *Let all assumptions of* 4. (*and* 5.) *be valid and let the convergence on* $\mathfrak{G}$ *fulfil the condition*

$$(\alpha_n = \alpha \ (n \in N)) \Rightarrow (\alpha_n \to \alpha)\,.$$

Suppose, further, that $Q \subset Y$, $Q + Q \subset Q$ *and let* φ *be an additive mapping of* Q *into* $\mathfrak{G}$ *such that the relations* $y \in Q$, $\{h_n\} \in \mathfrak{P}$, $h_n \in Q$, $yh_n = h_n \ (n \in N)$ *imply* $\varphi(h_n) \to 0$. *Then* φ *is continuous.*

Proof. Assume that $y_n \in Q$, $y \in Q$, $y_n \to y$ and put $h_n = y_n + y$. Then $\{h_n\} \in \mathfrak{P}$, $h_n \in Q$, $yh_n = h_n$, so that by hypothesis $\varphi(h_n) \to 0$. Since $y_n h_n = y_n + y_n = 0$, $y_n + h_n = y$, we have $\varphi(y) = \varphi(y_n) + \varphi(h_n)$, whence $\varphi(y_n) \to \varphi(y)$.

Remark. In the papers [1] and [2], Z is the class of all measurable sets and A is the class of all bounded sets with finite perimeter in the r-dimensional Euclidean space; $\mathfrak{G}$ is the additive group of all real numbers. (Of course, ab is the intersection $a \cap b$ and $a + b$ the symmetrical difference $(a - b) \cup (b - a)$ of sets $a, b \in$ Z.) The convergence on $\mathfrak{G}$ is defined in the usual way; the convergence on Z is defined in two different manners.

7. *If* $x_n \to x \in Z$, $a \in A$, *then* $x_n \vee a \to x \vee a$.

Proof. Put $y_n = x_n + ax_n$, $y = x + ax$. Then $y_n \to y$, $ay = 0$, so that $x_n \vee a = y_n + a \to y + a = x \vee a$.

8. *The sets* A, $Z - A$ *are closed.*

(The proof may be left to the reader.)

9. *For each* $P \subset Z$ *we have* $A \cap \mathbf{u}P = \mathbf{u}(A \cap P)$.

Proof. Put $F = (Z - A) \cup \mathbf{u}(A \cap P)$ and suppose that $x_n \in F$, $x_n \to x$. If $x \in Z - A$, then, clearly, $x \in F$; if $x \in A$, then $x_n = x + (x + x_n) \in A$, whence $x_n \in \mathbf{u}(A \cap P)$, $x \in \mathbf{u}(A \cap P)$, $x \in F$. We see that $\mathbf{u}F = F$. Since $P \subset F$, we have $\mathbf{u}P \subset F$; therefore $A \cap \mathbf{u}P \subset A \cap F \subset \mathbf{u}(A \cap P)$. Evidently $\mathbf{u}(A \cap P) \subset \mathbf{u}A \cap \mathbf{u}P$ so that, by 8, $\mathbf{u}(A \cap P) \subset A \cap \mathbf{u}P$.

10. *If* $P \subset A$, $Q \subset Z$, *then* $P\,\mathbf{u}Q \subset \mathbf{u}(PQ)$, $P \vee \mathbf{u}Q \subset \mathbf{u}(P \vee Q)$.

Proof. Choose an $x \in P$ and construct the set F of all y with $xy \in \mathbf{u}(PQ)$. Evidently $Q \subset F$. If $y_n \in F$, $y_n \to y$, we have $xy_n \in \mathbf{u}(PQ)$, $xy_n \to xy \in \mathbf{u}(PQ)$, whence $y \in F$. It follows that $\mathbf{u}Q \subset \mathbf{u}F = F$, $P\,\mathbf{u}Q \subset \mathbf{u}(PQ)$. The assertion 7 yields similarly the second inclusion.

11. *If* $a \in A$, $aP \subset P \subset Z$, *then* $aA \cap \mathbf{u}P \subset \mathbf{u}(aA \cap P)$.

Proof. Put $Q = A \cap P$ and choose an $x \in aA \cap \mathbf{u}P$. We have $x \in A \cap \mathbf{u}P = \mathbf{u}Q$ (see 9), whence $x = ax \in a\,\mathbf{u}Q \subset \mathbf{u}(aQ)$ (see 10); clearly $aQ \subset aA \cap P$.

12. *If* $P, Q \subset A$, *then* $\mathbf{u}P\,\mathbf{u}Q \subset \mathbf{u}(PQ)$, $\mathbf{u}P \vee \mathbf{u}Q \subset \mathbf{u}(P \vee Q)$.

Proof. Put $Q_1 = \boldsymbol{u}Q$. From 10 we infer that $PQ_1 \subset \boldsymbol{u}(PQ)$, whence $\boldsymbol{u}(PQ_1) \subset \subset \boldsymbol{u}(PQ)$; according to 8 we get $Q_1 \subset A$ and so, by 10, $Q_1 \boldsymbol{u}P \subset \boldsymbol{u}(Q_1 P)$. It follows that $\boldsymbol{u}P\, \boldsymbol{u}Q \subset \boldsymbol{u}(PQ)$. The second inclusion can be proved similarly.

13. *Suppose that $P, Q \subset Z$, $P + P \subset P$, $PQ \subset Q \subset P$ and that $y_1 + y_2 \in Q$ whenever $y_1, y_2 \in Q$, $y_1 y_2 = 0$. Then $Q + Q \subset Q$.*

Proof. Let $x_1, x_2 \in Q$; put $y_i = x_i + x_1 x_2$. Then $x_1 x_2 \in PQ \subset P$, $y_i \in P$, $y_i = = x_i y_i \in PQ \subset Q$, $y_1 y_2 = 0$ and so $x_1 + x_2 = y_1 + y_2 \in Q$.

14. *If D is an ideal in A, then $\boldsymbol{u}D$ is an ideal in A as well.*

Proof. By 8 we have $\boldsymbol{u}D \subset A$ and from 10 we get $A\, \boldsymbol{u}D \subset \boldsymbol{u}(AD) \subset \boldsymbol{u}D$. If $y_1, y_2 \in \boldsymbol{u}D$, $y_1 y_2 = 0$, then, by 12, $y_1 + y_2 = y_1 \vee y_2 \in \boldsymbol{u}(D \vee D) \subset \boldsymbol{u}D$ and on account of 13 (where we put $P = A$, $Q = \boldsymbol{u}D$) we obtain $\boldsymbol{u}D + \boldsymbol{u}D \subset \boldsymbol{u}D$.

15. *Let $Q, R \subset Z$, $Q \subset \boldsymbol{u}R$. Let the relations $x \in Q$, $y \in \boldsymbol{u}R$, $x + y \in A$, $xy = y$ imply that $y \in Q$ and let φ be such a continuous mapping of Q into $\mathfrak{G}$ that $\varphi(x) = 0$ for each $x \in Q \cap R$. Then $\varphi(x) = 0$ for each $x \in Q$.*

Proof. Put $T = \{t \in Q;\ \varphi(t) = 0\}$, $F = T \cup (\boldsymbol{u}R - Q)$. Suppose that $x_n \in F$, $x_n \to x$. If $x \in Z - Q$, then evidently $x \in \boldsymbol{u}R - Q \subset F$. Let now $x \in Q$. Since $x_n \in \boldsymbol{u}R$, $x + x_n \in A$, $xx_n = x_n$, we have, by assumption, $x_n \in Q$, whence $x_n \in T$, $\varphi(x_n) = 0$, $\varphi(x_n) \to \varphi(x)$ and so $\varphi(x) = 0$, $x \in T \subset F$. Thus we get $\boldsymbol{u}F = F$. From $Q \cap R \subset T$ we deduce that $R \subset F$; as $Q \subset \boldsymbol{u}R$, we obtain $Q \subset F$ and, consequently, $Q \subset T$.

16. *Suppose that $R, C \subset Z$, $AR \subset R$, $AC \subset C$, $b \in A \cap \boldsymbol{u}R$, $bA \cap R \subset C$. Let φ be a continuous mapping of bA into $\mathfrak{G}$ and let ψ be a continuous mapping of C into $\mathfrak{G}$. If $\varphi(x) = \psi(x)$ for each $x \in bA \cap R$, then $\varphi(x) = \psi(x)$ for each $x \in bA \cap C$.*

Proof. Put $Q = bA \cap C$. We have $Q \subset bA \subset A\,\boldsymbol{u}R$ and, according to 10, $A\,\boldsymbol{u}R \subset \boldsymbol{u}(AR)$; hence $Q \subset \boldsymbol{u}(AR) \subset \boldsymbol{u}R$. Further, $QA \subset bA \cap CA \subset Q \subset A$; the relations $x \in Q$, $x + y \in A$, $xy = y$ imply therefore that $y = x + (x + y) \in A$, $y = xy \in QA \subset Q$. Now we apply 15.

17. In 18–23, M is such a subring of Z that $AM \subset M$ and μ is a continuous additive mapping of M into $\mathfrak{G}$.

Remark. In 19, we shall construct a set B such that $A \cap M \subset B \subset A$ and a mapping β of B into $\mathfrak{G}$ which coincides with μ on $A \cap M$. Let now f be a function defined on some subset of the r-dimensional Euclidean space E_r; let M be the class of all sets $m \subset E_r$ such that the Lebesgue integral $\mu(m)$ of f over m converges and let $A, Z, \mathfrak{G}$ have the same meaning as in the remark in 6. Then for $b \in B - M$ the number $\beta(b)$ is a certain improper integral of f over b (see [1]).

18. *The sets $A \cap M$, $A \cap \boldsymbol{u}M$ are ideals in A.*

Proof. The set $D = A \cap M$ is clearly a ring; since $AD \subset AA \cap AM \subset A \cap M$, D is an ideal in A. By 9, $A \cap \boldsymbol{u}M = \boldsymbol{u}D$ and, on account of 14, $\boldsymbol{u}D$ is an ideal in A as well.

19. Let B be the set of all $b \in A \cap \mathbf{u}M$ with the following property: There exists a continuous mapping φ of bA into $\mathfrak{S}$ that coincides with μ on $bA \cap M$. According to 16, where we write $R = M$, $C = bA$, φ is determined by this condition in a unique way. We may therefore define a mapping β of B into $\mathfrak{S}$ by means of the relation $\beta(b) = \varphi(b)$, where φ has the mentioned property.

20. *We have* $A \cap M \subset B \subset A$ *and* $\beta(b) = \mu(b)$ *for each* $b \in A \cap M$.

Proof. If $b \in A \cap M$, then $bA \subset MA \subset M$ and we may choose $\varphi(x) = \mu(x)$ $(x \in \in bA)$.

Remark 1. We have $A \cap M = B \cap M$ and the equality $\beta(b) = \mu(b)$ holds whenever both sides have a meaning.

Remark 2. If $M \subset A$, then β is an extension of μ.

21. *Suppose that* $AR \subset R \subset M$, $b \in A \cap \mathbf{u}R$. *Let* φ *be a continuous mapping of* bA *into* $\mathfrak{S}$ *that coincides with* μ *on* $bA \cap R$. *Then* $b \in B$ *and* $\beta(b) = \varphi(b)$.

Proof. According to 16, where we write $C = M$, φ coincides with μ on $bA \cap M$.

22. *The set* B *is an ideal in* A; *the mapping* β *is continuous and additive.*

Proof. Choose $a \in A$, $b \in B$ and take the mapping φ of 19. Clearly $ab \in A$, $abA \subset \subset bA$ and, by 10, $ab \in A\,\mathbf{u}M \subset \mathbf{u}(AM) \subset \mathbf{u}M$. Hence it follows easily that $ab \in B$ and

$$\beta(ab) = \varphi(ab)\,. \tag{4}$$

First of all we obtain

$$AB \subset B\,. \tag{5}$$

If, further, $b_n \to b$, then $b_n \in A$, $b_n = b_n b$ and, by (4), $\beta(b_n) = \varphi(b_n) \to \varphi(b) = \beta(b)$. This proves the continuity of β.

Take now $b_1, b_2 \in B$ with $b_1 b_2 = 0$. By 18, $A \cap \mathbf{u}M$ is an ideal in A and so $b_1 + + b_2 \in A \cap \mathbf{u}M$. For each $x \in (b_1 + b_2) A$ put $\psi(x) = \beta(b_1 x) + \beta(b_2 x)$. The mapping ψ is evidently continuous. If $x \in ((b_1 + b_2) A) \cap M$, then $b_i x \in A \cap M$; it follows from 20 and from the additivity of μ that $\psi(x) = \mu(b_1 x) + \mu(b_2 x) = = \mu((b_1 + b_2) x) = \mu(x)$. Thus we get $b_1 + b_2 \in B$, $\beta(b_1 + b_2) = \psi(b_1 + b_2) = = \beta(b_1) + \beta(b_2)$. According to (5) and 13 (where we write $P = A$, $Q = B$), B is an ideal in A.

23. $A \cap (Z(B + M)) = B$.

Proof. Suppose that $a \in A$, $z \in Z$, $b \in B$, $m \in M$ and that $a = z(b + m)$. If we put $b_1 = ab$, $m_1 = am$, we have $a = a(b + m) = b_1 + m_1$, $b_1 \in B \subset A$, $m_1 \in M$; since $m_1 = a + b_1 \in A$, we have, by 20, $m_1 \in A \cap M \subset B$, $a = b_1 + m_1 \in B$. It follows that $B \subset A \cap (Z(B + M)) \subset B$.

24. Let Ψ be the set of all mappings ψ with the following properties:

a) ψ maps a subring $M(\psi)$ of Z into $\mathfrak{G}$ and $AM(\psi) \subset M(\psi)$;

b) ψ is continuous and additive.

Now we attach to each $\psi \in \Psi$ a set $B(\psi)$ and a mapping $\beta(\psi)$ in the same way as we attached the set B and the mapping β to μ in 19. Using this notation we have, of course, $\beta = \beta(\mu)$, $M = M(\mu)$, $B = B(\mu) = M(\beta(\mu))$; according to 22, $\beta(\psi) \in \Psi$ for each $\psi \in \Psi$. For $x \in B(\psi)$ we write $(\beta(\psi))(x) = \beta(\psi, x)$.

If we say that a certain relation is valid, we understand, of course, that all expressions in this relation are meaningful. If we write, e.g., $\beta(\psi, x) = 0$, we assert at the same time that $\psi \in \Psi$, $x \in B(\psi)$.

If ω is a mapping of $\mathfrak{G}$ into $\mathfrak{G}$ and if $\alpha \in \mathfrak{G}$, we write $\omega\alpha$ instead of $\omega(\alpha)$. If, moreover, ζ is a mapping of an arbitrary set Y into $\mathfrak{G}$, then $\omega\zeta$ denotes the corresponding composed mapping (i.e. $(\omega\zeta)(x) = \omega\zeta(x)$ for each $x \in Y$).

25. *Suppose that $\psi, \psi_1, \psi_2 \in \Psi$, $b \in B(\psi_1) \cap B(\psi_2)$ and that $\psi(x) = \psi_1(x) + \psi_2(x)$ for each $x \in bA \cap M(\psi_1) \cap M(\psi_2)$. Then $\beta(\psi, b) = \beta(\psi_1, b) + \beta(\psi_2, b)$.*

Proof. Put $R = bA \cap M(\psi_1) \cap M(\psi_2)$, $P_i = bA \cap M(\psi_i)$ $(i = 1, 2)$. Evidently $AP_i \subset P_i \subset A$, whence $P_1P_2 \subset P_1 \cap P_2 = R$. It follows from 12 that

$$\mathbf{u}P_1 \cap \mathbf{u}P_2 \subset \mathbf{u}P_1\,\mathbf{u}P_2 \subset \mathbf{u}(P_1P_2) \subset \mathbf{u}R\,. \tag{6}$$

According to 11, we have $b \in bA \cap \mathbf{u}(M(\psi_i)) \subset \mathbf{u}P_i$ $(i = 1, 2)$ so that, by (6), $b \in \mathbf{u}R$. For each $x \in bA$ put $\varphi(x) = \beta(\psi_1, x) + \beta(\psi_2, x)$. The mapping φ is continuous and for each $x \in R$, by assumption, $\varphi(x) = \psi_1(x) + \psi_2(x) = \psi(x)$. From 21 we infer that $b \in B(\psi)$ and $\beta(\psi, b) = \varphi(b) = \beta(\psi_1, b) + \beta(\psi_2, b)$.

26. *Let ω be a continuous homomorphism of $\mathfrak{G}$ into $\mathfrak{G}$. Suppose that $\chi, \psi \in \Psi$, $b \in B(\psi)$ and that $\chi(x) = \omega\psi(x)$ for each $x \in bA \cap M(\psi)$. Then $\beta(\chi, b) = \omega\beta(\psi, b)$.*

Proof. Put $R = bA \cap M(\psi)$ and define $\varphi(x) = \omega\beta(\psi, x)$ for each $x \in bA$. Then φ is continuous and, by assumption, $\varphi(x) = \omega\psi(x) = \chi(x)$ for each $x \in R$. On account of 11, $b \in bA \cap \mathbf{u}(M(\psi)) \subset \mathbf{u}R$ and, according to 21, where we write $\mu = \chi$, we have $\beta(\chi, b) = \varphi(b) = \omega\beta(\psi, b)$.

27. *Let ω be a continuous automorphism of $\mathfrak{G}$ such that the inverse mapping ω^{-1} is continuous as well. Suppose that $\chi, \psi \in \Psi$ and that $\chi(x) = \omega\psi(x)$ for each $x \in A \cap \cap M(\psi)$. Then $B(\psi) = B(\chi) \cap \mathbf{u}(M(\psi))$.*

Proof. According to 26 we have $B(\psi) \subset B(\chi)$; clearly $B(\psi) \subset \mathbf{u}(M(\psi))$. Choose now a $b \in B(\chi) \cap \mathbf{u}(M(\psi))$ and for each $x \in bA$ define $\varphi(x) = \omega^{-1}\beta(\chi, x)$. Then φ is continuous and $\varphi(x) = \omega^{-1}\chi(x) = \psi(x)$ for each $x \in bA \cap M(\psi)$, so that $b \in B(\psi)$.

28. *If $\psi \in \Psi$, then $\beta(\beta(\psi)) = \beta(\psi)$.*

Proof. If we put $\chi = \beta(\psi)$, we have $B(\chi) \subset \mathbf{u}(M(\chi))$, $M(\chi) = B(\psi) \subset \mathbf{u}(M(\psi))$

and so $B(\chi) \subset \mathbf{u}(M(\psi))$. Now we apply 26 and 27 (where we put $\omega\alpha = \alpha$ for each $\alpha \in \mathfrak{G}$).

29. *Let ω be a continuous automorphism of $\mathfrak{G}$ such that the inverse mapping ω^{-1} is continuous as well. Then $\beta(\omega\psi) = \omega\beta(\psi)$ for each $\psi \in \Psi$.*

Proof. Apply 26 and 27.

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Резюме

НЕПРЕРЫВНЫЕ АДДИТИВНЫЕ ОТОБРАЖЕНИЯ

ЯРОСЛАВ ХОЛЕЦ (Jaroslav Holec) и ЯН МАРЖИК (Jan Mařík), Прага

Пусть Z — кольцо Буля, A — подкольцо Z и $\mathfrak{G}$ — абелева группа. Предположим, что на $\mathfrak{G}$ и на Z определена сходимость со следующими свойствами:

1) Если $x_n \to x \in Z$, то $xx_n = x_n$, $x + x_n \in A$ для всякого n.

2) Если $x_n \to x \in Z$, $a \in A$, $z \in Z$, $xz = 0$, то $ax_n \to ax$, $x_n + ax_n \to x + ax$, $x_n + z \to x + z$.

3) Если $\alpha_n \to \alpha \in \mathfrak{G}$, $\beta_n \to \beta \in \mathfrak{G}$, то $\alpha_n - \beta_n \to \alpha - \beta$.

4) Если $\alpha_n = 0$ для всякого n и $\alpha_n \to \alpha$, то $\alpha = 0$.

Для $P, Q \subset Z$ положим $PQ = \{xy;\, x \in P,\, y \in Q\}$. Пусть Ψ — множество всех отображений ψ, удовлетворяющих следующим условиям:

а) Область определения $M(\psi)$ отображения ψ является подкольцом в Z и $AM(\psi) \subset M(\psi)$, $\psi(M(\psi)) \subset \mathfrak{G}$.

в) Отображение ψ непрерывно и аддитивно.

Каждому $\psi \in \Psi$ поставим в соответствие отображение $\beta(\psi) \in \Psi$, совпадающее с ψ на $A \cap M(\psi)$; $M(\beta(\psi))$ содержится в замыкании $F(\psi)$ множества $A \cap \cap M(\psi)$ и если $b \in F(\psi) - M(\beta(\psi))$, то β нельзя продолжить непрерывным образом на bA. Положим $\beta(\psi, x) = (\beta(\psi))(x)$ $(x \in M(\beta(\psi)))$. Если $\psi, \psi_1, \psi_2 \in \Psi$ и если $\psi_1(x) + \psi_2(x) = \psi(x)$ для $x \in M(\psi_1) \cap M(\psi_2)$, то $\beta(\psi_1, x) + \beta(\psi_2, x) = \beta(\psi, x)$ для $x \in M(\beta(\psi_1)) \cap M(\beta(\psi_2))$. Эти результаты используются в дальнейшей работе для продолжения отображений $\psi \in \Psi$.