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# AN INEQUALITY FOR TRACES OF MATRIX FUNCTIONS[1])

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**1.** M. FIEDLER recently gave an inequality for traces of matrices [1]. H. SCHWERDTFEGER, reporting on this paper at the University of Wisconsin, suggested that, in Fiedler's theorem, the inverse function might be replaced by an arbitrary non-constant matrixmonotone function [2]. I found to my surprise that the function may be still more general. The result is as follows:

**Theorem 1.** *Let $A, H$ be n-by-n hermitian matrices, and $[a,b]$ a real interval containing the spectra of $A$ and $A + H$. Let $f$ be a real-valued function on $[a,b]$ such that the divided difference $f^{[1]}(t, u) = [f(t) - f(u)]/[t - u]$ $(t \neq u)$ satisfies*

$$m \leqq f^{[1]}(t, u) \leqq M \tag{1}$$

*for $t, u \in [a, b]$. Then the hermitian matrices $f(A)$ and $f(A + H)$ satisfy*

$$m \operatorname{tr} H^2 \leqq \operatorname{tr} \{H(f(A + H) - f(A))\} \leqq M \operatorname{tr} H^2 . \tag{2}$$

I will prove this theorem in § 2. Then in § 3 I will discuss some particularly useful special cases: Fiedler's original theorem, and a Lipschitz condition for matrix functions which is applicable to matrix analysis. The final section concerns weakening of the restrictions on $A$, $H$, and $f$.

**2.** I will write $x^*$ for the linear functional determined by any vector $x$. The inner product of $x$ with $y$ will be written $x^*y$; whereas $yx^*$ means an operator, namely, $(yx^*)\, z = (x^*z)\, y$, for any $z$.

Thus the spectral decomposition for $A$ and $A + H$ may be written

$$A = \sum_{i=1}^{n} t_i x_i x_i^* , \quad A + H = \sum_{i=1}^{n} u_i y_i y_i^* , \tag{3}$$

where $\{x_i\}$ and $\{y_i\}$ are orthonormal bases, while the $t_i$ and the $u_i$ are numbers between $a$ and $b$. By definition,

$$f(A) = \sum_{i=1}^{n} f(t_i)\, x_i x_i^* , \quad f(A + H) = \sum_{i=1}^{n} f(u_i)\, y_i y_i^* .$$

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I will occasionally use the notation $\|B\|_2 = (\mathrm{tr}\,(B^*B))^{\frac{1}{2}}$. The notation $\|B\|$ will mean the "bound norm" of $B$.

The proof of the theorem is very short and follows familiar lines [2]. Define numbers $h_{ij} = x_i^* H y_j$ (note these are not the usual matrix elements in either representation). Since $H = \sum u_j y_j y_i^* - \sum t_i x_i x_j^*$, we compute

$$(4) \qquad h_{ij} = (u_j - t_i)\, x_i^* y_j \,.$$

In a similar manner, we obtain

$$x_i^*(f(A + H) - f(A))\, y_j = (f(u_j) - f(t_i))\, x_i^* y_j \,.$$

Now we must estimate

$$\mathrm{tr}\,\{H(f(A + H) - f(A))\} = \sum_{ij} y_j^*\, H x_i x_i^*(f(A + H) - f(A))\, y_j =$$

$$(5) \qquad = \sum \overline{h_{ij}}[f(u_j) - f(t_i)]\, x_i^* y_j = \sum |h_{ij}|^2 \frac{f(u_j) - f(t_i)}{u_j - t_i}$$

(substituting (4)). In the last line the summation is extended over only those pairs $(i, j)$ such that $u_j \neq t_i$. Each such term is $|h_{ij}|^2$ times a difference quotient which, by the hypothesis (1), lies between $m$ and $M$. But terms with $t_i = u_j$ have also $h_{ij} = 0$ by (4), so (5) is between $m \sum |h_{ij}|^2$ and $M \sum |h_{ij}|^2$. Since $H$ is hermitian, $\mathrm{tr}\, H^2 = \sum |h_{ij}|^2 = \|H\|_2^2$.

This proves the theorem.

**3.** In particular, suppose $f$ is the function $f(t) = -t^{-1}$ for $t \in [0, b]$. If $A$ and $A + H$ are both positive-definite then the theorem applies. Let us discuss only the first inequality. For $t_i$ and $u_j$ as above, $t_i \in ]0, \|A\|]$ and $u_j \in ]0, \|A + H\|]$. Hence $f^{[1]}(t_i, u_j) = (t_i u_j)^{-1} \geqq \|A\|^{-1} \,.\, \|A + H\|^{-1}$. This gives

$$\mathrm{tr}\,\{H(f(A + H) - f(A))\} \geqq \|A\|^{-1}\, \|A + H\|^{-1}\, \|H\|_2^2 \,,$$

which is Fiedler's result in different notation, except that it does not include conditions for the equality to hold. Thus, with this reservation, Fiedler's theorem is a special case of Theorem 1. By slightly modifying the proof, the following theorem is obtained, which seems to be the most natural generalization of Fiedler's Corollary 2.

**Theorem 2.** *Let $A$, $H$ be hermitian matrices, and $[a, b]$ a real interval containing the spectra of $A$ and $A + H$. Let $f$ be a strictly monotone increasing real function on $[a, b]$. Then*

$$(6) \qquad \mathrm{tr}\,\{H(f(A + H) - f(A))\} \geqq 0 \,,$$

*with equality only if $H = 0$.*

Again, Fiedler's case is $f(t) = -t^{-1}$ and $a = 0$.

To prove (6), one again uses (5). Each term in the last sum in (5) is $\geqq 0$, so (6) is immediate. For equality to hold in (6) – that is, in (5) – $h_{ij}$ must be 0 for all the terms with $u_j \neq t_i$. But if this is assumed we conclude that $H = \sum h_{ij} x_i y_j^*$ must be 0, for we know by (4) that $h_{ij}$ is zero for the other terms, those with $u_j = t_i$. The proof is complete.

Thus the function need not have divided differences bounded strictly above zero, and it need not be matrix-monotone. The latter circumstance seemed less surprising to me when I reflected that if $H \geqq 0$ and $f$ is monotone (not necessarily matrix-monotone) then $\operatorname{tr} f(A + H) \geqq \operatorname{tr} f(A)$. This more-or-less familiar theorem is an immediate consequence of Weyl's theorem on monotonicity of eigenvalues.

Note that conditions for equality in Theorem 1 can also be supplied easily.

As noted in the introduction, there is a Lipchitz condition of a sort which results form Theorem 1.

**Corollary.** *Let $A$, $H$, $a$, $b$ be as in Theorem 1. Let $f$ be a real-valued function on $[a, b]$ satisfying the Lipchitz condition $|f(t) - f(u)| \leqq M \cdot |t - u|$ there. Then*

$$|\operatorname{tr} \{H(f(A + H) - f(A))\}| \leqq M \operatorname{tr} H^2 .$$

Proof. Take $m = -M$ in Theorem 1.

**4.** Here is a more general version of the theorem; the restrictions on $A$, $H$ and on $f$ have both been relaxed, but the statement of the theorem has become more clumsy. $A$ and $H$ are no longer required to be hermitian, or even diagonable. I use the notation $\sigma(A)$ for the spectrum of any $A$.

**Theorem 3.** *Let $A$, $H$ be n-by-n complex matrices, $H \neq 0$. Let $f$ be a complex-valued function such that $f(A)$ and $f(A + H)$ are defined. Assume, for a suitable closed convex subset $\mathscr{K}$ of the complex plane, that $f^{[1]}(t, u) \in \mathscr{K}$ for all $t \in \sigma(A)$ and $u \in \sigma(A + H)$, $t \neq u$. Then*

$$\|H\|_2^{-2} \operatorname{tr} \{H^*(f(A + H) - f(A))\} \in \mathscr{K} . \tag{7}$$

First let me deal with the case where both $A$ and $A + H$ are diagonable, that is, are similar to normal matrices; for in that case all goes as in Theorem 1.

In place of the spectral decomposition (3) we now have this weaker statement: There exist bases $\{x_i\}$, $\{x'_i\}$, $\{y_i\}$, $\{y'_i\}$ and numbers $\{t_i\}$, $\{u_i\}$ $(i = 1, \ldots, n)$ such that

$$x_i'^* x_j = \delta_{ij} , \quad y_i'^* y_j = \delta_{ij} , \tag{8}$$

$$A = \sum t_i x_i x_i'^* , \quad A + H = \sum u_i y_i y_i'^* ; \tag{9}$$

by definition $f(A) = \sum f(t_i)\, x_i x_i'^*$, etc.

Every closed convex set $\mathscr{K}$ of complex numbers is characterized by a real function $h$ in the following way: a complex number $\zeta$ is in $\mathscr{K}$ if and only if, for all $\Theta$, $\operatorname{Re}(e^{-i\Theta}\zeta) \geqq \geqq h(\Theta)$. Thus the hypothesis involving $\mathscr{K}$ in the present theorem may be expressed

$\operatorname{Re}(e^{-i\Theta}f^{[1]}(t, u)) \geqq h(\Theta)$. The argument involving (5) is essentially unaltered: if $h_{ij} = x_i'^* H y_j$, then $\overline{h_{ij}} = y_j'^* H^* x_i$, and so

$$e^{-i\Theta} \operatorname{tr}\{H^*(f(A+H) - f(A))\} = e^{-i\Theta} \sum y_j'^* H^* x_i x_i'^* (f(A+H) - f(A))\, y_j = \\ = \sum |h_{ij}|^2\, e^{-i\Theta} f^{[1]}(t_i, u_j)\ ;$$

dividing by $\sum |h_{ij}|^2 = \|H\|_2^2$ and taking real parts, and using the same argument as above for the terms with $t_i = u_j$, shows that the number $\zeta$ in (7) satisfies $\operatorname{Re}(e^{-i\Theta}\zeta) \geqq \geqq h(\Theta)$, which was to be proved.

Now let $A$ and $A + H$ be allowed to be non-diagonable. To use the customary definitions of $f(A)$ [4,3] we must assume that, for each $t \in \sigma(A)$, a value has been assigned not only to $f(t)$, but also to $f'(t), \ldots, f^{(k-1)}(t)$, where $k$ is the degree of $(\lambda - t)$ in the minimal polynomial $m(\lambda)$ of $A$. Similarly for each $u \in \sigma(A + H)$. If $f(s)$ was given values for any other points $s$ of the complex plane, they would not affect hypotheses or conclusion of Theorem 3. We can suit our convenience, accordingly, by supposing $f$ is a polynomial having the assigned values (with its derivatives up to the orders which enter) at the points of the spectra of $A$ and $A + H$. Also, if there is a point $s$, common to the spectra of $A$ and $A + H$, at which $f'(s)$ is not yet assigned, we can require our interpolating polynomial to satisfy $f'(s) \in \mathscr{K}$. The reason we want to do this is so that we can assert $f^{[1]}(t, u) \in \mathscr{K}$ for all cases when $t \in \sigma(A)$ and $u \in \sigma(A + H)$; for the polynomial $f^{[1]}$ is extended to equal arguments by $f^{[1]}(s, s) = f'(s)$.

We can now assert that $f(B)$ has been defined as a continuous function of $B$, using the usual topology for the space of matrices.

With these understandings I proceed to extend Theorem 3 by continuity.

For any $\varepsilon > 0$ let $\mathscr{K}_\varepsilon$ denote the set of all complex $\zeta$ at distance $\varepsilon$ or less from $\mathscr{K}$; it is a closed convex set. Because $f^{[1]}$ is now continuous and everywhere defined, and because $f^{[1]}(t, u) \in \mathscr{K}$ for $t \in \sigma(A)$ and $u \in \sigma(A + H)$, there is a neighborhood of $A$, say $\mathscr{U}_\varepsilon$, such that, for $B \in \mathscr{U}_\varepsilon$, we have $f^{[1]}(t, u) \in \mathscr{K}_\varepsilon$ for $t \in \sigma(B)$ and $u \in \sigma(B + H)$. That is, all $B \in \mathscr{U}_\varepsilon$ satisfy the hypotheses of the theorem for $\mathscr{K}_\varepsilon$.

Now $\mathscr{U}_\varepsilon$ is a manifold. The subset of matrices with all $n$ eigenvalues simple, is an open dense set. Hence the set of non-diagonable $B$ in $\mathscr{U}_\varepsilon$ is nowhere dense; likewise the set of $B$ with $B + H$ non-diagonable is nowhere dense; hence so is their union. But for $B$ and $B + H$ diagonable, Theorem 3 is already established; it gives the conclusion that the number

$$\|H\|_2^{-2} \operatorname{tr}\{H^*(f(B+H) - f(B))\}$$

is in $\mathscr{K}_\varepsilon$ for a set of $B$ dense in $\mathscr{U}_\varepsilon$. But then it is in $\mathscr{K}_\varepsilon$ for all $B \in \mathscr{U}_\varepsilon$. In particular for $B = A$, it is in $\bigcap \mathscr{K}_\varepsilon = \mathscr{K}$, which was to be proved.

It would be interesting to find a more "elementary" proof – perhaps to avoid continuity arguments altogether.

**Corollary.** *Theorem* 3 *remains true if the word "closed" is omitted from its statement.*

Proof. Every convex set is the union of an increasing sequence of closed convex sets; the rest is easy.

Added in proof: G. MINTY has called my attention to his definition of numerical range of non-linear functions on vector spaces. The result of the present paper may be regarded as a theorem about such numerical ranges.

If $\Phi$ is a non-linear operator in a Hilbert space with vectors $X. Y, \ldots$, then Minty defines its numerical range as the set of all complex numbers

$$X^*(\Phi(Y + X) - \Phi(Y))/X^*X$$

for all $X, Y\,(X \neq 0)$, Let in particular the Hilbert space be that of all $n$-by-$n$ matrices, under the norm $\| \;\|_2$; and let $\Phi$ be the non-linear operator obtained by extending a numerical function $f$ to matrix arguments. Then Theorem 3 and Corollary above say that the numegical range of $\Phi$ is contained in the convex hull of the range of $f$[1]. To be exact, they say more, for they allow for the case where $f$ is not defined on the whole complex plane and $\Phi$ has a correspondingly restricted domain.

*References*

[1] *M. Fiedler:* Poznámka o positivně definitních maticích. Časopis Pěst. Mat. *85* (1960), 75—77.

[2] *K. Löwner:* Über monotone Matrixfunktionen. Math. Z. *38* (1934), 177—216.

[3] *R. F. Rinehart:* The equivalence of definitions of a matric function. Amer. Math. Monthly *62* (1955), 395—414.

[4] *H. Schwerdtfeger:* Les fonctions de matrices. I. Les fonctions univalentes. Actualités Sci. Industr. no. 649, Paris, Hermann, 1938.

## Резюме

## ОДНО НЕРАВЕНСТВО ДЛЯ СЛЕДОВ ФУНКЦИЙ МАТРИЦ

ЧАНДЛЕР ДЭЙВИС (Chandler Davis), Торонто, Канада

Главным результатом работы является следующая теорема:

*Если $A$ и $H$ — симметричные матрицы, а $f$ — действительная функция, определенная на некотором открытом интервале, содержащем спектры матриц $A$, $A + H$ и такая, что на этом интервале имеют место неравенства*

$$m \leqq \frac{f(t) - f(u)}{t - u} \leqq M \quad (t \neq u),$$

*то справедливо соотношение*

$$m \operatorname{tr} H^2 \leqq \operatorname{tr} \{H(f(A + H) - f(A))\} \leqq M \operatorname{tr} H^2 .$$

Приводятся некоторые следствия этой теоремы, а также некоторые результаты более общего характера.