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ON CONVERGENCE SPACES AND THEIR SEQUENTIAL ENVELOPES

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In this paper closure topological structures of convergence spaces ($\mathscr{L}$-spaces of M. Fréchet) and their Cartesian products are investigated. By means of continuous functions on L the sequentially regular convergence spaces and their sequential envelopes $\sigma_e(L)$ are defined. The existence of σ-envelopes is proved and topological and convergence properties of such spaces are studied. Several convergence spaces are constructed as examples illustrating some properties of convergence topology.

The notion of convergence on an abstract point set L was axiomatically introduced by M. FRÉCHET [3]. By convergence on L we mean a system $\mathfrak{L}$ of elements $(\{x_n\}, x)$ where $\{x_n\}$ is a sequence of points and x a point of L fulfilling axioms $(\mathscr{L}_0)$, $(\mathscr{L}_1)$ and $(\mathscr{L}_2)$. The closure of a set A consists of all points x such that $(\{x_n\}, x) \in \mathfrak{L}$ where $\bigcup x_n \subset A$. Two distinct convergences on L can induce the same convergence topology. This fact leads to a classification within the system of all convergences on a given set L. In section 2 it is proved [Theorem 1] that in each class of convergences there is a largest convergence $\mathfrak{L}^*$ which is characterized by axiom $(\mathscr{L}_3)$. There is a one-to-one order preserving mapping on the system of all convergence topologies onto the system of all largest convergences on L [Corollary 1].

In section 3 convergence Cartesian space is defined by means of coordinatewise convergence (8). A Cartesian convergence is largest if and only if each coordinate convergence is largest (it is a Cartesian property). The convergence topology on the Cartesian product $\mathbf{X}\{(L_n, \mathfrak{L}_n, \lambda_n) : n = 1, 2, \ldots, n_0\}$ of a finite number n_0 of spaces does not depend on the choice of convergences in the classes $[\mathfrak{L}_n]$ of equivalent convergences. However, the example (p. 84) shows that this need not be true if the number of spaces is infinite [Theorem 6]. The convergence Euclidean space of dimension $P(I)$ is defined as convergence Cartesian space $\mathbf{X}\{(L_\alpha, \mathfrak{L}, \lambda) : \alpha \in I\}$ where L_α denotes for each $\alpha \in I$ the set of real numbers and $\mathfrak{L}$ the usual convergence on it.

In section 4 some properties of sequentially regular spaces are studied and the location of such spaces in the scheme of classification of convergence spaces is given. It is proved that the sequential regularity is a Cartesian property provided that $\mathfrak{L}_\alpha = \mathfrak{L}^*_\alpha$, $\alpha \in I$. [Theorem 10]. Sequentially regular spaces can actually be treated as subspaces of convergence Euclidean spaces [Theorem 11].

A sequentially regular space S is a sequential envelope $\sigma_e(L)$ of a convergence subspace L of S if S is the smallest closed set in S containing L, and the largest sequentially regular space such that each continuous function f on L has a continuous extension $\bar{f}$ on S. In section 5 a criterion for a sequentially regular space S to be a sequential envelope of a subspace L is given [Theorem 13]. Each sequentially regular space has sequential envelopes which are homeomorphic to each other [Theorem 14 and Corollary 5]. A sequentially regular space L_{11} is constructed such that $L_{11} \neq \sigma_e(L_{11})$.

In section 6 it is proved that each system of point sets for which the well known convergence of sets applies is a sequentially regular space. The study of convergence topological properties of systems of sets is essentially the same as the study of the convergence topological structure of the convergence cube vertex space [Theorem 16].

It should be noticed that the definition of a $\langle 0, 1\rangle$ sequential envelope $\sigma_e(L)$ of a completely regular topological convergence space L (which is sequentially regular) is to a certain extent analogous to the definition of Stone-Čech compactification $\beta(L)$. The properties of Stone-Čech compactification $\beta(L)$, however, can substantially differ from those of the sequential envelope $\sigma_e(L)$.

1

In this paper we call the T_1 closure space a point set P and a map u on the system of all subsets of P into itself fulfilling two axioms:

(C_1) $uA = A$ for any finite set $A \subset P$;

(C_2) $u(A \cup B) = uA \cup uB$ for $A \subset P$ and $B \subset P$.

It will[1]) be denoted by (P, u) or simply[2]) by P. The T_1 closure space (P, u) is called the topological T_1 space if the axiom

(F) $u(uA) = uA$ for $A \subset P$

is true.

The map u is called the T_1 closure topology. If it has the property (F), we speak of T_1 topology. The set uA is called the u-closure or simply closure of the set A. If $A = uA$, the set A is closed; it is open if its complement is closed.

From (C_1) and (C_2) it immediately follows that $A \subset B$ implies $A \subset uA \subset uB$.

A subset $U(x)$ of a T_1 closure space (P, u) is a u-neighbourhood of a point x if $x \in P - u(P - U(x))$.

It is easy to prove that $x \in U(x)$ and that the intersection of any two u-neighbourhoods of the same point x is its u-neighbourhood as well. From the definition

[1]) Each T_1 closure space is a gestufter Raum in the sense of F. Hausdorff [5].

[2]) When no confusion seems possible we shall suppress the symbols of topologies and convergences.

of a neighbourhood of the point x it follows that $x \in uA$ if and only if $A \cap U(x) \neq \emptyset$ for each neighbourhood $U(x)$ of x.

Let u and v be two T_1 closure topologies on the same point set P. We say that u is weaker than v (or that v is stronger than u) in symbols $u < v$, if $uA \subset vA$ for each $A \subset P$. The binary relation $<$ partially orders the system of all T_1 closure topologies on the set P. The discrete topology is the weakest element in it.

Now, we mention the definition of a continuous map φ on a T_1 closure space (P, u) into a T_1 closure space (Q, v). The map φ is continuous [5] on P if

$$(1) \qquad \varphi(uA) \subset v\,\varphi(A) \quad \text{for each} \quad A \subset P\,.$$

It is easy to prove that φ is continuous on P if and only if

(2) for each point $a \in P$ and every v-neighbourhood $V(\varphi(a))$ of the point $\varphi(a) \in Q$, there is a u-neighbourhood $U(a) \subset P$ such that $\varphi(U(a)) \subset V(\varphi(a))$.

A continuous map φ on a T_1 closure space P onto a T_1 closure space Q is a homeomorphism if it is one-to-one and φ^{-1} is also continuous.

2

Let L be a point set. Denote by N the set of all naturals. A map φ on N into L such that $\varphi(n) = x_n$ is called a (simple) sequence and denoted by $\{x_n\}_{n=1}^{\infty}$ or simply $\{x_n\}$. A map ψ on $N \times N$ into L such that $\psi(m, n) = x_{mn}$ will be called a double sequence and denoted by $\{x_{mn}\}_{m,n=1}^{\infty}$ or simply $\{x_{mn}\}$; a (simple) sequence $\{x_{mn_m}\}_{m=1}^{\infty}$ (in this case $\{n_m\}$ need not be increasing) is called a cross-sequence of $\{x_{mn}\}$. A cross-subsequence is a subsequence of a cross-sequence.

Let L be a point set. Let $\mathfrak{L}$ be a set of pairs $(\{x_n\}, x)$ where $\{x_n\}$ is a sequence of points $x_n \in L$ and x a point of L. We say that $\mathfrak{L}$ is a convergence on L if the following axioms are true [3]:

($\mathscr{L}_0$) If $(\{x_n\}, x) \in \mathfrak{L}$ and $(\{x_n\}, y) \in \mathfrak{L}$, then $x = y$.

($\mathscr{L}_1$) If $x_n = x$ for each natural n, then $(\{x_n\}, x) \in \mathfrak{L}$.

($\mathscr{L}_2$) If $(\{x_n\}, x) \in \mathfrak{L}$ and $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$,then $(\{x_{n_i}\}, x) \in \mathfrak{L}$.

The set L with a convergence $\mathfrak{L}$ on it is called the $\mathscr{L}$-space (Fréchet) and designated by $(L, \mathfrak{L})$.

Instead of $(\{x_n\}, x) \in \mathfrak{L}$ we shall[2]) write $\mathfrak{L} - \lim x_n = x$ and say that the sequence $\mathfrak{L}$-converges to the limit x. A sequence of points is totally $\mathfrak{L}$-divergent if there is no $\mathfrak{L}$-convergent subsequence of points in it.

By means of the convergence $\mathfrak{L}$ on an $\mathscr{L}$-space $(L, \mathfrak{L})$ the closure of a set is defined as follows [5]:

The closure λA of a set $A \subset L$ is the set of all points $x \in L$ such that $x = \mathfrak{L} - \lim x_n$, all x_n being points of A.

It is easy to see that λ is a T_1 closure topology. The T_1 closure space (L, λ) will be called the convergence space and denoted by $(L, \mathfrak{L}, \lambda)$; λ will be called the convergence topology.

From the definition of neighbourhoods in T_1 closure spaces it follows that a set $U(x)$ is a λ-neighbourhood of a point x in a convergence space $(L, \mathfrak{L}, \lambda)$ if

$$(3) \qquad \mathfrak{L} - \lim x_n = x \text{ implies } x_n \in U(x) \text{ for nearly all } n.$$

Now we shall construct several convergence spaces which will be used as examples illustrating some convergence and closure topological properties.

Let A be the set of points x_{mn} and B the set of points x_m and x_0, m and n being naturals. Let $\{m_i\}$ and $\{n_i\}$ denote subsequences of the sequence of all naturals.

The convergence space $(L_1, \mathfrak{L}_1, \lambda_1)$: $L_1 = B$ and $\mathfrak{L}_1$ contains elements of two kinds, viz. all $(\{y_n\}, x)$ such that $y_n = x$ for nearly all n, where $x \in L_1$ and all $(\{y_n\}, x_0)$ such that there does not exist a constant subsequence $\{x\}$ of $\{y_n\}$ for any $x \neq x_0$.

The convergence space $(L_2, \mathfrak{L}_2, \lambda_2)$: $L_2 = B$ and $\mathfrak{L}_2$ consists of elements of two kinds: of $(\{x\}, x)$ for $x \in L_2$ and of $(\{x_{m_i}\}, x_0)$ for each $\{m_i\}$.

The convergence space $(L_3, \mathfrak{L}_3, \lambda_3)$: $L_3 = A \cup B$ and $\mathfrak{L}_3$ is the set of elements $(\{x\}, x)$ for each $x \in L_3$, of $(\{x_{m_i}\}, x_0)$ and of elements $(\{x_{mn_i}\}_{i=1}^{\infty}, x_m)$ for each $\{m_i\}$ and $\{n_i\}$ and for each $m = 1, 2, \dots$.

The convergence space $(L_4, \mathfrak{L}_4, \lambda_4)$: $L_4 = A \cup (x_0)$ and $\mathfrak{L}_4$ is the convergence of elements $(\{x\}, x)$ where $x \in L_4$, and of $(\{x_{mn_i}\}, x_0)$ for each $\{n_i\}$ and for each $m = 1, 2, \dots$.

The convergence space $(L_5, \mathfrak{L}_5, \lambda_5)$: L_5 is the set of all real numbers and $\mathfrak{L}_5$ is the ordinary convergence on it, i.e. $(\{x_n\}, x) \in \mathfrak{L}_5$ whenever $|x_n - x| \to 0$.

The convergence space $(L_6, \mathfrak{L}_6, \lambda_6)$: L_6 is the set of all real numbers and $\mathfrak{L}_6$ consists of all elements $(\{x_n\}, x)$ with the property: $\sum_{n=1}^{\infty} |x_n - x| < \infty$.

Lemma 1. *Let $(L, \mathfrak{L}, \lambda)$ and $(L, \mathfrak{M}, \mu)$ be convergence spaces. Then the two following statements are equivalent:*

(a) $\lambda < \mu$.

(b) If $(\{x_n\}, x) \in \mathfrak{L}$ then $(\{x_{n_i}\}, x) \in \mathfrak{M}$ for a suitable subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

Proof. If $\lambda < \mu$ and $\mathfrak{L} - \lim x_n = x$, then $x \in \lambda \bigcup x_n \subset \mu \bigcup x_n$; consequently there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\mathfrak{M} - \lim x_{n_i} = x$. On the other hand if $x \in \lambda A$, then from (b) it instantly follows that $x \in \mu A$ i.e. $\lambda < \mu$.

If $\mathfrak{L} \subset \mathfrak{M}$, then the condition (b) of Lemma 1 is obviously fulfilled so that $\lambda < \mu$. If however $\lambda < \mu$, we cannot conclude that $\mathfrak{L} \subset \mathfrak{M}$. As a matter of fact, we have $\lambda_1 = \lambda_2$, i.e. $\lambda_1 < \lambda_2$, but the convergence $\mathfrak{L}_1$ is not contained in the convergence $\mathfrak{L}_2$. In order to find the necessary and sufficient conditions let us notice that the inclusion $\subset$ partially orders the system $\mathbf{\mathfrak{L}}$ of all convergences defined on the same point set L. Let us define the equivalence relation $\sim$ in the system $\mathbf{\mathfrak{L}}$:

If $\mathfrak{L} \in \mathbf{\mathfrak{L}}$ and $\mathfrak{M} \in \mathbf{\mathfrak{L}}$, then $\mathfrak{L} \sim \mathfrak{M}$ if $\lambda = \mu$.

In such a way every class $[\mathfrak{L}]$ is a partially ordered class containing all convergences on L which induce the same convergence topology λ in L.

Theorem 1. *In every class $[\mathfrak{L}]$ of convergences on L there is a largest element $\mathfrak{L}^* \in [\mathfrak{L}]$.*

Proof. Denote $\mathfrak{L}^0 = \bigcup_{\mathfrak{N} \in [\mathfrak{L}]} \mathfrak{N}$ and prove[3]) that $\mathfrak{L}^0 = \mathfrak{L}^*$. If $(\{x_n\}, x) \in \mathfrak{L}^0$ and $(\{x_n\}, y) \in \mathfrak{L}^0$ then $(\{x_n\}, x) \in \mathfrak{N}'$ and $(\{x_n\}, y) \in \mathfrak{N}''$ where $\mathfrak{N}'$ and $\mathfrak{N}''$ are suitable convergences in the class $[\mathfrak{L}]$. Denote by λ, ν' and ν'' the respective topologies. Then $\lambda = \nu' = \nu''$ and $y \in \nu'' \bigcup x_n (= \nu' \bigcup x_n)$ so that $(\{x_{n_i}\}, y) \in \mathfrak{N}'$ for a suitable subsequence $\{x_{n_i}\}$ of $\{x_n\}$. By $(\mathscr{L}_2)$ and $(\mathscr{L}_0)$ we have $x = y$. Therefore the axiom $(\mathscr{L}_0)$ is fulfilled in $\mathfrak{L}^0$. Evidently $\mathfrak{L}^0$ also satisfies the axioms $(\mathscr{L}_1)$ and $(\mathscr{L}_2)$.

It remains to prove that $\lambda^0 = \lambda$. First notice that $\mathfrak{L} \subset \mathfrak{L}^0$ so that $\lambda < \lambda^0$. Now suppose that $x \in \lambda^0 A$; then there is a sequence $\{x_n\}$, $x_n \in A$, such that $(\{x_n\}, x) \in \mathfrak{N}'''$ for a suitable $\mathfrak{N}''' \in [\mathfrak{L}]$. Therefore $x \in \nu''' \bigcup x_n (\subset \nu''' A = \lambda A)$, ν''' being a topology induced by $\mathfrak{N}'''$; hence $\lambda^0 < \lambda$. Therefore $\mathfrak{L}^0 = \mathfrak{L}^*$.

F. Hausdorff [5] has defined the maximal $\mathscr{L}$-space the convergence of which cannot be extended without having changed its closure topology. He proved that the maximal $\mathscr{L}$-space satisfies the following axiom [1]:

$(\mathscr{L}_3)$ If each subsequence $\{x_{n_i}\}$ of a sequence $\{x_n\}$ contains a subsequence $\{x_{n_{i_k}}\}$ converging to a point x, then the whole sequence $\{x_n\}$ converges to x.

It is clear that the maximal $\mathscr{L}$-space $(L, \mathfrak{L})$ is defined by the largest convergence in our sense, i.e. $\mathfrak{L} = \bigcup_{\mathfrak{N} \in [\mathfrak{L}]} \mathfrak{N}$.

Now we shall prove that each convergence satisfying condition $(\mathscr{L}_3)$ is the largest one. This assertion is contained in the following

Theorem 2. *Let $(L, \mathfrak{L}, \lambda)$ be a convergence space. Then the conditions* I), II), III) *are equivalent:*

I) *$\mathfrak{L}$ is a largest convergence on L.*

II) *$\mathfrak{L}$ fulfills axiom $(\mathscr{L}_3)$.*

III) *$\mathfrak{L} - \lim x_n = x$ if the following property is fulfilled: each λ-neighbourhood of x contains nearly all x_n.*

Proof. I) $\Rightarrow$ II) by Hausdorff. Prove II) $\Rightarrow$ III). Suppose III) is not true; let x be a point and $\{x_n\}$ a sequence of points of L fulfilling the property mentioned in III) however not $\mathfrak{L}$-converging to x. Then by $(\mathscr{L}_3)$, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$

[3]) The convergences will be denoted by the German capitals $\mathfrak{L}, \mathfrak{M}, \mathfrak{N}, \mathfrak{P}, \mathfrak{T}, \mathfrak{E}, \mathfrak{G}, \ldots$ and the respective convergence topologies usually by the Greek letters $\lambda, \mu, \nu, \pi, \tau, \varepsilon, \gamma, \ldots$. The largest convergence and the respective topology will be usually denoted by the asterisk.

which is totally $\mathfrak{L}$-divergent or which $\mathfrak{L}$-converges to $y \neq x$ and such that $x \neq x_{n_i}$ for each i. Then from (3) it follows that $L - \bigcup_i x_{n_i}$ is a λ-neighbourhood of x not containing nearly all x_n. It remains to prove that III) $\Rightarrow$ I). Suppose III) is true. Denote by $\mathfrak{T}$ the set of all elements $(\{x_n\}, x)$ having the property mentioned in III). If $\mathfrak{N} \in [\mathfrak{L}]$ and $(\{x_n\}, x) \in \mathfrak{N}$ then, in view of (3), $(\{x_n\}, x) \in \mathfrak{T}$ so that $\bigcup_{\mathfrak{N} \in [\mathfrak{L}]} \mathfrak{N} \subset \mathfrak{T}$. On the other hand, from our supposition it follows that $\mathfrak{T} \subset \mathfrak{L}$. Hence $\mathfrak{L} = \bigcup_{\mathfrak{N} \in [\mathfrak{L}]} \mathfrak{N}$. and $\mathfrak{L} = \mathfrak{L}^*$ by Theorem 1.

Let us notice that neither the convergence $\mathfrak{L}_2$ nor $\mathfrak{L}_6$ is the largest one. On the other hand, $\mathfrak{L}_1$ and $\mathfrak{L}_5$ are largest convergences and $\mathfrak{L}_2^* = \mathfrak{L}_1$, $\mathfrak{L}_6^* = \mathfrak{L}_5$ so that $\lambda_2 = \lambda_1$ and $\lambda_6 = \lambda_5$.

Corollary 1. *Let $(L, \mathfrak{L}, \lambda)$ and $(L, \mathfrak{M}, \mu)$ be convergence spaces. Then $\lambda < \mu$ if and only if $\mathfrak{L}^* \subset \mathfrak{M}^*$.*

Proof. If $\lambda < \mu$ and $(\{x_n\}, x) \in \mathfrak{L}^*$, then $x \in \lambda \bigcup x_{n_i} \subset \mu \bigcup x_{n_i}$ for each subsequence $\{x_{n_i}\}$ of $\{x_n\}$. Therefore outside of each μ-neighbourhood of x there is at most a finite number of x_n. Hence $(\{x_n\}, x) \in \mathfrak{M}^*$, by Theorem 2. The converse part of the proof follows immediately from Lemma 1.

From Corollary 1 it can be deduced that there is a one-to-one order preserving correspondence between the system of all largest convergences (this system is partially ordered relative to the ordering $\subset$) and the system of all convergence topologies on the same point set L. As a matter of fact, if $\mathfrak{L}^* \neq \mathfrak{M}^*$, then $\mathfrak{L}^* - \mathfrak{M}^* \neq \emptyset$ or $\mathfrak{M}^* - \mathfrak{L}^* \neq \emptyset$. Consequently, by Corollary 1, either $\lambda < \mu$ or $\mu < \lambda$ is false so that $\lambda \neq \mu$. From the same Corollary it follows that this one-to-one correspondence preserves the order.

Let (P, u) be a closure space. Let $\mathfrak{T}$ be the set of all elements $(\{x_n\}, x)$ such that each u-neighbourhood of x contains nearly all x_n. Then $\mathfrak{T}$ evidently fulfills [4] axioms $(\mathscr{L}_1)$ and $(\mathscr{L}_2)$. It is easy to see that $(\mathscr{L}_3)$ is also fulfilled. Therefore in the case when the axiom $(\mathscr{L}_0)$ is also valid – for example if any two distinct points can be separated by u-neighbourhoods – $\mathfrak{T}$ is the largest convergence and we get a convergence space $(P, \mathfrak{T}, \tau)$ such that $\tau < u$; clearly $\tau = u$ whenever $x \in uA$ implies that $(\{x_n\}, x) \in \mathfrak{T}$ for a suitable sequence of points $x_n \in A$. In such a case the closure topology u is a convergence topology and (P, u) is a convergence space.

Let $(L, \mathfrak{L}, \lambda)$ and $(M, \mathfrak{M}, \mu)$ be convergence spaces. Let $L \subset M$. Then $(L, \mathfrak{L}, \lambda)$ is a subspace of $(M, \mathfrak{M}, \mu)$ whenever $\lambda A = L \cap \mu A$ for each $A \subset L$. If a convergence space $(M, \mathfrak{M}, \mu)$ is given and if P is a subset of M, then it is possible to define a convergence $\mathfrak{P}$ on P in different ways to get a subspace $(P, \mathfrak{P}, \pi)$ of $(M, \mathfrak{M}, \mu)$; for example to define $\mathfrak{P}$ as a subsystem $\mathfrak{M}_P \subset \mathfrak{M}$ consisting of all elements $(\{x_n\}, x) \in \mathfrak{M}$ such that $x_n \in P$ and $x \in P$. It is clear that $\mathfrak{M}_P$ is largest if $\mathfrak{M}$ is the largest one.

Lemma 2. *Let $(L, \mathfrak{L}, \lambda)$ and $(M, \mathfrak{M}, \mu)$ be convergence spaces. Let $L \subset M$. Then $(L, \mathfrak{L}, \lambda)$ is a subspace of $(M, \mathfrak{M}, \mu)$ if and only if the following conditions 1° and 2° are satisfied:*

1° *If $(\{x_n\}, x) \in \mathfrak{L}$, then $(\{x_{n_i}\}, x) \in \mathfrak{M}$ for a suitable subsequence $\{x_{n_i}\}$ of $\{x_n\}$.*

2° *If $(\{y_n\}, y) \in \mathfrak{M}$, $y \in L$, $y_n \in L$, then $(\{y_{n_k}\}, y) \in \mathfrak{L}$, $\{y_{n_k}\}$ being a suitable subsequence of $\{y_n\}$.*

Proof. Let $(L, \mathfrak{L}, \lambda)$ be a subspace of $(M, \mathfrak{M}, \mu)$. Let $(\{x_n\}, x) \in \mathfrak{L}$. Then $x \in \lambda \bigcup x_n = L \cap \mu \bigcup x_n$ so that 1° is true. If $(\{y_n\}, y) \in \mathfrak{M}$, $y \in L$, $y_n \in L$, then $y \in L \cap \mu \bigcup y_n$ so that 2° is also true.

Now, let 1° and 2° be satisfied and let A be a subset of L. From 1° it follows that $x \in \lambda A$ implies $x \in \mu A$ so that $\lambda A \subset L \cap \mu A$. On the other hand, if $y \in L \cap \mu A$, then $y \in \lambda A$, by 2°. Hence $\lambda A = L \cap \mu A$.

F. Hausdorff [5] defined the continuity of a map $y = \varphi(x)$ on an $\mathscr{L}$-space E onto an $\mathscr{L}$-space H by means of the following property

$$(4) \qquad \lim x_n = x \quad \text{implies} \quad \lim \varphi(x_n) = \varphi(x) \quad \text{for each} \quad x \in E$$

and proved that (4) implies (1). The following example, however, shows that (4) need not be implied by (1):

Let $\varphi(x) = x$ be the identical map of $(L_5, \mathfrak{L}_5, \lambda_5)$ onto $(L_6, \mathfrak{L}_6, \lambda_6)$. Then $\mathfrak{L}_5 - \lim (1/n) = 0$ whereas $\{\varphi(1/n)\}_{n=1}^{\infty}$ does not $\mathfrak{L}_6$-converge in $(L_6, \mathfrak{L}_6, \lambda_6)$ at all.

For this reason the following definition[4]) seems to be useful:

A map $y = \varphi(x)$ of a convergence space $(L, \mathfrak{L}, \lambda)$ into a convergence space $(M, \mathfrak{M}, \mu)$ is sequentially continuous if

(5) $\mathfrak{L} - \lim x_n = x$ implies $\mathfrak{M} - \lim \varphi(x_{n_i}) = \varphi(x)$ for a suitable subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

From this definition it follows that the properties (1), (2) and (5) in convergence spaces are equivalent. (1) implies (5): if $(\{x_n\}, x) \in \mathfrak{L}$ then $\varphi(x) \in \mu\varphi(\bigcup x_n)$, by (1), so that there is an element $(\{\varphi(x_{n_i})\}, \varphi(x)) \in \mathfrak{M}$. Now, suppose (5) is true. Then $A \subset L$ and $x \in \lambda A$ implies that there are elements $(\{x_n\}, x) \in \mathfrak{L}$ and $(\{\varphi(x_{n_i})\}, \varphi(x)) \in \mathfrak{M}$, where $\bigcup x_n \subset A$ so that $\varphi(x) \in \mu \bigcup \varphi(x_{n_i}) \subset \mu\varphi(A)$ i.e. $\varphi(\lambda A) \subset \mu\varphi(A)$. Consequently (1) is true.

It is easy to be proved [6] that (5) can be replaced by (4) whenever $\mathfrak{M}$ is the largest convergence in $[\mathfrak{M}]$.

In this paper all three equivalent definitions of continuity of a map in convergence spaces will be used.

According to E. Čech [2], to any T_1 closure topology v there corresponds a modified topology $u(v)$. The topological modification $u(v)$ of v is characterized as the weakest T_1

[4]) See [6] p. 85 the footnote under the line.

topology among all topologies which are stronger than the given T_1 closure topology v. Now, we shall investigate the structure of the topological modification $u(\lambda)$ of a convergence topology λ. First define [5], for each ordinal ξ, the correspondence λ^ξ on the system of all subsets A of a convergence space $(L, \mathfrak{L}, \lambda)$ into itself as follows: $\lambda^0 A = A$, $\lambda^1 A = \lambda A$, $\lambda^\xi A = \lambda(\lambda^{\xi-1} A)$ if $\xi - 1$ exists and $\lambda^\xi A = \bigcup_{\eta<\xi} \lambda^\eta A$ if $\xi - 1$ does not exist. Using the method of transfinite induction we easily prove that the map λ^ξ fulfills both axioms (C_1) and (C_2); consequently λ^ξ is a T_1 closure topology. Clearly

$$(6) \qquad \lambda^0 A \subset \lambda^1 A \subset \ldots \subset \lambda^\xi A \subset \ldots$$

for each $A \subset L$. Therefore $\eta < \xi$ implies that λ^η is weaker than λ^ξ.

The T_1 closure topology λ^ξ need not be a convergence topology. As a matter of fact, the T_1 closure topology λ_3^2 on L_3 fails to be a convergence topology because $x_0 \in \lambda_3^2 \bigcup\bigcup x_{mn}$, but there is no sequence of points in $\bigcup\bigcup x_{nn}$ such that each λ_3^2-neighbourhood of x_0 would contain nearly all of them.

Theorem 3. *Let $(L, \mathfrak{L}, \lambda)$ be a convergence space. Then λ^α, $\alpha \geqq 2$, is a convergence topology on L if and only if λ fulfills the axiom* (F).

Proof. Let λ^α, $\alpha \geqq 2$, be a convergence topology on L. Let $A \subset L$; it suffices to prove that $\lambda^2 A \subset \lambda A$. Let $x_0 \in \lambda^2 A$. Because λ^α is a convergence topology and $x_0 \in \lambda^\alpha A$ there is a sequence $\{x_n\}$ of points $x_n \in A$ converging to x_0. Consequently $\lambda^\alpha B = x_0 \cup B$, where $B = \bigcup_{n=1}^{\infty} x_n$, by $(\mathscr{L}_0)$ and $(\mathscr{L}_2)$. Since $\lambda < \lambda^\alpha$, we have $B \subset \subset \lambda B \subset x_0 \cup B$. Hence $x_0 \in \lambda B$; otherwise $B = \lambda B$ so that $B = \lambda^\alpha B$ which is not possible. Therefore $x_0 \in \lambda A$.

The converse assertion follows immediately from the fact that $\lambda A = \lambda^2 A = \lambda^\alpha A$ for each $A \subset L$ whenever the convergence topology λ fulfills (F).

Now, we shall show that λ^{ω_1} is[5]) the topological modification of λ.

Theorem 4. *Let $(L, \mathfrak{L}, \lambda)$ be a convergence space. Then λ^{ω_1} is the weakest T_1 topology of all T_1 topologies which are stronger than λ.*

Proof. From (6) and with regard to the property of the set of all countable ordinals it follows that if $(\{x_n\}, x) \in \mathfrak{L}$, $\bigcup x_n \subset \lambda^{\omega_1} A$, then $x \in \lambda^{\omega_1} A$ so that $\lambda\lambda^{\omega_1} A = \lambda^{\omega_1} A$. Consequently $\lambda^{\omega_1}\lambda^{\omega_1} A = \lambda^{\omega_1} A$. Thus (L, λ^{ω_1}) is a topological space[6]).

If v is a topology on L such that $\lambda < v$ then $A \subset L$ and $\lambda^\eta A \subset vA$ implies $\lambda^{\eta+1} A \subset \subset v^2 A (= vA)$ for each ordinal η. From this it easily follows that $\lambda^\xi A \subset vA$ for each ordinal ξ. Therefore $\lambda^{\omega_1} A \subset vA$ and so $\lambda^{\omega_1} < v$.

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[5]) ω_1 is the first uncountable ordinal.

[6]) From this it follows that each T_1 closure topology λ^ξ, $\xi \geqq \omega_1$, is identical with the T_1 topology λ^{ω_1}, λ being a convergence topology. Consequently there is no sense in constructing T_1 closure topologies λ^ξ for $\xi > \omega_1$.

It is worth noting that, by Theorem 3, $\lambda \neq \lambda^2$ if and only if the topological modification λ^{ω_1} of λ fails to be a convergence topology.

Let $(L, \mathfrak{L}, \lambda)$ be a convergence space, x a point in L and A a subset in L such that $x \in \lambda^{\omega_1} A$. Then there is the least ordinal ϑ with the property $x \in \lambda^\vartheta A$. Evidently ϑ is an isolated ordinal. It will be denoted by $\vartheta(x, A)$ or simply by ϑ and called the order of the point x relative to the set A in the space L. Now, denote by $\vartheta(x)$ the least ordinal such that $\vartheta(x) \geqq \vartheta(x, A)$ for each $A \subset L$ such that $x \in \lambda^{\omega_1} A$ and call $\vartheta(x)$ the order of the point x in the convergence space L.

Lemma 3. *Let $(L, \mathfrak{L}, \lambda)$ be a convergence space. Then the order of a point x relative to a set $A \subset L$ such that $x \in \lambda^{\omega_1} A$ is a topological property.*

Proof. Let h be a homeomorphism mapping the convergence space $(L, \mathfrak{L}, \lambda)$ onto a convergence space $(M, \mathfrak{M}, \mu)$. Suppose $h(\lambda^\eta A) = \mu^\eta h(A)$ for all ordinals $\eta < \xi$. It follows that $h(\lambda^\xi A) = \mu\mu^{\xi - 1} h(A)$ for isolated $\xi \neq 0$ and $h(\bigcup_{\eta < \xi} \lambda^\eta A) = \bigcup_{\eta < \xi} \mu^\eta h(A)$ for non-isolated ξ. In both cases

$$h(\lambda^\xi A) = \mu^\xi h(A) . \tag{7}$$

Hence $\vartheta(x, A) = \vartheta(h(x), h(A))$ whenever $x \in \lambda^{\omega_1} A$.

Let $(L, \mathfrak{L}, \lambda)$ be a convergence space. Let us define a set $A \subset L$ to be λ^ξ-dense in a set $B \subset L$ whenever $\lambda^\xi A = B$. Then we have

Corollary 2. *Let $(L, \mathfrak{L}, \lambda)$ be a convergence space. Then λ^ξ-density is a topological property for each ordinal ξ.*

Proof follows instantly from (7).

Let $(L, \mathfrak{L}, \lambda)$ be a convergence space. It can happen that $\lambda^{\omega_1} = \lambda^\xi$ for a countable ordinal ξ. For example $\lambda_1^{\omega_1} = \lambda_1$ or $\lambda_3^{\omega_1} = \lambda_3^2$. If there is a subset $A \subset L$ such that $\lambda^{\omega_1} A \neq \lambda^\xi A$ for each $\xi < \omega_1$, then the power of $\lambda^{\omega_1} A$ and consequently also of L must be uncountable. Now we are going to construct a countable convergence space $(L_7, \mathfrak{L}_7, \lambda_7)$ such that $\lambda_7^{\omega_1} \neq \lambda_7^\xi$ for each $\xi < \omega_1$.

The convergence space L_7 consists of all rational numbers. We define the convergence $\mathfrak{L}_7$ as follows:

Well-order the irrationals and define P_ξ to be a one-to-one sequence of rationals converging (in the usual sense) to the ξ-th irrational, $\xi < \omega_1$. For each $\xi < \omega_1$ and $\eta \leqq \xi$ let $P_{\xi\eta}$ be subsets of P_ξ such that

(1) $P_{\xi 0} \subset P_{\xi 1} \subset \ldots \subset P_{\xi\eta} \subset \ldots \subset P_{\xi\xi}$;

(2) $P_{\xi\eta}$ and $P_{\xi\eta+1} - P_{\xi\eta}$ are infinite sets for each $\eta < \xi$;

(3) $P_{\xi\xi} = P_\xi$ and $P_{\xi\eta} = \bigcup_{\zeta < \eta} P_{\xi\zeta}$ for non-isolated ordinal η.

Let $\xi < \omega_1$. Define $P_{\xi, -1} = \emptyset$. Using the method of transfinite induction we shall define sequences S_x^ξ of points in P_ξ as follows: Suppose that, for each ordinal $\eta < \alpha$, where $\alpha < \xi$, we have just assigned to each point $x \in P_{\xi\eta+1} - P_{\xi\eta}$ a one-to-one sequence S_x^ξ of points of $P_{\xi\eta}$ such that

$(+)$ $y \in P_{\xi,\eta+1} - P_{\xi\eta}$ and $x \neq y$ implies that S_x^ξ and S_y^ξ have no points in common.

If α is an isolated ordinal, then assign to each point $x \in P_{\xi\alpha+1} - P_{\xi\alpha}$ a one-to-one sequence S_x^ξ of points of $P_{\xi\alpha} - P_{\xi\alpha-1}$ with the property $(+)$, for $\eta = \alpha$. If α is a limiting ordinal, then choose a sequence of isolated ordinals $\alpha_1 < \alpha_2 < \ldots$ such that $\lim \alpha_n = \alpha$ and assign to each point $x \in P_{\xi\alpha+1} - P_{\xi\alpha}$ a one-to-one sequence $S_x^\xi = \{x_n\}$ such that $x_n \in P_{\xi\alpha_n} - P_{\xi\alpha_n-1}$ and such that $(+)$ holds for $\eta = \alpha$.

In such a way to each point $x \in P_{\xi\xi} - P_{\xi 0}$ there corresponds a sequence S_x^ξ of points such that

$(++)$ $y \in P_{\xi\xi} - P_{\xi 0}$ and $x \neq y$ implies that S_x^ξ and S_y^ξ have at most a finite number of points in common.

The convergence $\mathfrak{L}_7$ consists of all elements $(\{x\}, x)$, where $x \in L_7$, and of all elements $(S_x'^\xi, x)$ where $x \in P_{\xi\xi} - P_{\xi 0}$, $\xi < \omega_1$, $S_x'^\xi$ being a subsequence of S_x^ξ. The axioms $(\mathscr{L}_1)$ and $(\mathscr{L}_2)$ evidently hold true. Now, if $(\{x_n\}, x) \in \mathfrak{L}_7$, $(\{x_n\}, y) \in \mathfrak{L}_7$ and if $\{x_n\}$ is a constant sequence, then evidently $x = y$. If $\{x_n\}$ is not constant, then $S_x'^\xi = \{x_n\} = S_y'^\zeta$ for suitable ordinals ξ and ζ. Since $\beta < \gamma < \omega_1$ implies that $P_{\beta\beta} \cap P_{\gamma\gamma}$ contains at most a finite number of points, we have $\xi = \zeta$. Therefore $x = y$, by $(++)$.

The convergence space $(L_7, \mathfrak{L}_7, \lambda_7)$ has the following property: If $\alpha < \beta < \omega_1$ then $\lambda_7^\alpha P_{\beta 0} = P_{\beta\alpha} \neq P_{\beta\beta} = \lambda_7^{\omega_1} P_{\beta 0}$. Consequently $\lambda_7^{\omega_1} \neq \lambda_7^\xi$ for each $\xi < \omega_1$.

Now, we are going to construct a countable convergence space containing a point of order ω_1. For this purpose add to the set L_7 a new element x^* and denote $L_8 = L_8 \cup (x^*)$. Define the convergence $\mathfrak{L}_8$ on L_8 : $(\{x_n\}, x) \in \mathfrak{L}_8$ whenever $(\{x_n\}, x) \in \mathfrak{L}_7$ or $x = x_n = x^*$ for all $n \in N$, or $x = x^*$ and $\{x_n\}$ is a one-to-one sequence of points such that $\bigcup x_n \subset P_{\xi\xi} - P_{\xi\xi-1}$ where ξ denotes isolated countable ordinals. It is easy to show that $(L_8, \mathfrak{L}_8, \lambda_8)$ is a convergence space such that $x^* \in \lambda_8^{\xi+1} P_{\xi 0} - \lambda_8^\xi P_{\xi 0}$, i.e. $\vartheta(x^*, P_{\xi 0}) = \xi + 1$ for each isolated ordinal $\xi < \omega_1$. Consequently $\vartheta(x^*) = \omega_1$.

3

Let I be a non-void set of indexes and $(L_\alpha, \mathfrak{L}_\alpha, \lambda_\alpha)$, $\alpha \in I$, convergence spaces. Let $L = \mathbf{X}\{L_\alpha : \alpha \in I\}$ be the Cartesian product of the sets L_α. The Cartesian T_1 closure topology in L is defined by Cartesian neighbourhoods as follows: $U(x_\alpha)$ is a Cartesian neighbourhood of a point $(x_\alpha) \in L$ if it is a Cartesian product of λ_α-neighbourhoods $U(x_\alpha) \subset L_\alpha$, where $U(x_\alpha) = L_\alpha$ except for at most a finite number of indexes $\alpha \in I$. Now we shall define a convergence $\mathfrak{L}$ in the Cartesian product in the following manner:

$$(8) \qquad (\{(x_\alpha^n)\}, (x_\alpha)) \in \mathfrak{L} \text{ whenever } (\{x_\alpha^n\}, x_\alpha) \in \mathfrak{L}_\alpha \text{ for each } \alpha \in I\,.$$

It is easy to see that all three axioms $(\mathscr{L}_0)$, $(\mathscr{L}_1)$ and $(\mathscr{L}_2)$ are fulfilled. The convergence $\mathfrak{L}$ will be called the Cartesian convergence; it induces a Cartesian convergence topology λ. The space $(L, \mathfrak{L}, \lambda)$ will be called convergence Cartesian space or convergence Cartesian product and denoted by $\mathbf{X}\{(L_\alpha, \mathfrak{L}_\alpha, \lambda_\alpha) : \alpha \in I\}$.

From (3) and (8) it follows that the Cartesian convergence topology is weaker than the Cartesian T_1 closure topology [7].

Theorem 5. *The Cartesian convergence* $\mathfrak{L}$ *in a convergence Cartesian space* $(L, \mathfrak{L}, \lambda) = \mathbf{X}\{(L_\alpha, \mathfrak{L}_\alpha, \lambda_\alpha) : \alpha \in I\}$ *is largest if and only if each convergence* $\mathfrak{L}_\alpha$ *is largest on* L_α, $\alpha \in I$.

Proof. Let $\mathfrak{L} = \mathfrak{L}^*$ and $\gamma \in I$. Let $\{x^n\}_{n=1}^\infty$ be a sequence of points $x^n \in L_\gamma$ and x a point of L_γ. Choose a point $(x_\alpha) \in L$ such that $x_\gamma = x$ and suppose that each subsequence of $\{x^n\}$ contains a subsequence $\mathfrak{L}_\gamma$-converging to x. Then the same property holds for the sequence of points $(x_\alpha^n) \in L$ and the point $(x_\alpha) \in L$ where $x_\alpha^n = x_\alpha$ for $\alpha \neq \gamma$ and $x_\gamma^n = x^n$. Consequently $\mathfrak{L} - \lim (x_\alpha^n) = (x_\alpha)$ by $(\mathscr{L}_3)$. Now from (8) it follows that $\mathfrak{L}_\gamma - \lim x_\gamma^n = x_\gamma$, i.e. $\mathfrak{L}_\gamma - \lim x^n = x$. Hence $\mathfrak{L}_\gamma = \mathfrak{L}_\gamma^*$, by Theorem 2.

Now let $\mathfrak{L}_\alpha = \mathfrak{L}_\alpha^*$ for each $\alpha \in I$. Let $\{(x_\alpha^n)\}_{n=1}^\infty$ be any sequence of points $(x_\alpha^n) \in L$ and (x_α) a point of L such that in each subsequence there is a sequence $\mathfrak{L}$-converging to (x_α). In view of (8) the same property holds for the sequence $\{x_\alpha^n\}_{n=1}^\infty$ and the point $x_\alpha \in L_\alpha$ for each $\alpha \in I$. Then $\mathfrak{L}_\alpha = \mathfrak{L}_\alpha^*$ implies that $\mathfrak{L}_\alpha - \lim x_\alpha^n = x_\alpha$, $\alpha \in I$, so that $\mathfrak{L} - \lim (x_\alpha^n) = (x_\alpha)$, by (8). Consequently, according to Theorem 2, we have $\mathfrak{L} = \mathfrak{L}^*$.

Remark. In each convergence space $(L, \mathfrak{L}, \lambda)$ both convergences $\mathfrak{L}$ and $\mathfrak{L}^*$ induce the same topology λ, so that $\lambda = \lambda^*$. If $(L, \mathfrak{L}, \lambda) = \mathbf{X}\{(L_\alpha, \mathfrak{L}_\alpha, \lambda_\alpha) : \alpha = 1, 2, \ldots, k\}$ and $(L, \mathfrak{L}', \lambda') = \mathbf{X}\{(L_\alpha, \mathfrak{L}_\alpha^*, \lambda_\alpha) : \alpha = 1, 2, \ldots, k\}$, k being a natural, then $\lambda = \lambda' = \lambda^*$ as well. As a matter of fact, let $A \subset L$ and $(z_\alpha) \in \lambda A$; then $\mathfrak{L} - \lim (z_\alpha^n) = (z_\alpha)$ for a suitable sequence of points $(z_\alpha^n) \in A$ and $\mathfrak{L}_\alpha - \lim z_\alpha^n = z_\alpha$ for each $\alpha = 1, 2, \ldots, k$. Since $\mathfrak{L}_\alpha \subset \mathfrak{L}_\alpha^*$, we have $\mathfrak{L}_\alpha^* - \lim z_\alpha^n = z^\alpha$ for each $\alpha = 1, 2, \ldots, k$; consequently $(z_\alpha) \in \lambda' A$ and so $\lambda A \subset \lambda' A$. On the other hand, if $(t_\alpha) \in \lambda' A$ then $\mathfrak{L}_\alpha^* - \lim t_\alpha^n = t_\alpha$ for each $\alpha = 1, 2, \ldots, k$, where $\{(t_\alpha^n)\}_{n=1}^\infty$ is a suitable sequence of points of A. From $(\mathscr{L}_3)$, $(\mathscr{L}_2)$ and because $\alpha \leqq k$, it follows that there is a subsequence $\{(t_\alpha^{n_i})\}_{i=1}^\infty$ which $\mathfrak{L}$-converges to the point (t_α). Consequently $(t_\alpha) \in \lambda A$ and we have $\lambda A \supset \lambda' A$. Therefore $\lambda = \lambda' = \lambda^*$.

If $(L, \mathfrak{L}, \lambda) = \mathbf{X}\{(L_\alpha, \mathfrak{L}_\alpha, \lambda_\alpha) : \alpha \in I\}$ and $(L, \mathfrak{L}', \lambda') = \mathbf{X}\{(L_\alpha, \mathfrak{L}_\alpha^*, \lambda_\alpha) : \alpha \in I\}$, where I is infinite, then it might happen that $\lambda \neq \lambda'$. This is shown by the following example[7]):

Let $I = N$. Let each L_α, $\alpha \in N$, consist of numbers 0 and $1/n$, where $n \in N$. Let $\mathfrak{L}_\alpha$ be the set of elements $(\{y_n\}, y)$ such that $y \in L_\alpha$ and $y_n = y$ for each $n \in N$ or $y = 0$ and $\{y_n\}$ is a subsequence of $\{1/n\}_{n=\alpha}^\infty$. In such a way we get convergences $\mathfrak{L}_\alpha$ and $\mathfrak{L}_\alpha^*$ on L_α. Let $(L, \mathfrak{L}, \lambda) = \mathbf{X}\{(L_\alpha, \mathfrak{L}_\alpha, \lambda_\alpha) : \alpha \in N\}$ and $(L, \mathfrak{L}', \lambda') = \mathbf{X}\{(L_\alpha, \mathfrak{L}_\alpha^*, \lambda_\alpha) : \alpha \in N\}$.

[7]) Another example was given by V. Koutník: Let each L_α, $\alpha \in N$, consist of two numbers 0 and 1. Let $(\{y_n\}, y) \in \mathfrak{L}_\alpha$ whenever $y \in L_\alpha$ and $y_n = y$ for each $n \in N$. Then $(0) \in \lambda' B - \lambda B$ where B is the set of all $(z_\alpha^n) \in L$ such that $z_\alpha^n = 1$ for $n = \alpha$ and $z_\alpha^n = 0$ for $n \neq \alpha$.

Denote A the set of points $(z_\alpha^n) \in L$ where $z_\alpha^n = 1/n$, $\alpha \in N$, $n \in N$; let (0) be the point $(z_\alpha) \in L$ where $z_\alpha = 0$ for each $\alpha \in N$. It is clear that $(0) \in \lambda' A - \lambda A$.

It is worth noting that in this example $\mathfrak{L} \subset \mathfrak{L}^* \subset \mathfrak{L}' = \mathfrak{L}'^*$ and that $\mathfrak{L} \neq \mathfrak{L}^* \neq \mathfrak{L}'$. Hence $\lambda < \lambda' \neq \lambda$. From this it follows that the convergence topology in the Cartesian product can depend on whether or not the convergences $\mathfrak{L}_\alpha$ on the spaces L_α are largest or not.

Let $(L, \mathfrak{L}, \lambda)$ be a convergence space. Let us classify all elements of the largest convergence $\mathfrak{L}^*$ (with regard to $\mathfrak{L}$) into three classes as follows:

Each element $(\{x_n\}_{n=1}^\infty, x)$ of $\mathfrak{L}^* - \mathfrak{L}$ such that $(\{x_n\}_{n=n_0}^\infty, x) \in \mathfrak{L}$ for a suitable natural n_0 will be called the element of the first kind. Each element of $\mathfrak{L}^* - \mathfrak{L}$ which is not of the first kind will be called the element of the second kind and each element of $\mathfrak{L}$ the element of the third kind.

Let $(\{x_n\}, x)$ be an element of the first kind. Let m_0 be the smallest natural such that $(\{x_n\}_{n=m_0}^\infty, x) \in \mathfrak{L}$. Evidently $m_0 > 1$. We call $(\{x_n\}, x)$ the element of the first kind in the strict sense if each subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $n_1 < m_0$, is of the first kind.

Notice that $\mathfrak{L}_2^* - \mathfrak{L}_2$ contains elements of the first kind in the strict sense whereas $\mathfrak{L}_6^* - \mathfrak{L}_6$ contains only elements of the second kind.

Theorem 6. *Let*

$$(L, \mathfrak{L}, \lambda) = \mathbf{X}\{(L_\alpha, \mathfrak{L}_\alpha, \lambda_\alpha) : \alpha \in I\} \text{ and } (L, \mathfrak{L}', \lambda') = \mathbf{X}\{(L_\alpha, \mathfrak{L}_\alpha^*, \lambda_\alpha) : \alpha \in I\}$$

be convergence Cartesian spaces. If there is at most a countable number of $\alpha \in I$ such that $\mathfrak{L}_\alpha \neq \mathfrak{L}_\alpha^*$ and at most a finite number of $\alpha \in I$ such that $\mathfrak{L}_\alpha^*$ contains elements of the first kind, then $\lambda = \lambda'$. If there is an infinite number of $\alpha \in I$ such that $\mathfrak{L}_\alpha^*$ contains elements of the first kind in the strict sense or if there is a subset $I_0 \subset I$ of power $\geqq 2^{\aleph_0}$ such that $\mathfrak{L}_\alpha \neq \mathfrak{L}_\alpha^*$ for each $\alpha \in I_0$, then $\lambda \neq \lambda'$.

Proof. Denote by I_1 a subset of I containing all α for which $\mathfrak{L}_\alpha \neq \mathfrak{L}_\alpha^*$ and by I_0 a subset of I consisting of all α such that $\mathfrak{L}_\alpha^*$ contains elements of the first kind. Evidently $I_0 \subset I_1$. If I_1 is finite, then the proof of the first assertion follows instantly from what has just been proved above in the Remark. Consequently we can put $I_1 = N$.

Let $A \subset L$ and let (x_α) be a point of $\lambda' A$; then there is a sequence $\{({}^1x_\alpha^n)\}_{n=1}^\infty$ of points in A which $\mathfrak{L}'$-converges to (x_α).

In order to prove that $(x_\alpha) \in \lambda A$ let us suppose (mathematical induction) that for each natural $i < k$, where $k > 1$ is a natural, we have just chosen subsequences $\{({}^{i+1}x_\alpha^n)\}_{n=1}^\infty$ of $\{({}^ix_\alpha^n)\}_{n=1}^\infty$ such that

$$(+++) \qquad (\{{}^{i+1}x_i^n\}, x_i) \in \mathfrak{L}_i\,.$$

For $i = k - 1$ we have the subsequence $\{({}^kx_\alpha^n)\}_{n=1}^\infty$ of $\{({}^{k-1}x_\alpha^n)\}_{n=1}^\infty$ which $\mathfrak{L}'$-converges to the point (x_α). Consequently, by (8), $\mathfrak{L}_k^* - \lim\limits_n {}^kx_k^n = x_k$. In view of $(\mathscr{L}_3)$, there is a subsequence $\{({}^{k+1}x_\alpha^n)\}_{n=1}^\infty$ of $\{({}^k x_\alpha^n)\}_{n=1}^\infty$ such that $\mathfrak{L}_k - \lim_n {}^{k+1}x_k^n = x_k$, so that $(+++)$ also holds for $i = k$.

In such a way we get a sequence $\{({}^{n+1}x_\alpha^n)\}_{n=1}^\infty$ such that $\{({}^{n+1}x_\alpha^n)\}_{n=k}^\infty$ is a subsequence of the sequence $\{({}^{k+1}x_\alpha^n)\}_{n=1}^\infty$ consequently $\{{}^{n+1}x_k^n\}_{n=k}^\infty$ is a subsequence of $\{{}^{k+1}x_k^n\}_{n=1}^\infty$, $k = 1, 2, \ldots$. Denote by I_0' the set of all indexes $\alpha \in I$ such that $(\{{}^{n+1}x_\alpha^n\}_{n=1}^\infty, x_\alpha)$ is the element of the first kind. Then $I_0' \subset I_0$ is finite so that there is a natural m_0 such that $k \in I_0'$ implies $(\{{}^{n+1}x_k^n\}_{n=m_0}^\infty, x_k) \in \mathfrak{L}_k$. Since for $\alpha \in I - I_0'$ no element $(\{{}^{n+1}x_n^\alpha\}_{n=1}^\infty, x_\alpha)$ of $\mathfrak{L}_\alpha^*$ is of the first kind, we have $(\{({}^{n+1}x_\alpha^n)\}_{n=m_0}^\infty, (x_\alpha)) \in \mathfrak{L}$. Therefore $(x_\alpha) \in \lambda A$. Consequently $\lambda' A = \lambda A$.

Now, suppose that there is an infinite countable set $I_2 \subset I$ such that each $\mathfrak{L}_\alpha^*$, $\alpha \in I_2$, contains an element $(\{x_\alpha^n\}_{n=1}^\infty, x_\alpha)$ of the first kind in the strict sense. Denote $I_2 = N$. Since $\mathfrak{L}_\alpha \neq \mathfrak{L}_\alpha^*$ for each $\alpha \in N$, there is a point $y_\alpha \in L_\alpha$, $y_\alpha \neq x_\alpha$, $\alpha \in N$. Choose a point $(t_\alpha) \in L$ such that $t_\alpha = x_\alpha$ for $\alpha \in N$. Let (t_α^n) be points of L such that for $\alpha \in I - N : t_\alpha^n = t_\alpha$ for each natural n whereas, for $\alpha \in N : t_\alpha^n = y_\alpha$ whenever $n \leqq \alpha$ and $t_\alpha^n = x_\alpha^{n-\alpha}$ if $n > \alpha$. It is easy to show that $(t_\iota) \in \lambda' \bigcup_{n=1}^\infty (t_\alpha^n) - \lambda \bigcup_{n=1}^\infty (t_\alpha^n)$.

Now, let I_1 be the set of all indexes $\alpha \in I$ such that $\mathfrak{L}_\alpha \neq \mathfrak{L}_\alpha^*$. Suppose that $P(I_1) \geqq 2^{\aleph_0}$. Then there is a one-to-one map φ on the system of all subsequences $\{n_i\}$ of the sequence of all naturals onto $I_1' \subset I_1$. For each $\alpha \in I_1'$ choose an element $(\{x_\alpha^n\}_{n=1}^\infty, x_\alpha) \in \mathfrak{L}_\alpha^* - \mathfrak{L}_\alpha$ and define the element $(\{y_\alpha^n\}_{n=1}^\infty, x_\alpha)$ as follows: Let $\alpha \in I_1'$. If $\varphi(\{n_i\}) = \alpha$ then put $y_\alpha^n = x_\alpha^i$ if $n = n_i$ and $y_\alpha^n = x_\alpha$ if $n \neq n_i$ for each i. If $\alpha \in I - I_1'$, then choose $x_\alpha \in L_\alpha$ and put $y_\alpha^n = x_\alpha$, for each natural n. From axiom $(\mathscr{L}_3)$ it easily follows that $(\{y_\alpha^n\}_{n=1}^\infty, x_\alpha) \in \mathfrak{L}_\alpha^*$ for each $\alpha \in I$.

Denote $B = \bigcup_{n=1}^\infty (y_\alpha^n)$. Since $(\{y_\alpha^n\}_{n=1}^\infty, x_\alpha) \in \mathfrak{L}_\alpha^*$, $\alpha \in I$, the point (x_α) belongs to $\lambda' B$. On the other hand, let $\{(y_\alpha^{n_i})\}_{i=1}^\infty$ be a subsequence of $\{(y_\alpha^n)\}_{n=1}^\infty$. Denote $\varphi(\{n_i\}) = \beta$. Since $(\{x_\beta^n\}_{n=1}^\infty, x_\beta)$ does not belong to $\mathfrak{L}_\beta$ and because $x_\beta^i = y_\beta^{n_i}$, it follows that $\{y_\beta^{n_i}\}$ does not $\mathfrak{L}_\beta$-converge to x_β. Therefore $(x_\alpha) \in \lambda' B - \lambda B$.

In [13] I called a ϱ-point any point x of a convergence space $(L, \mathfrak{L}, \lambda)$ having the following property: there is a one-to-one double sequence of points $x_n^m \in L$ such that $\mathfrak{L} - \lim_n x_n^m = x$ for each natural m, but there is no cross-subsequence $\mathfrak{L}$-converging to x. For example the point x_0 is a ϱ-point in the space $(L_4, \mathfrak{L}_4, \lambda_4)$. It is easy to see that both convergence spaces $(L_2, \mathfrak{L}_2, \lambda_2)$ and $(L_4, \mathfrak{L}_4, \lambda_4)$ are topological spaces although their convergence Cartesian product $L_2 \times L_4$ fails to be a topological space[8]). As a matter of fact, if we denote by λ the convergence topology in $L_2 \times L_4$, then $(x_0, x_0) \in \lambda\lambda A - \lambda A$ where $A = \bigcup_{m=1}^\infty \bigcup_{n=1}^\infty (x_m, x_{mn})$. The following problem arises: Does the assumption that $\mathbf{X}\{(L_\alpha, \mathfrak{L}_\alpha, \lambda_\alpha) : \alpha = 1, 2\}$ is not whereas $(L_\alpha, \mathfrak{L}_\alpha, \lambda_\alpha)$, $\alpha = 1, 2$, are topological spaces imply the existence of a ϱ-point in L_1 or in L_2?

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[8]) Cf. the example in [7] p. 22.

The convergence Cartesian product $(E, \mathfrak{C}, \varepsilon) = \mathbf{X}\{(L_\alpha, \mathfrak{L}_5, \lambda_5) : L_\alpha = L_5, \alpha \in I\}$ will be called the convergence Euclidean space of the dimension $P(I)$, P denoting the power of I. Since $\mathfrak{L}_5 = \mathfrak{L}_5^*$ from Theorem 5 it follows that $\mathfrak{C} = \mathfrak{C}^*$. If I is the set of real numbers (i.e. $I = L_5$), then the elements of E are real valued functions of the real argument $x \in I$ and the convergence $\mathfrak{C}$ is the convergence of functions at each point $x \in I$. If we denote by D the set of all continuous functions on $(L_5, \mathfrak{L}_5, \lambda_5)$, then $\varepsilon^{\omega_1} D$ is the set of all Baire functions. Consequently the convergence Cartesian space $(E, \mathfrak{C}, \varepsilon)$ of dimension $\geqq 2^{\aleph_0}$ does not fulfill the axiom of closed closure (F), i.e. $\varepsilon \neq \varepsilon^{\omega_1}$.

Let $(E, \mathfrak{C}, \varepsilon)$ be a convergence Euclidean space. Let C be the set of all points $(x_\alpha) \in E$ such that $x_\alpha \in \langle 0, 1 \rangle$ and C_0 the set of all points $(x_\alpha) \in E$ such that $x_\alpha = 0$ or $= 1$, $\alpha \in I$. Put $\mathfrak{C} = \mathfrak{C}_C$ and $\mathfrak{C}_0 = \mathfrak{C}_{C_0}$ (see p. 79). Then $(C, \mathfrak{C}, \gamma)$ and $(C_0, \mathfrak{C}_0, \gamma_0)$ are subspaces of $(E, \mathfrak{C}, \varepsilon)$. We call $(C, \mathfrak{C}, \gamma)$ the convergence cube space and $(C_0, \mathfrak{C}_0, \gamma_0)$ the convergence cube vertex space.

The relation between the convergence topology γ_0 of the convergence cube vertex space $(C_0, \mathfrak{C}_0, \gamma_0)$ of the dimension $P(I)$ and the usual Cartesian topology u on C_0 is described in the following:

Statement. $u = \gamma_0$ *if and only if* $P(I) \leqq \aleph_0$.

Proof. It is well known [6] that in the case when $P(I) = \aleph_0$ the spaces (C_0, γ_0), (C_0, u) and the Cantor discontinuum are homeomorphic.

Now, let $P(I) > \aleph_0$. Let A be the set of all points (z_α) of C_0 such that $z_\alpha = 1$ for at most countable number of α. Denote by $I(z_\alpha)$, where $(z_\alpha) \in A$, the set of all $\alpha \in I$ such that $z_\alpha = 1$. If $(z_\alpha^n) \in A$, $n = 1, 2, \ldots$, and $\mathfrak{C}_0 - \lim (z_\alpha^n) = (y_\alpha)$, then $z_\alpha^n = 0$ for each $\alpha \in I - \bigcup_{n=1}^{\infty} I(z_\alpha^n)$ and each n. Consequently $(y_\alpha) \in A$ so that $A = \gamma_0 A$. Since $P(I) > \aleph_0$, the point (t_α) of C_0, where $t_\alpha = 1$ for $\alpha \in I$, is not contained in A. On the other hand, each u-neighbourhood of (t_α) contains points of A so that

$$(t_\alpha) \in uA - \gamma_0 A.$$

We have just proved that $A = \gamma_0 A = \gamma_0^{\omega_1} A \neq uA$. From this and since $\varepsilon \neq \varepsilon^{\omega_1}$ it can be deduced that in convergence Euclidean space E of the dimension $\geqq 2^{\aleph_0}$ we have $\varepsilon \neq \varepsilon^{\omega_1} \neq u \neq \varepsilon$, u being the usual Cartesian topology on E.

4

A convergence space is said to be separated if any two distinct points of it can be separated by two disjoint neighbourhoods. A convergence space $(L, \mathfrak{L}, \lambda)$ is regular if for each point $x \in L$ and each λ-neighbourhood $U(x)$ of x there is a λ-neighbourhood $V(x)$ of x such that $\lambda\, V(x) \subset U(x)$. It is not difficult to construct topological convergence spaces which are not separated and those which are separated but not regular.

The usual definition of completely regular space is not suitable for spaces in which axiom (F) is not fulfilled. This is shown by the following consideration: Let (P, v) be a T_1 closure space not fulfilling axiom (F). Then there is a set $A \subset P$ and a point $x_0 \in vvA - vA$. If f is any real valued function continuous on P such that $f(x) = 0$ for each $x \in vA$, then also $f(x_0) = 0$ because every v-neighbourhood of x_0 contains points of vA. In particular, if f is a continuous function on a convergence space $(L, \mathfrak{L}, \lambda)$ such that $f(x) = 0$ for each $x \in B$, $B \subset L$, then $f(x) = 0$ for each $x \in \lambda^{\omega_1}B$.

Instead of complete regularity we are going to introduce the notions of sequential and α sequential regularity in convergence spaces as follows:

Let $(L, \mathfrak{L}, \lambda)$ be a convergence space. Denote by α a property such that it can be decided whether or not any real valued continuous function f on L has the property α or not.

A convergence space $(L, \mathfrak{L}, \lambda)$ is called a sequentially regular [α sequentially regular] space if for each point $x_0 \in L$ and each sequence of points $x_n \in L$ no subsequence of which $\mathfrak{L}$-converges to x_0, there is a real valued continuous function f on L [having the property α] such that the sequence $\{f(x_n)\}$ does not converge to $f(x_0)$.

If $\mathfrak{L}^*$ is a largest convergence, then the definition of sequential regularity can be simplified; this is shown in the following

Lemma 4. *A convergence space $(L, \mathfrak{L}, \lambda)$ is sequentially [α sequentially] regular if and only if for each point $x_0 \in L$ and each sequence of points $x_n \in L$ not $\mathfrak{L}^*$-converging to x_0 there is a real valued continuous function f on L [having the property α] such that $\{f(x_n)\}$ does not converge to $f(x_0)$.*

As a matter of fact, if $(L, \mathfrak{L}, \lambda)$ is sequentially [α sequentially] regular, $\mathfrak{L}^*$ the largest convergence in $[\mathfrak{L}]$, $x_0 \in L$ and $\{x_n\}$ a sequence not $\mathfrak{L}^*$-converging to x_0 then, by Theorem 2, there is a subsequence $\{x_{n_i}\}$ no subsequence of which $\mathfrak{L}$-converges to x_0 so that, by the definition above there is a continuous function f on L [having the property α] such that $\{f(x_{n_i})\}$ and consequently $\{f(x_n)\}$ does not converge to $f(x_0)$. The proof of the converse is evident.

In the sequel we shall consider two special properties $\alpha' : 0 \leqq f(x) \leqq 1$ for each $x \in L$ and: $f(x) = 0$ or $= 1$ for each $x \in L$. In this case instead of α' we shall sometimes[9]) write $\langle 0, 1 \rangle$ and $\{0, 1\}$. From the definition it immediately follows that each α sequentially regular space is sequentially regular as well. Notice that the convergence space $(L_3, \mathfrak{L}_3, \lambda_3)$ is $\{0, 1\}$ sequentially regular.

Now, we are going to determine the location of sequentially regular and $\{0, 1\}$ sequentially regular spaces in the classification of the convergence spaces. First of all we shall prove the following

Theorem 7. *Each sequentially regular space is separated; its topological modification is also separated.*

[9]) In [9] $\langle 0, 1 \rangle$ sequential regularity is called „halbe Regularität". See also [10].

Proof. Let a and b be two distinct points of a sequentially regular space $(L, \mathfrak{L}, \lambda)$. In view of $(\mathscr{L}_0)$ the constant sequence $\{b\}$ does not $\mathfrak{L}$-converge to the point a. Consequently there is a continuous function f on L such that $f(a) \neq f(b)$. Hence $\{x \in L: |f(x) - f(a)| < \delta\}$ and $\{x \in L: |f(x) - f(b)| < \delta\}$, where $\delta = \frac{1}{2}|f(a) - - f(b)|$, are two disjoint λ-neighbourhoods of the points a and b. Moreover, both neighbourhoods are λ-open in L so that they are also λ^{ω_1}-neighbourhoods of the points a and b in the topological space (L, λ^{ω_1}).

Theorem 8. *Let $(L, \mathfrak{L}, \lambda)$ be a sequentially regular space fulfilling the first axiom of countability. Then it is regular.*

Proof. Suppose, on the contrary, that there is a point $x_0 \in L$ and a λ-neighbourhood $U(x_0)$ such that the λ-closure of each λ-neighbourhood of x_0 has points common with $L - U(x_0)$. Denote by V_n the elements of a countable strictly monotone complete system of λ-neighbourhoods of the point x_0 and choose points $x_n \in \lambda V_n - U(x_0)$. Then, by Theorem 2, the sequence $\{x_n\}$ does not $\mathfrak{L}^*$-converge to x_0. Since each λ-neighbourhood of x_n contains points of V_n, it is easy to deduce that $\lim f(x_n) = f(x_0)$ for each continuous function f on L. By Lemma 4 this is a contradiction.

Let us notice that the convergence $2^{\aleph_0}$ dimensional Euclidean space $E_{2^{\aleph_0}}$ fails to be regular [15]. Nevertheless $E_{2^{\aleph_0}}$ is sequentially regular. As a matter of fact, let f_0 and f_n, $n \in N$, be elements of $E_{2^{\aleph_0}}$ such that no subsequence of $\{f_n\}$ converges to f_0. Then there is a real number a such that $\{f_n(a)\}$ does not converge to $f_0(a)$. The function φ such that $\varphi(f) = f(a)$ for each $f \in E_{2^{\aleph_0}}$ is sequentially continuous on $E_{2^{\aleph_0}}$ and such that the sequence $\{\varphi(f_n)\}$ does not converge to $\varphi(f_0)$.

In [8] I constructed a regular convergence space Q such that each sequentially continuous function f on Q is constant. Therefore every regular convergence space need not be sequentially regular. In [11] it is proved that a sequentially regular space need not be $\{0, 1\}$ sequentially regular.

Denoting by $\mathscr{L}$, S, R, sR, $\{0, 1\}$ sR the general, separated, regular, sequentially regular, $\{0, 1\}$ sequentially regular convergence space and by $\rightarrow$ the direction of the specialization then we get the scheme of the classification of convergence spaces as follows:

$$\mathscr{L} \rightarrow S \begin{array}{l} \nearrow R \\ \searrow sR \rightarrow \{0, 1\}\, sR\,. \end{array}$$

Remark. We have shown that the sequentially regular space $E_{2^{\aleph_0}}$ is neither regular nor topological space. Under the supposition that $2^{\aleph_0} = \aleph_1$ I constructed the topological $\{0, 1\}$ sequentially regular space L_9 which is not regular.

Let $\{x_{mn}\}$ be a one-to-one double sequence of points. Denote by $\mathfrak{A}$ the system of all cross-sequences $\{x_{mn_m}\}_{m=1}^{\infty}$ of $\{x_{mn}\}$. Define a binary relation $\prec$ in $\mathfrak{A}$ as follows: $\{x_{mn_m}\} \prec \{x_{mp_m}\}$ whenever $n_m < p_m$ for nearly all n. Then the ordering $\prec$ directs the system $\mathfrak{A}$ so that $\mathfrak{A}$ is partially ordered.

Suppose that $2^{\aleph_0} = \aleph_1$. Then from a result of W. SIERPIŃSKI [14] it follows that there is a completely ordered subsystem $\mathfrak{A}_0 \subset \mathfrak{A}$ of elements

$$A_0 \prec A_1 \prec \ldots A_\xi \prec \ldots$$

$\xi < \omega_1$, which is cofinal in $\mathfrak{A}$.

Now, let L_9 be the set of all points x_{mn}, x_0 and of points y_ξ, $\xi < \omega_1$. Define a convergence $\mathfrak{L}_9$ on L_9: each constant sequence $\{x\}$ $\mathfrak{L}_9$-converges to x, $x \in L_9$; each subsequence of $\{x_{mn}\}_{n=1}^{\infty}$ $\mathfrak{L}_9$-converges to x_0 for every $m = 1, 2, \ldots$; each subsequence of A_ξ $\mathfrak{L}_9$-converges to y_ξ, $\xi < \omega_1$.

The convergence space $(L_9, \mathfrak{L}_9, \lambda_9)$ is a topological space. As a matter of fact, each point x_{mn} is isolated and its order $\vartheta(x_{mn}) = 0$ whereas each point $z \in L_9$, $z \neq x_{mn}$, has order $\vartheta(z) = 1$. Hence $A \subset L_9$ implies $\lambda_9\lambda_9 A = \lambda_9 A$.

The space L_9 is not regular: The set $U(x_0) = (x_0) \cup \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} x_{mn}$ is a neighbourhood of x_0, by (3). Now, let $V(x_0)$ be any neighbourhood of x_0 such that $V(x_0) \subset U(x_0)$. Since $\mathfrak{L}_9 - \lim_n x_{mn} = x_0$, then, by (3), there is a sequence $\{p_m\}$ of naturals such that $x_{mn} \in V(x_0)$, $n > p_m$, $m = 1, 2, \ldots$. Since $\mathfrak{A}_0$ is cofinal in $\mathfrak{A}$ there is a cross-sequence $A_\eta \succ \{x_{mp_m}\}$; hence $y_\eta \in \lambda_9\, V(x_0) - U(x_0)$.

The space L_9 is $\{0, 1\}$ sequentially regular: Let z be a point and $\{z_n\}$ a sequence of points of L_9 not containing z such that no subsequence of $\{z_n\}$ $\mathfrak{L}_9$-converges to z. It suffices to prove that there is a closed-open set $B \subset L$ containing z and not containing any z_n. If $\vartheta(z) = 0$, we put $B = (z)$. If $z = y_\xi$, then we put $B = \lambda_9 A_\xi - \bigcup_{n=1}^{\infty} z_n$. Now, let $z = x_0$. Since no subsequence of $\{z_n\}$ converges to x_0, there are naturals p_m, $m = 1, 2, \ldots$, such that $\bigcup z_n \cap \bigcup_{m=1}^{\infty} \bigcup_{n \geq p_m} x_{mn} = \emptyset$. Denote by ϱ a countable ordinal such that $\xi < \varrho$ for each point y_ξ of $\bigcup z_n$ and such that $\{x_{mp_m}\} \prec A_\varrho$. Denote $A_\varrho = \{x_{mr_m}\}_{m=1}^{\infty}$ and put $s_m = \max\{p_m, r_m\}$); then the set $B = (x_0) \cup \bigcup_{\varrho \leq \xi} y_\xi \cup \bigcup_{m=1}^{\infty} \bigcup_{n \geq s_m} x_{mn}$ is open by (3). In order to prove that B is closed, suppose $\{t_n\}$ is a one-to-one sequence of points in B converging to a point $t \in L_9$. Then either $t = x_0$ so that $t \in B$, or $t = y_{\xi_0}$ for a suitable ξ_0. If $\xi_0 < \varrho$ then $A_{\xi_0} \prec A_\varrho$ which would contradict the fact $t_n \in A_{\xi_0}$ and $t_n \in B$ for nearly all n. Consequently $\varrho \leq \xi_0$ and $t \in B$.

Theorem 9. *Sequential regularity is a topological and hereditary property.*

Proof. Let $(L, \mathfrak{L}, \lambda)$ be a sequentially regular space; let h be a homeomorphism on L onto a convergence space $(M, \mathfrak{M}, \mu)$. Let y_0 and y_n, $n \in N$, be points of M such that no subsequence of $\{y_n\}$ $\mathfrak{M}$-converges to y_0. Then no subsequence of $\{h^{-1}(y_n)\}$ $\mathfrak{L}$-converges to $h^{-1}(y_0)$; since L is sequentially regular, there is a continuous function f on L such that $\{f(h^{-1}(y_n))\}$ does not converge to $f(h^{-1}(y_0))$. Then fh^{-1} is a continuous function on M such that $\{fh^{-1}(y_n)\}$ does not converge to $fh^{-1}(y_0)$. Consequently $(M, \mathfrak{M}, \mu)$ is a sequentially regular space.

Now, let $(P, \mathfrak{P}, \pi)$ be a convergence subspace of a sequentially regular space $(L, \mathfrak{L}, \lambda)$. Let z_0 be a point of P and $\{z_n\}$ a sequence of points $z_n \in P$ no subsequence of which $\mathfrak{P}$-converges to z_0. By Lemma 2 no subsequence of $\{z_n\}$ $\mathfrak{L}$-converges to z_0; consequently there is a continuous function f on L such that $\{f(z_n)\}$ does not converge to $f(z_0)$. Hence $(P, \mathfrak{P}, \pi)$ is a sequentially regular space, the partial function f_P being continuous on P.

Theorem 10. *The convergence Cartesian product $(L, \mathfrak{L}, \lambda)$ of convergence spaces $(L_\alpha, \mathfrak{L}^*_\alpha, \lambda_\alpha)$, $\alpha \in I$, is sequentially regular if and only if each space L_α, $\alpha \in I$, is sequentially regular.*

Proof. Suppose that each $(L_\alpha, \mathfrak{L}^*_\alpha, \lambda_\alpha)$ is sequentially regular. By Theorem 5 we have $\mathfrak{L} = \mathfrak{L}^*$. If $\{(z^n_\alpha)\}_{n=1}^{\infty}$ is a sequence of points of L not $\mathfrak{L}^*$-converging to a point $(z_\alpha) \in L$ then, by (8), there is an index $\gamma \in I$ such that the sequence $\{z^n_\gamma\}_{n=1}^{\infty}$ does not $\mathfrak{L}^*_\gamma$-converge to z_γ in L_γ. Since L_γ is sequentially regular, there exists a continuous function g on L_γ such that $\{g(z^n_\gamma)\}_{n=1}^{\infty}$ does not converge to $g(z_\gamma)$. Therefore the function f such that $f((x_\alpha)) = g(x_\gamma)$ is sequentially continuous on L and such that $\{f((z^n_\alpha))\}_{n=1}^{\infty}$ does not converge to $f((z_\alpha))$.
The proof of the converse follows immediately from Theorem 9.

Theorem 11. *The convergence space $(L, \mathfrak{L}, \lambda)$ is sequentially regular if and only if it is homeomorphic to a subspace of a convergence Euclidean space of the dimension $\leqq 2^{\aleph_0 P(L)}$.*

Proof. The sufficiency of the condition follows instantly from[10]) Theorems 9 and 10.

Now, suppose that $(L, \mathfrak{L}, \lambda)$ is a sequentially regular space. Denote by $\mathfrak{F}(L)$ the family of all continuous functions f_α, $\alpha \in I$, on L. Let $(E, \mathfrak{E}, \varepsilon) = \mathbf{X}\{(L_\alpha, \mathfrak{L}_5, \lambda_5) : L_\alpha = L_5, \alpha \in I\}$ be the $P(I)$-dimensional convergence Euclidean space, where $P(I) = P(\mathfrak{F}(L))$. Since $P(\mathfrak{F}(L)) \leqq (2^{\aleph_0})^{P(L)}$, we have $P(I) \leqq 2^{\aleph_0 P(L)}$. Prove that the map

$$(9) \qquad \varphi(x) = (f_\alpha(x)) \text{ where } x \in L, (f_\alpha(x)) \in E \text{ and } f_\alpha \in \mathfrak{F}(L)$$

is a homeomorphism[11]) on L onto the subspace $\varphi(L) \subset E$. As a matter of fact, if x and y are two distinct points of L, then $f_\gamma(x) \neq f_\gamma(y)$ for a suitable index $\gamma \in I$, L being sequentially regular. Hence φ is one-to-one map of L onto $\varphi(L)$. Now, if $\mathfrak{L} - \lim x_n = x$ in L, then $\lim f_\alpha(x_n) = f_\alpha(x)$ for each $\alpha \in I$, f_α being continuous on L and $\mathfrak{L}_5 = \mathfrak{L}^*_5$. Hence $\mathfrak{E} - \lim \varphi(x_n) = \varphi(x)$; thus φ is continuous on L.

On the other hand, if $\mathfrak{E} - \lim (z^n_\alpha) = (z_\alpha)$, where $(z_\alpha) \in \varphi(L)$ and $(z^n_\alpha) \in \varphi(L)$, $n \in N$, then there is a subsequence $\{(z^{n_i}_\alpha)\}_{i=1}^{\infty}$ of $\{(z^n_\alpha)\}$ such that $\{\varphi^{-1}((z^{n_i}_\alpha))\}$ $\mathfrak{L}$-converges to

[10]) Notice that $(L_5, \mathfrak{L}_5, \lambda_5)$ is a sequentially regular space such that $\mathfrak{L}_5 = \mathfrak{L}^*_5$ and that the convergence space containing only two distinct points a and b with the usual largest convergence is $\{0, 1\}$ sequentially regular.

[11]) The map defined in (9) will be called a special homeomorphism on L and will be denoted by the thick letter $\boldsymbol{\varphi}$.

the point $\varphi^{-1}((z_\alpha))$ in L; otherwise there would be a continuous function $f_\delta \in \mathfrak{F}(L)$ on L such that the sequence of real numbers $\{f_\delta(\varphi^{-1}((z_\alpha^n)))\}_{n=1}^{\infty}$ would not converge to the real number $f_\delta(\varphi^{-1}((z_\alpha)))$, L being sequentially regular. Since $f_\delta(\varphi^{-1}((z_\alpha^n))) = z_\delta^n$ and $f_\delta(\varphi^{-1}((z_\alpha)) = z_\delta$, the sequence $\{z_\delta^n\}_{n=1}^{\infty}$ would not converge to z_δ; this would contradict the assumption that $\mathfrak{C} - \lim (z_\alpha^n) = (z_\alpha)$. Consequently, the inverse map φ^{-1} is also continuous, by (5).

Remark. If we replace the postulate of sequential regularity by the assumption that the space is α' sequentially regular, then Theorems 9 and 10 remain true and Theorem 11 can[10]) be formulated as follows [10]:

The convergence space $(L, \mathfrak{L}, \lambda)$ is $\langle 0, 1\rangle$ sequentially [$\{0, 1\}$ sequentially] regular if and only if it is homeomorphic to a subspace of the convergence cube space [convergence cube vertex space] of the dimension $\leqq 2^{\aleph_0 P(L)}$.

The proofs of the theorems are analogous to the proof above and involve no difficulties; therefore they may be omitted.

5

Let $(L, \mathfrak{L}, \lambda)$ and $(M, \mathfrak{M}, \mu)$ be convergence spaces. Let $(L, \mathfrak{L}, \lambda)$ be a subspace of $(M, \mathfrak{M}, \mu)$. Let f be a continuous function on $(L, \mathfrak{L}, \lambda)$ and g a continuous function on $(M, \mathfrak{M}, \mu)$. If $x \in L$ implies $f(x) = g(x)$, then g is called the continuous extension of f.

Lemma 5. *Let $(L, \mathfrak{L}, \lambda)$ and $(M, \mathfrak{M}, \mu)$ be convergence spaces. Let $(L, \mathfrak{L}, \lambda)$ be a subspace of $(M, \mathfrak{M}, \mu)$ such that $\mu^{\omega_1}L = M$. Then each continuous function f on L has at most one continuous extension g on M.*

Proof. Suppose, on the contrary, that there are two different continuous extensions g and g' of a continuous function f on L onto M. Then there exists a point $x_0 \in M$ of the least possible order $\vartheta = \vartheta(x_0, L)$ in M such that $g(x_0) \neq g'(x_0)$. Since $f(x) = g(x) = g'(x)$, $x \in L$, then $\vartheta > 0$. Consequently, there is a sequence $\{x_n\}$ of points $x_n \in \mu^{\vartheta - 1}L$ which $\mathfrak{M}$-converges to x_0. Since g and g' are real valued continuous functions on M and $\mathfrak{L}_5 = \mathfrak{L}_5^*$, it follows that $g(x_0) = \lim g(x_n) = \lim g'(x_n) = g'(x_0)$; this is a contradiction.

Definition. Let $(L, \mathfrak{L}, \lambda)$ and $(S, \mathfrak{S}, \sigma)$ be sequentially regular spaces. Denote by α a property such that for each real valued continuous function f on L it can be decided whether it has property α or not. Then $(S, \mathfrak{S}, \sigma)$ is called the *sequential* [α sequential] *envelope* (abbr. σ-envelope) of $(L, \mathfrak{L}, \lambda)$ if the following conditions are satisfied:

$\sigma0)$ $(L, \mathfrak{L}, \lambda)$ is a subspace of $(S, \mathfrak{S}, \sigma)$.

$\sigma1)$ L is σ^{ω_1}-dense in S.

σ2) Each continuous function f on $(L, \mathfrak{L}, \lambda)$ [with the property α] has a continuous extension $\bar{f}$ onto $(S, \mathfrak{S}, \sigma)$.

σ3) There is no sequentially regular space $(S', \mathfrak{S}', \sigma')$ containing $(S, \mathfrak{S}, \sigma)$ as a proper subspace and fulfilling σ1) and σ2) relative to L and S'.

Theorem 12. *Let $(L, \mathfrak{L}, \lambda)$ and $(M, \mathfrak{M}, \mu)$ be sequentially regular spaces. Let L be a subspace of M such that $\mu^{\omega_1}L = M$. Let φ be a special homeomorphism on L into the convergence Euclidean space $(E, \mathfrak{E}, \varepsilon)$ of dimension $P(\mathfrak{F}(L))$. Then each continuous function on L can be extended to a continuous function on M if and only if there is a homeomorphism h on M into $\varepsilon^{\omega_1}\varphi(L)$ such that $h(x) = \varphi(x)$, $x \in L$.*

Proof. Let φ be a special homeomorphism on L onto the subset of E:

$$\varphi(L) = \{(f_\alpha(x)) \in E : f_\alpha \in \mathfrak{F}(L),\ x \in L,\ \alpha \in I\}.$$

Suppose that each continuous function $f \in \mathfrak{F}(L)$ can be extended to a continuous function $g \in \mathfrak{F}(M)$. According to Lemma 5 there is a one-to-one correspondence on $\mathfrak{F}(L)$ onto $\mathfrak{F}(M)$. Denote by g_α the corresponding continuous extension of f_α, $\alpha \in I$. Consequently there is a special homeomorphism ψ on M onto the subset of E:

$$\psi(M) = \{(g_\alpha(x)) \in E : g_\alpha \in \mathfrak{F}(M),\ x \in M,\ \alpha \in I\}$$

such that $\psi(x) = \varphi(x)$, $x \in L$.

Now, assume that for all ordinals $\xi < \eta$, where $0 \leqq \xi$, $\eta \leqq \omega_1$, we have just proved that $\psi(\mu^\xi L) \subset \varepsilon^\xi \varphi(L)$. If η is isolated, then $\psi(\mu^\eta L) \subset \varepsilon\psi(\mu^{\eta-1}L) \subset \varepsilon^\eta\varphi(L)$, by (1). If η is not isolated, then $\psi(\mu^\eta L) = \psi(\bigcup_{\xi<\eta} \mu^\xi L) \subset \bigcup_{\xi<\eta} \varepsilon^\xi\varphi(L) = \varepsilon^\eta\varphi(L)$. Thus, by transfinite induction we have proved that

$$\psi(M) \subset \varepsilon^{\omega_1}\varphi(L) \tag{10}$$

With respect to (10) it is sufficient to put $h(x) = \psi(x)$, $x \in M$.

Now, let φ be a special homeomorphism on L onto $\varphi(L) \subset E$. Let h be a homeomorphism on M into $\varepsilon^{\omega_1}\varphi(L)$ such that $h(x) = \varphi(x)$, $x \in L$. Let f be any continuous function on L. Then there is an index $\alpha_0 \in I$ such that $f = f_{\alpha_0}$. Define a projection mapping: $p((z_\alpha)) = z_{\alpha_0}$ for each $(z_\alpha) \in \varepsilon^{\omega_1}\varphi(L)$. The function ph is continuous on M, by (4), and $h(x) = \varphi(x)$, $x \in L$, implies $ph(x) = f(x)$, $x \in L$. Consequently ph is a continuous extension of f.

Corollary 3. *Let $(L, \mathfrak{L}, \lambda)$ and $(M, \mathfrak{M}, \mu)$ be sequentially regular spaces. Let L be a subspace of M such that $\mu^{\omega_1}L = M$. Let φ be a special homeomorphism on L into the convergence Euclidean space $(E, \mathfrak{E}, \varepsilon)$ of dimension $P(\mathfrak{F}(L))$. Then each continuous function on L can be extended to a continuous function on M if and only if there is a special homeomorphism ψ on M into $\varepsilon^{\omega_1}\varphi(L)$ such that $\psi(x) = \varphi(x)$, $x \in L$.*

The proof follows instantly from Theorem 12 and from (10) by putting $h(x) = \psi(x)$; $x \in M$.

Theorem 13. *Let $(L, \mathfrak{L}, \lambda)$ be a sequentially regular space and $(S, \mathfrak{S}, \sigma)$ a convergence space. Let L be a subspace of S. Let φ be a special homeomorphism on L into the convergence Euclidean space $(E, \mathfrak{E}, \varepsilon)$ of dimension $P(\mathfrak{F}(L))$. Then $(S, \mathfrak{S}, \sigma)$ is a sequential envelope of $(L, \mathfrak{L}, \lambda)$ if and only if there is a homeomorphism h on S onto $\varepsilon^{\omega_1} \varphi(L)$ such that $h(x) = \varphi(x)$, $x \in L$.*

Proof. Let $(S, \mathfrak{S}, \sigma)$ be a σ-envelope of $(L, \mathfrak{L}, \lambda)$. Let φ be a special homeomorphism on L onto the subset of E:

$$\varphi(L) = \{(f_\alpha(x)) \in E : f_\alpha \in \mathfrak{F}(L),\ x \in L,\ \alpha \in I\} .$$

By σ1), σ2) and Corollary 3 there is a special homeomorphism ψ on S into $\varepsilon^{\omega_1} \varphi(L)$ such that $\psi(x) = \varphi(x)$, $x \in L$. Suppose, on the contrary, that there is a point $b \in \varepsilon^{\omega_1} \varphi(L) - \psi(S)$ with the least possible order $\vartheta = \vartheta(b, \varphi(L))$ in E. Add a new element a to the set S, put $h'(x) = \psi(x)$ for $x \in S$, $h'(a) = b$ and define a convergence $\mathfrak{S}'$ on the set $S' = S \cup (a)$ as follows:

If x is a point and $\{x_n\}$ a sequence of points of S', then $(\{x_n\}, x) \in \mathfrak{S}'$ whenever $\mathfrak{E} - \lim h'(x_n) = h'(x)$. In such a way we get a convergence space $(S', \mathfrak{S}', \sigma')$. Evidently $h'(x)$, $x \in S'$, is a homeomorphism on S' onto the convergence subspace $h'(S') \subset \varepsilon^{\omega_1} \varphi(L)$. Hence, from Theorem 11 it follows that $(S', \mathfrak{S}', \sigma')$ is a sequentially regular space. It is easy to see that $(\{x_n\}, x) \in \mathfrak{S}$ implies $(\{x_n\}, x) \in \mathfrak{S}'$ and $(\{y_n\}, y) \in \mathfrak{S}'$, $y \in S$, $y_n \in S$ implies $(\{y_{n_i}\}, y) \in \mathfrak{S}$, $\{y_{n_i}\}$ being a suitable subsequence of $\{y_n\}$. Therefore, by Lemma 2, $(S, \mathfrak{S}, \sigma)$ and consequently also $(L, \mathfrak{L}, \lambda)$, are subspaces of $(S', \mathfrak{S}', \sigma')$. From Lemma 3 it follows that the order $\vartheta(a, L)$ in $(S', \mathfrak{S}', \sigma')$ equals the order $\vartheta = \vartheta(b, h'(L))$ in $(E, \mathfrak{E}, \varepsilon)$. Hence $a \in \sigma'^{\vartheta} L$. and so $S' = \sigma'^{\omega_1} L$. Since $h'(x) = \psi(x) = \varphi(x)$, $x \in L$, then in view of Theorem 12, each continuous function on L can be continuously extended onto S'. Consequently all conditions σ0), σ1) and σ2) are satisfied with respect to the spaces L and S'. This contradicts the property σ3). Therefore there is a special homeomorphism ψ on S onto

$$\psi(S) = \varepsilon^{\omega_1} \varphi(L) . \tag{11}$$

Consequently we can put $h(x) = \psi(x)$, $x \in S$.

Now, suppose that a special homeomorphism φ on L onto $\varphi(L) \subset E$ is given and that there is a homeomorphism h on S onto $\varepsilon^{\omega_1} \varphi(L)$ with properties mentioned in Theorem 13. Since $h(S) = \varepsilon^{\omega_1} \varphi(L)$ and $\varphi(L) = h(L)$, then $S = \sigma^{\omega_1} L$ by Corollary 2. Consequently, σ1) is true. In view of Theorems 11 and 12 the property σ2) is also satisfied. The validity of σ3) remains to be proved.

Suppose, on the contrary, that there is a sequentially regular space $(\bar{S}, \bar{\mathfrak{S}}, \bar{\sigma})$ containing $(S, \mathfrak{S}, \sigma)$ as a proper subspace and fulfilling σ1) and σ2) with regard to the spaces L and $\bar{S}$. Then there is a least ordinal ϑ such that $\bar{\sigma}^{\vartheta} L - S \neq \emptyset$; choose a point $\bar{a} \in \bar{\sigma}^{\vartheta} L - S$. Then $\vartheta = \vartheta(\bar{a}, L)$ in $\bar{S}$ and $L \subset S$ implies that $\vartheta > 0$ and $\bar{\sigma}^{\vartheta - 1} L - S = \emptyset$. By σ2) and Theorem 12 there is a homeomorphism $\bar{h}$ on $\bar{S}$ into $\varepsilon^{\omega_1} \varphi(L)$ such that $\bar{h}(x) = \varphi(x)$, $x \in L$. We shall prove that $\bar{h}(x) = h(x)$ for each $x \in \sigma^{\vartheta - 1} L$.

Assume that $\bar{h}(x) = h(x)$ for each $x \in \sigma^{\xi} L$ and each $\xi < \zeta$, where $0 < \zeta \leqq \vartheta - 1$. If ζ is a limiting ordinal there is nothing to be proved. If ζ is an isolated ordinal and x_0 a point of $\sigma^{\zeta} L$, then there is a sequence $\{x_n\}$ of points $x_n \in \sigma^{\zeta-1} L$ which $\mathfrak{S}$-converges to x_0. Then $h(x_0) = \mathfrak{E} - \lim h(x_n) = \mathfrak{E} - \lim \bar{h}(x_n) = \bar{h}(x_0)$.

Now, choose a sequence $\{t_n\}$ of points $t_n \in S \cap \bar{\sigma}^{\vartheta-1} L$ which $\bar{\mathfrak{S}}$-converges to the point $\bar{a}$. Then $\mathfrak{E} - \lim \bar{h}(t_n) = \bar{h}(\bar{a}) \in \varepsilon^{\omega_1} \varphi(L)$. Since $\bar{\sigma}^{\vartheta-1} L - S = \emptyset$, we have $\sigma^{\vartheta-1} L = S \cap \bar{\sigma}^{\vartheta-1} L$; consequently $\mathfrak{E} - \lim h(t_n) = \bar{h}(\bar{a})$. Since $h(S) = \varepsilon^{\omega_1} \varphi(L)$ then $\bar{h}(\bar{a}) \in h(S)$, $h^{-1}(\bar{h}(\bar{a})) \in S$ and $\mathfrak{S} - \lim t_n = h^{-1}(\bar{h}(\bar{a}))$. By Lemma 2 there is a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ which $\bar{\mathfrak{S}}$-converges to the point $h^{-1}(\bar{h}(\bar{a}))$. Since $h^{-1}(\bar{h}(\bar{a})) \neq \bar{a}$ and because the axiom $(\mathscr{L}_0)$ is true, we have a contradiction.

Corollary 4. *Let $(L, \mathfrak{L}, \lambda)$ and $(S, \mathfrak{S}, \sigma)$ be sequentially regular spaces. Let L be a subspace of S. Let φ be a special homeomorphism on L into the convergence Euclidean space $(E, \mathfrak{E}, \varepsilon)$ of dimension $P(\mathfrak{F}(L))$. Then S is a sequential envelope of L if and only if there is a special homeomorphism ψ on S onto $\varepsilon^{\omega_1} \varphi(L)$ such that $\psi(x) = \varphi(x)$, $x \in L$.*

The proof follows immediately from (11) and Theorem 13 by putting $h(x) = \psi(x)$, $x \in S$.

Theorem 14. *Let $(L, \mathfrak{L}, \lambda)$ be a sequentially regular space. Then there exists a sequential envelope $(S, \mathfrak{S}, \sigma)$ of $(L, \mathfrak{L}, \lambda)$.*

Proof. Let φ be a special homeomorphism on L onto $\varphi(L) \subset E$. Choose a set S containing L as a subset such that $S - L$ and $\varepsilon^{\omega_1} \varphi(L) - \varphi(L)$ have the same power. Then there is a one-to-one map s on S onto $\varepsilon^{\omega_1} \varphi(L)$ such that $s(x) = \varphi(x)$ for each $x \in L$. Define the convergence $\mathfrak{S}$ on S as follows: $(\{x_n\}, x) \in \mathfrak{S}$ whenever $(\{s(x_n)\}, s(x)) \in \mathfrak{E}$, where $x \in S$ and $x_n \in S$. In such a way we get a convergence space $(S, \mathfrak{S}, \sigma)$. The map s is a homeomorphism on $(S, \mathfrak{S}, \sigma)$ and $(L, \mathfrak{L}, \lambda)$ is a subspace of $(S, \mathfrak{S}, \sigma)$. Consequently $(S, \mathfrak{S}, \sigma)$ is a σ-envelope of $(L, \mathfrak{L}, \lambda)$, by Theorem 13.

Theorem 15. *Let $(S^{(i)}, \mathfrak{S}^{(i)}, \sigma^{(i)})$ be a sequential envelope of a sequentially regular space $(L^{(i)}, \mathfrak{L}^{(i)}, \lambda^{(i)})$, $i = 1, 2$. Let h_0 be a homeormorphism on $L^{(1)}$ onto $L^{(2)}$. Then there is a homeomorphism h on $S^{(1)}$ onto $S^{(2)}$ such that $h(x) = h_0(x)$ for each $x \in L^{(1)}$.*

Proof. Let φ_1 be a special homeomorphism on $L^{(1)}$ onto

$$\varphi_1(L^{(1)}) = \{(f_\alpha(x)) \in E : f_\alpha \in \mathfrak{F}(L^{(1)}),\ x \in L^{(1)},\ \alpha \in I\} .$$

Since $g \in \mathfrak{F}(L^{(2)})$ if and only if $g = f h_0^{-1}$ where $f \in \mathfrak{F}(L^{(1)})$, then there is a one-to-one correspondence on $\mathfrak{F}(L^{(1)})$ onto $\mathfrak{F}(L^{(2)})$ such that $g_\alpha = f_\alpha h_0^{-1}$, $\alpha \in I$. Consequently $\varphi_2(y) = (g_\alpha(y))$, $y \in L^{(2)}$, is a special homeomorphism onto

$$\varphi_2(L^{(2)}) = \{(f_\alpha h_0^{-1}(y)) \in E : f_\alpha h_0^{-1} \in \mathfrak{F}(L^{(2)}),\ y \in L^{(2)},\ \alpha \in I\} .$$

Since $f_\alpha h_0^{-1}(y) = f_\alpha(x)$ for each $y = h_0(x)$, $x \in L^{(1)}$, we have $\varphi_1(x) = \varphi_2(h_0(x))$, $x \in L^{(1)}$; therefore $\varphi_1(L^{(1)}) = \varphi_2(L^{(2)})$ and $\varepsilon^{\omega_1} \varphi_1(L^{(1)}) = \varepsilon^{\omega_1} \varphi_2(L^{(2)})$.

According to Theorem 13 there is a homeomorphism h_i on $S^{(i)}$ onto $\varepsilon^{\omega_1} \varphi_1(L^{(i)})$ such that $h_i(x) = \varphi_i(x)$, $x \in L^{(i)}$, $i = 1, 2$. Then $h = h_2^{-1} h_1$ is a homeomorphism on $S^{(1)}$ onto $S^{(2)}$ such that $h(x) = h_0(x)$ for each $x \in L^{(1)}$.

Corollary 5. *Let $(S_1, \mathfrak{S}_1, \sigma_1)$ and $(S_2, \mathfrak{S}_2, \sigma_2)$ be sequential envelopes of a sequentially regular space $(L, \mathfrak{L}, \lambda)$. Then there is a homeomorphism h on S_1 onto S_2 such that $h(x) = x$ for each $x \in L$.*

Proof. If suffices to be put in Theorem 15: $L^{(i)} = L$, $\mathfrak{L}^{(i)} = \mathfrak{L}$, $\lambda^{(i)} = \lambda$, $i = 1, 2$, and $h_0(x) = x$ for each $x \in L$.

Now, let us show that there exists a sequentially regular space $(L, \mathfrak{L}, \lambda)$ such that $\sigma(L) \neq L$. We are going to construct such a space.

Let L_{10} be the set of all pairs (α, k), α being an ordinal $\leqq \omega_1$ and k a natural. Define the convergence $\mathfrak{L}_{10}$ on L_{10} as follows: $(\{z\}, z) \in \mathfrak{L}_{10}$ for each $z \in L_{10}$, $(\{(\alpha_n, 1)\}, (\alpha, 1)) \in \mathfrak{L}_{10}$ whenever $\lim \alpha_n = \alpha$. $(\{(\alpha_n, k)\}_{n=1}^{\infty}, (\omega_1, k)) \in \mathfrak{L}_{10}$ whenever $k > 1$ and $\alpha_n \neq \alpha_m$ for $n \neq m$. $(\{(\alpha, k_j)\}_{j=1}^{\infty}, (\alpha, 1)) \in \mathfrak{L}_{10}$ for each $\alpha \leqq \omega_1$ and each $k_1 < k_2 < \ldots$. In such a way we get a convergence space $(L_{10}, \mathfrak{L}_{10}, \lambda_{10})$. Notice that each point (α, k), $\alpha \neq \omega_1$, $k \neq 1$, is isolated in L_{10} and that the set $\bigcup_{\alpha \leqq \omega_1} (\alpha, k)$, where $k \neq 1$, is closed-open. Now, let $\alpha < \omega_1$; arrange all ordinals $\xi \leqq \alpha$ in a one-to-one sequence $\{\xi_n\}_{n=1}^{\infty}$ and then choose any sequence of naturals $p_1 < p_2 < \ldots$. Evidently, the set $\bigcup_{n=1}^{\infty} \bigcup_{\substack{k > p_n \\ k=1}} (\xi_n, k)$ is closed-open in L_{10}. It will be denoted by $[\alpha; \{\xi_n; p_n\}]$.

Prove that $(L_{10}, \mathfrak{L}_{10}, \lambda_{10})$ is sequentially regular. Suppose that (α_0, k_0) is a point and $\{(\alpha_n, k_n)\}_{n=1}^{\infty}$ a sequence of points of L_{10} such that no subsequence of it $\mathfrak{L}_{10}$-converges to (α_0, k_0). It suffices to find a closed-open set G such that $(\alpha_n, k_n) \in G$ for infinitely many naturals n whereas $(\alpha_0, k_0) \in L_{10} - G$. If there is a natural $k' > 1$ such that $k_n = k'$ for each $n \in N'$ where N' is an infinite subset of naturals, then put $G = \bigcup_{n \in N'} (\alpha_n, k') - (\alpha_0, k_0)$ if $(\alpha_0, k_0) = (\omega_1, k')$ (in this case G is a finite set of isolated points) or $G = \bigcup_{\alpha \leqq \omega_1} (\alpha, k') - (\alpha_0, k_0)$ if $(\alpha_0, k_0) \neq (\omega_1, k')$. Hence we may suppose that either $k_n = 1$ for all n or $k_n \neq k_m$ for $n \neq m$.

First suppose that $k_0 > 1$. Then we put $G = L_{10} - (\alpha_0, k_0)$ if $\alpha_0 \neq \omega_1$ or $G = L_{10} - \bigcup_{\alpha \leqq \omega_1} (\alpha, k_0)$ if $\alpha_0 = \omega_1$. Now assume that $k_0 = 1$. Denote by β the least ordinal such that $\alpha_n \leqq \beta$ for each $n \in N''$ where N'' is an infinite subset of naturals. If $\alpha_0 < \beta$ then put $G = L_{10} - [\alpha_0; \{\xi_n, p_n\}]$. If $\alpha_0 > \beta$ then put $G = \bigcup_{n \in N''} (\alpha_n, k_n) \cup [\beta; \{\xi_n, p_n\}]$. In the remaining case $\alpha_0 = \beta$ there is a subsequence $\{(\alpha_{n_i}, k_{n_i})\}$ of $\{(\alpha_n, k_n)\}$ such that $\alpha_{n_i} < \beta$ and $k_{n_i} > 1$ for all i so that $G = \bigcup_{i=1}^{\infty} (\alpha_{n_i}, k_{n_i})$ is a closed-open set not containing the point $(\alpha_0, 1)$.

Now, prove that each continuous function f on the subset $L_{11} = L_{10} - (\omega_1, 1)$ can be extended to a continuous function on L_{10}. Define the convergence $\mathfrak{L}_{11}$ on L_{11}

as follows: $(\{x_n\}, x) \in \mathfrak{L}_{11}$ whenever $\bigcup x_n \subset L_{11}$, $x \in L_{11}$ and $(\{x_n\}, x) \in \mathfrak{L}_{10}$. Then $(L_{11}, \mathfrak{L}_{11}, \lambda_{11})$ is a sequentially regular subspace of $(L_{10}, \mathfrak{L}_{10}, \lambda_{10})$, by Theorem 9, and $\lambda_{10}L_{11} = L_{10}$. Let f be a continuous function on L_{11}. Let $k > 1$ be a natural. Since $\mathfrak{L}_{11} - \lim (\alpha_n, k) = (\omega_1, k)$ if $\alpha_1 < \alpha_2 < \ldots$, then there is a countable ordinal γ_k such that $\gamma_k \leqq \alpha < \omega_1$ implies $f(\alpha, k) = f(\omega_1, k)$. From the property of the set of all countable ordinals it follows that there exists a countable ordinal γ exceeding all γ_k, $k = 2, 3, \ldots$, and a real number c such that $\gamma \leqq \alpha < \omega_1$ implies $f(\alpha, 1) = c$. Since $\mathfrak{L}_{11} - \lim\limits_k (\gamma, k) = (\gamma, 1)$, then $\lim\limits_k f(\gamma, k) = c$. Consequently $f(\gamma, k) = f(\omega_1, k)$, $k = 1, 2, \ldots$, implies $\lim\limits_k f(\omega_1, k) = c$. Now, define $\bar{f}(x) = f(x)$, $x \in L_{11}$ and $\bar{f}(\omega_1, 1) = c$. Then $\bar{f}$ is a continuous extension onto $(L_{10}, \mathfrak{L}_{10}, \lambda_{10})$ of the function f. From property $\sigma 3)$ it follows that $\sigma_e(L_{11}) \neq L_{11}$.

Remark. If we replace the assumption that the sequential envelope S is sequentially regular by the postulate that it is a convergence space, then Theorem 13 need not be true. This is shown by the following example:

Add a new element $(\omega_1, 0)$ to the set L_{10}, denote $L_{12} = L_{10} \cup (\omega_1, 0)$ and define the convergence $\mathfrak{L}_{12}$ on L_{12} as folows: $(\{x\}, x) \in \mathfrak{L}_{12}$ for each $x \in L_{12}$.

If $(\omega_1, 0) \neq x \neq (\omega_1, 1)$, then $(\{x_n\}, x) \in \mathfrak{L}_{12}$ whenever $(\{x_n\}, x) \in \mathfrak{L}_{10}$.
If $(\omega_1, 0) = x$, then $(\{x_n\}, x) \in \mathfrak{L}_{12}$ if $\{x_n\}$ is a subsequence of $\{(\omega_1, 2n)\}_{n=1}^{\infty}$.
If $(\omega_1, 1) = x$, then $(\{x_n\}, x) \in \mathfrak{L}_{12}$ if $\{x_n\}$ is a subsequence of $\{(\omega_1, 2n - 1)\}_{n=1}^{\infty}$.

The convergence space $(L_{12}, \mathfrak{L}_{12}, \lambda_{12})$ is not sequentially regular[12]. The sequentially regular space $(L_{11}, \mathfrak{L}_{11}, \lambda_{11})$ is a subspace of $(L_{12}, \mathfrak{L}_{12}, \lambda_{12})$ such that $\lambda_{12}L_{11} = L_{12}$ and that each continuous function on L_{11} has a continuous extension[12] on L_{12}. From Theorem 11 it follows that the space L_{12} is not homeomorphic to any subspace of a convergence Euclidean space $(E, \mathfrak{E}, \varepsilon)$.

6

Let X be an abstract point set and $\mathbf{X}$ the system of all its subsets. The following definition of the convergence $\mathfrak{X}$ on the system $\mathbf{X}$ is well-known: $(\{A_n\}, A) \in \mathfrak{X}$ whenever $A = \bigcup\limits_{k=1}^{\infty} \bigcap\limits_{n=k}^{\infty} A_n = \bigcap\limits_{k=1}^{\infty} \bigcap\limits_{n=k}^{\infty} A_n$. The convergence $\mathfrak{X}$ fulfills [12] all four axioms of convergence $(\mathscr{L}_0) - (\mathscr{L}_3)$. Consequently $(\mathbf{X}, \mathfrak{X}, \xi)$ is a convergence space; it will be called the convergence system of sets.

Theorem 16. *Let X be a set and $\mathbf{X}$ the system of all subsets of X. Then the convergence system $(\mathbf{X}, \mathfrak{X}, \xi)$ is homeomorphic to the convergence cube vertex space $(C_0, \mathfrak{C}_0, \gamma_0)$ of dimension $P(X)$.*

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[12]) The easy proof may be omitted.

Proof. Put $I = X$ and define $\varphi(A) = (f_x(A))$, $x \in I$, $A \in \mathbf{X}$, whereby $f_x(A) = 0$ or $=1$ according to whether $x \in X - A$ or $x \in A$. Then φ is a one-to-one map on $\mathbf{X}$ onto C_0.

Now, let $\mathfrak{X} - \lim A_n = A$ and $x \in X$; then $x \in A$ if and only if $\lim f_x(A_n) = 1$ and $x \in (X - A)$ if and only if $\lim f_x(A_n) = 0$. Consequently $\mathfrak{C}_0 - \lim \varphi(A_n) = \varphi(A)$.

Conversely, suppose that $\mathfrak{C}_0 - \lim (z_x^n) = (z_x)$ in C_0 and denote $B_n = \varphi^{-1}((z_x^n))$ and $B = \varphi^{-1}((z_x))$. If $x \in B$, then $z_x = 1$ and $\lim z_x^n = z_x$ implies $x \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} B_n$ and so $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} B_n$; on the other hand, if $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} B_n$, then $z_x^n = 1$ for infinitely many n and since $\lim z_x = z_x$, we have $z_x = 1$ so that $x \in B$. Therefore $B = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} B_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} B_n$. Hence $\mathfrak{X} - \lim \varphi^{-1}((z_x^n)) = \varphi^{-1}((z_x))$.

Let $(\mathbf{X}, \mathfrak{X}, \xi)$ be a convergence system of all subsets of a set X. Let $\mathbf{A}$ be a convergence subsystem of $(\mathbf{X}, \mathfrak{X}, \xi)$. From Theorems 16 and 11 it follows that $(\mathbf{A}, \mathfrak{X}_{\mathbf{A}}, \xi_{\mathbf{A}})$ is a $\{0, 1\}$ sequentially regular space. Since $\mathbf{A}$ is homeomorphic to a subset of the convergence cube vertex space, it follows that the study of convergence topological properties of convergence systems of sets is essentially the same as the study of the convergence structure of the convergence cube vertex space.

From this consideration it can be deduced that each $\{0, 1\}$ sequentially regular space is homeomorphic to a subspace of a convergence system of all subsets of a point set X of a suitable cardinal. For example the convergence space $(L_i, \mathfrak{L}_i^*, \lambda_i)$, $i = 1, 2, 3, 4, 9, 10, 11$, may be realized by convergence systems of sets.

The following questions might be of interest: What is the relation between the sigma ring $\sigma(\mathbf{E})$ of a ring $\mathbf{E}$ of sets and α sequential envelopes $\sigma_e(\mathbf{E})$ of the ring $\mathbf{E}$? What is the relation between the system of all Baire functions and α sequential envelopes of the system of all real-valued continuous functions?

Remark. In [10] I used the notion of $\langle 0, 1\rangle$ sequential regularity instead of sequential regularity. It is worth noting that both notions are the same. As a matter of fact, a convergence space $(L, \mathfrak{L}, \lambda)$ is sequentially regular if and only if it is $\langle 0, 1\rangle$ sequentially regular. It is clear that $\langle 0, 1\rangle$ sequentional regularity implies the sequential regularity. On the other hand, suppose that L is sequentially regular; let $\{x_n\}$ be a sequence of points and x_0 a point of L such that no subsequence of it converges.to x_0. Then there is a continuous function f on L such that $\{f(x_n)\}$ does not converge to $f(x_0)$ so that $|f(x_n) - f(x_0)| > c$ for a suitable $c > 0$ and for infinitely many n. Now, it suffices to put $g(x) = (1/c)\ |f(x) - f(x_0)|$ for all $x \in L$ such that $|f(x) - f(x_0)| \leqq c$ and $g(x) = 1$, if $|f(x) - f(x_0)| > c$. Consequently, L is $\langle 0, 1\rangle$ sequentially regular.

It should be noticed that the definition of a $\langle 0, 1\rangle$ sequential envelope $\sigma_e(L)$ of a topological space L which is a completely regular convergence space is to a certain extent analogous to the definition of Stone-Čech compactification $\beta(L)$, viz. as to the conditions σ0), σ1) and σ2). However, both envelopes $\beta(L)$ and $\sigma_e(L)$ can differ substantially from each other. For instance, let P be an infinite isolated space. Since $\beta(P)$ is compact, we have $P \neq \beta(P)$. On the other hand, it is easy to see that $\sigma_e(P) = P$. From this example it follows that the sequential envelope of a topological space need not be compact, not even countably compact.

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Резюме

О ПРОСТРАНСТВАХ СХОДИМОСТИ И ИХ СЕКВЕНЦИОНАЛЬНЫХ ОБОЛОЧКАХ

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Сходимостью $\mathfrak{L}$ на множестве L мы разумеем подмножество декартова произведения $\mathfrak{U} \times L$, где $\mathfrak{U}$ – система всех последовательностей $\{x_n\}$ точек $x_n \in L$ удовлетворяющих аксиомам Фреше $(\mathscr{L}_0)$, $(\mathscr{L}_1)$ и $(\mathscr{L}_2)$. Замыкание λA множества $A \subset L$ – это множество всех точек $x \in L$, к которым существует такая последо-

вательность точек $x_n \in A$, что $(\{x_n\}, x) \in \mathfrak{L}$. Отображение λ является секвенциональной топологией, удовлетворяющей аксиомам (C_1) и (C_2). Сходимости $\mathfrak{L}$ и $\mathfrak{M}$ на множестве L эквивалентны, если соответствующие секвенциональные топологии тождественны. Доказывается, что в каждом классе взаимно эквивалентных сходимостей на L существует самая большая сходимость, характеризованная аксиомой $(\mathscr{L}_3)$ „Быть наибольшей сходимостью" — это топологическое и продуктивное свойство (т.е. свойство декартова произведения, в котором сходимость определена по компонентам). На примерах показано, что в случае, когда сходимость в компонентах декартова произведения заменена эквивалентными сходимостями, может измениться и декартова топология сходимости.

Понятию вполне регулярного топологического пространства соответствует следующее понятие секвенционально регулярного пространства сходимости:

Пространство сходимости L является секвенционально регулярным, если к каждой точке $x_0 \in L$ и к каждой последовательности точек $x_n \in L$, причем никакая выбранная из нее последовательность не сходится к точке x_0, существует действительная непрерывная функция f такая, что последовательность чисел $f(x_n)$ не сходится к $f(x_0)$. Пространство сходимости L является секвенциально регулярным тогда и только тогда, когда существует гомеоморфное отображение пространства L в евклидово пространство сходимости (т.е. декартово произведение сходимости, каждая компонента которого представляет множество действительных чисел с обыкновенной сходимостью).

Произведена классификации пространств сходимости.

Секвенционально регулярное пространство S является секвенциональной оболочкой секвенционально регулярного пространства $L \subset S$, если S — наименьшее замкнутое множество в S, содержащее L, и одновременно наибольшее-секвенционально регулярное пространство, обладающее тем свойством, что каждую непрерывную функцию f на L можно непрерывно расширить на S. Доказано, что к каждому секвенционально регулярному пространству L существует секвенциональная оболочка $\sigma_e(L)$. Приведены критерии, по которым можно установить, если $S = \sigma_e(L)$. Построен пример секвенционально регулярного пространства $L \neq \sigma_e(L)$.

В заключение работы внимание обращено на системы множеств с обыкновенной сходимостью множеств. Каждая такая система является секвенционально регулярным пространством сходимости и, следовательно, существует к ней секвенциональная оболочка.