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TOPOLOGIES ON PRODUCTS AND DECOMPOSITIONS OF TOPOLOGICAL SPACES[1])

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Some results concerning quotient-spaces are given. The connection between quotient-spaces and topologies, in which the closure of a set is not closed, is used. In particular, we investigate how complicated is the decomposition of a space, the quotient-space of which is a topological product of spaces.

I

It is well known that from various types of convergence we get in a natural way topologies, where the closure of a set need not to be a closed set. Such topologies, however, also appear in a natural way in connection with decompositions of topological spaces. Some of the properties of such topologies are in fact properties of quotient-spaces. For example, the notion of F-order (introduced in the present article) of a space P indicates, roughly said, how complicated may be a decomposition of a space with given quotient-space P.

A topological space is the couple (P, u), where P is a set and u is a mapping of the set $\exp P$ of all subsets of P into $\exp P$ such that $u\emptyset = \emptyset$, and $u(M_1 \cup M_2) = uM_1 \cup uM_2$ and $M_1 \subset uM_1$ for $M_1, M_2 \subset P$. We call u a topology on the set P and uM the closure of the set M. The requirement that $u(uM) = uM$ for every $M \subset P$, is called the axiom F in [1]; spaces satisfying this axiom are called F-spaces, their topologies are called F-topologies.

For topological spaces defined in such a general way we may introduce the usual notions without any change in their definition. In [1] the theory of such spaces was examined systematically. In [1] the T_1 axiom is assumed and for all definitions and theorems we shall use this axiom is not necessary as it is stated in [1] in examples to section 4. The knowledge of [1] is not necessary for reading of this paper as all needed definitions and results will be stated in full. Here we give only the definitions

[1]) Some results of the present article were presented at the Symposium on General topology and its relations to modern analysis and algebra in September 1961 in Prague. Some theorems without proof are contained in [2]. The present article was announced in [2] as reference [17] with title "Non-F-spaces".

and trivial facts which are needed for our considerations. Let (P, u) be a topological space. A set $M \subset P$ is called closed if $uM = M$; it is called open, if $P - M$ is closed. The set $\operatorname{Int} M = P - u(P - M)$ is called the interior of M (it is possible, of course, that $\operatorname{Int}(\operatorname{Int} M) \neq \operatorname{Int} M$). A set U is called a neighbourhood of a set M (or of a point x respectively) if $M \subset \operatorname{Int} U$ (or $x \in \operatorname{Int} U$ respectively). Clearly, requiring the axiom F or not, a point belongs to a closure of a set if and only if every neighbourhood has a non-void intersection with this set. If U is a neighbourhood of x, then $\operatorname{Int} U$ need not be a neighbourhood of x. Evidently, a set is open if and only if it is a neighbourhood of each of its points, in other words, if it is identical with its interior. A collection $\mathscr{U}$ of neighbourhoods of a point x is called complete, if for every neighbourhood V of x there exists an $U \in \mathscr{U}$ such that $U \subset V$. (The system of all open neighbourhoods of x need not be complete.) The most simple topology not satysfying axiom F, may be set up on a three-point set. Let $P = \{1, 2, 3\}$ and put $u\{1\} = \{1, 2\}$, $u\{2\} = \{2, 3\}$, $u\{3\} = \{3, 1\}$ and for $M \subset P$ let $uM = \bigcup_{x \in M} u\{x\}$. Evidently the only open sets are P and $\emptyset$. The set $\{1, 2\}$ is a neighbourhood of the point 2, the set $\{2, 3\}$ is a neighbourbood of the point 3 and so on.

Even if the axiom F is not requived, the topology can be defined by complete collections of neighbourhoods. In fact the following theorem holds (cf. e.g. [1], p. 62):

If for every element x of a non-void set P there is given a non-void collection $\mathscr{U}_x$ of subsets of P such that

1) $U \in \mathscr{U}_x \Rightarrow x \in U$,

2) if $U_1, U_2 \in \mathscr{U}_x$, then there exists an $U_3 \in \mathscr{U}_x$ with $U_3 \subset U_1 \cap U_2$, then there exists exactly one topology on P, for which $\mathscr{U}_x$ is a complete collection of neighbourhoods of x for every $x \in P$. In any case we can consider the T_1-axiom and the Hausdorff axiom for a space (P, u). A space in which each two different points have disjoint neighbourboods (not necessarilly open) is called a Hausdorff space. A space in which every finite set is closed is called a T_1-space; a space which is both T_1-space and an F-space is called an T_1F-space.

In the following there will frequently occur the situation in which more than one topology on a set P is considered simultaneously. We then use the notation u-open for sets open in (P, u) and similarly u-closed set, u-neighbourhood, complete collection of u-neighbourhoods and so on. If u and v are topologies on a set P, $x \in P$ and every v-neighbourhood of x is its u-neighbourhood, then we write $u_x \leqq v_x$. If $u_x \leqq v_x$ and $v_x \leqq u_x$, then we write $u_x = v_x$. If $u_x = v_x$ is not true, we write $u_x \neq v_x$. If $u_x \leqq v_x$, $u_x \neq v_x$, then we write $u_x < v_x$. If $u_x \leqq v_x$ for all $x \in P$, then we write $u \leqq v$, which is the usual notation of the fact that u is finer than v. If $u \leqq v$, $u \neq v$, we write $u < v$. If (P, u) is a space, $Q \subset P$, then u/Q denotes the topology on Q for which $(u/Q)\, M = Q \cap uM$ for every $M \subset Q$. Then $(Q, u/Q)$ is called a subspace of (P, u). If f is a mapping of a set P into a set R, $Q \subset P$, then $f \mid Q$ denotes the mapping of Q into R, for which $(f \mid Q)(x) = f(x)$ for every $x \in Q$.

II

Let (P, u) be a topological space. For $M \subset P$ we define $u^1M = uM$, $u^\alpha M = (\bigcup_{\beta<\alpha} u^\beta M)$ (α is an ordinal). Denote by $\varphi(M)$ the smallest ordinal α for which $u^\alpha M = u^{\alpha+1}M$. Let $\varphi = \sup_{M\subset P} \varphi(M)$. It is known that u^φ is an F-topology on P. In [1] this F-topology is called the F-modification of u, the ordinal number φ is called the order of u. Evidently $uM \subset u^\varphi M$ for $M \subset P$; hence if $u^\varphi M = M$, then $uM = M$. Conversely, if $uM = M$, then $u^\alpha M = M$ for every ordinal number α. Consequently, every set M is u-closed (or u-open respectively) if and only if it is u^φ-closed (or u^φ-open respectively). Consequently the collection of all u^φ-closed sets is identical with the collection of all u-closed sets. Hence the F-topology u^φ can also be defined by means of the collection of all u-closed sets. Clearly, u^φ is the finest of all F-topologies coarser than u; the equality $u = u^\varphi$ holds if and only if u is an F-topology.

Let f be a mapping of an F-space (Q, t) onto an F-space (P, v). We shall say that the mapping f is quotient map if and only if every set $M \subset P$ is v-closed if and only if the set $f^{-1}(M)$ is t-closed. In [1] the definition of an exactly continuous mapping is given: A mapping f of a space (Q, t) onto a space (P, v) is exactly continuous if and only if for every set $M \subset P$, $ftf^{-1}(M) = vM$. Evidently, every exactly continuous mapping of an F-space onto an F-space is a quotient map. The converse is not true For example, let (Q, t) be the space of all real numbers with its usual topology, let P be the set of all integers; we consider the following topology v on P: for $x \in P$ let $v\{x\}$ be the set of all integers not less than x; let $vM = \bigcup_{x\in M} v\{x\}$ for $M \subset P$. Let f be a mapping of (Q, t) onto (P, v), $f(x) = [x]$ ($[x]$ is the greatest integer not greater than x). Evidently f is a quotient map, but it is not exactly continuous. Thus $v\{0\}$ is the set of all non-negative integers, $f^{-1}(\{0\})$ is the half-open interval $\langle 0, 1)$, hence $ftf^{-1}(\{0\})$ is the two-point set $\{0, 1\}$, which is, of course, different from $v\{0\}$.

Let $\mathscr{P}$ be a decomposition of some set Q; let π be the mapping of Q onto $\mathscr{P}$, which to every $x \in Q$ assings the set $\mathbf{x} \in \mathscr{P}$ containing x; π is usually called the projection. A subset $P \subset Q$ which contains exactly one point from every $\mathbf{x} \in \mathscr{P}$, is usually called a system of representants of the decomposition $\mathscr{P}$. If (Q, t) is an F-space, $\mathscr{P}$ is a decomposition of the set Q, then it is usual to define the F-topology v on $\mathscr{P}$ in such a manner that the projection π of Q onto $\mathscr{P}$ is a quotient map of (Q, t) onto $(\mathscr{P}, v)$. The space $(\mathscr{P}, v)$ is then called the quotient-space. Now we define another topology u on $\mathscr{P}$ (this definition was formulated by prof. M. Katětov in a conversation): For $M \subset \mathscr{P}$ we put $uM = \pi t \pi^{-1}(M)$. This topology we call the fine quotient-topology and $(\mathscr{P}, u)$ the fine quotient-space. Evidently, $u \leqq v$ and in general u is not an F-topology. Clearly, the projection π is an exactly continuous mapping of the F-space (Q, t) onto $(\mathscr{P}, u)$. Evidently, a set $M \subset \mathscr{P}$ is v-closed if and only if it is u-closed; hence the quotient-topology is the F-modification of the fine quotient-topology.

Some theorems about fine quotient spaces, analogical to theorems about quotient-spaces, are satisfied. For example: The fine quotient-space is a T_1-space if and only if the decomposition is closed.

Now we shall show that every topology is a fine quotient-topology, in other words, it may be obtained by a decomposition of an F-space.

Theorem 1. *Let (P, u) be a topological space. Then there exists an F-space and its decomposition such that the fine quotient-space is homeomorphic with (P, u). In particular, it is possible to choose an F-space (Q, t) and its decomposition $\mathscr{P}$ such that P is a closed subset of (Q, t), P is a system of representants of the decomposition $\mathscr{P}$, $t/(Q - P)$ is discrete, and if π is the projection of Q onto $\mathscr{P}$, then $\pi \mid P$ is a homeomorphic mapping of (P, u) onto the fine quotient-space. Moreover, if (P, u) is a T_1-space (or a Hausdorff space) then (Q, t) is also a T_1-space (or a Hausdorff space, respectively).*

Proof. Let a space (P, u) be given. Let φ be a one-to-one mapping of the set P onto some set P', for which $P \cap P' = \emptyset$. Put $Q = P \cup P'$. Let $\mathscr{P}$ be the system of all sets $\{x\} \cup \varphi(\{x\})$, where x runs over P; let π be the projection of Q onto $\mathscr{P}$. Now we define an F-topology t on Q: for $M \subset Q$ define $M^+ = \varphi^{-1}(M \cap P')$ and put $tM = M \cup \operatorname{der} M^+$ (where $\operatorname{der} M^+$ denotes the set of all accumulations points of M^+). Evidently, for $x \in P'$, the set $\{x\}$ is a t-neighbourhood of x; for $x \in P$ the system $(\{x\} \cup \varphi(U - \{x\})\}$, where U runs over all u-neighbourhoods of x, is a complete collection of t-neighbourhoods of x. Evidently, (Q, t) is an F-space, P is a closed subst of (Q, t), $t/(Q - P)$ is discrete and if (P, u) is a T_1-space (or a Hausdorff space), then (Q, t) is also a T_1-space (or a Hausdorff space, respectively). Let v be the fine quotient-topology on $\mathscr{P}$. By definition of the fine quotient-topology, for $\mathscr{A} \subset \mathscr{P}$ there is $v\mathscr{A} = \pi t \pi^{-1} \mathscr{A}$. Denote $\psi = \pi \mid P$. We are to prove that $v\mathscr{A} = \psi u \psi^{-1} \mathscr{A}$. Denote $\psi^{-1}\mathscr{A} = A$. Evidently, $\pi^{-1}\mathscr{A} = A \cup \varphi(A)$, $(\pi^{-1}\mathscr{A})^+ = A$, $t(A \cup \varphi(A)) = A \cup \varphi(A) \cup (\operatorname{der} A) = uA \cup \varphi(A)$ and $\pi(uA \cup \varphi(A)) = \psi uA$.

Let (P, u) be a space. In [1] the following topology $\tilde{u}$ is defined: for $A \subset P$ let $\tilde{u}A = \bigcap_{X \subset P, uX \supset A} uX$.

Now we show that $\tilde{u}$ satisfies all the axioms for an F-topology: evidently $\tilde{u}\emptyset = \emptyset$, $\tilde{u}A \supset A$; clearly $\tilde{u}\tilde{u}A = \bigcap_{uX \supset \tilde{u}A} uX = \bigcap_{uX \supset A} uX = \tilde{u}A$; for $A, B \subset P$ clearly

$$\tilde{u}A \cup \tilde{u}B = \bigcap_{uX \supset A} uX \cup \bigcap_{uY \supset B} uY = \bigcap_{uX \supset A,\, uY \supset B} (uX \cup uY) = \bigcap_{uZ \supset A \cup B} uZ = \tilde{u}(A \cup B)\,.$$

In [1] this F-topology $\tilde{u}$ is called the F-reduction of u. Evidently $\tilde{u} \leqq u$. For $A \subset P$ obviously $\tilde{u}uA = uA$, consequently every set uA is $\tilde{u}$-closed; evidently the system $\{uX; X \subset P\}$ (or $\{P - uX; X \subset P\}$ respectively) is a base for $\tilde{u}$-closed sets (or $\tilde{u}$-open sets respectively). Hence if U runs over all u-neighbourhood of x, then $\operatorname{Int} U$ form a complete collection of $\tilde{u}$-neighbourhoods. Clearly $\tilde{u} = u$ if and only if u is an F-

topology. Evidently, if (P, u) is a T_1-space (or a Hausdorff space respectively), then $(P, \tilde{u})$ also such. Evidently, if (P, u) is the three-point space described in part I, then every subset of P is $\tilde{u}$-closed.

Lemma 1. *Let (P, u) be a space, let P be a system of representants of a decomposition $\mathcal{P}$ of a set Q; let π be the projection of Q onto $\mathcal{P}$; for $M \subset Q$ denote $\mathcal{R}M = \pi^{-1}\pi M$. Let t be an F-topology on Q such that P is a closed subset of (Q, t) and $t/(Q - P)$ is discrete. Then $\pi \mid P$ is a homeomorphic mapping of (P, u) onto the fine quotient space if and only if $uX = P \cap t\mathcal{R}X$ for every $X \subset P$.*

Proof. Let t be an F-topology on Q with described properties; let v be the fine quotient-topology on $\mathcal{P}$. Set $\psi = \pi \mid P$. Evidently, for $M \subset P$ there is $\psi^{-1}\pi M = \psi^{-1}\pi\mathcal{R}M = M$, $\pi^{-1}\psi M = \mathcal{R}M$. Clearly ψ is a homeomorphic mapping of (P, u) onto $(\mathcal{P}, v)$ if and only if $\psi uX = v\psi X$ for every $X \subset P$; by the definition of the fine quotient-topology, $v\psi X = \pi t\pi^{-1}(\psi X) = \pi t\mathcal{R}X$. But t is discreteon $Q - P$ and P is t-closed, and therefore $t\mathcal{R}X = \mathcal{R}X \cup (P \cap t\mathcal{R}X)$. Consequently $\psi^{-1}v\psi X = \psi^{-1}\pi t\mathcal{R}X = \psi^{-1}\pi[\mathcal{R}X \cup P) \cap t\mathcal{R}X)] = X \cup (P \cap t\mathcal{R}X) = P \cap t\mathcal{R}X$. Consequently $uX = \psi^{-1}v\psi X$ if and only if $uX = P \cap t\mathcal{R}X$.

Theorem 2. *Let (P, u) be a space, P a system of representants of a decomposition $\mathcal{P}$ of a set Q; let π be the projection of Q onto $\mathcal{P}$. Let $\mathbf{P}$ be the system of all those F-topologies on P which can be extended to the whole set Q in such a manner that $Q - P$ is a diserete subspace, P is closed in this topology and $\pi \mid P$ is a homeomorphic mapping of (P, u) onto the fine quotient-space.*

Then every element of $\mathbf{P}$ is finer than the F-reduction $\tilde{u}$ of u. If v and w are F-topologies on P, $v \in \mathbf{P}$, $v \leqq w \leqq \tilde{u}$, then $w \in \mathbf{P}$. Moreover, if every element of the decomposition $\mathcal{P}$ contains at least two points of Q, then every F-topology on P finer than $\tilde{u}$ belongs to $\mathbf{P}$.

Proof. Let (P, u), $\tilde{u}$, $\mathcal{P}$, Q, π, $\mathbf{P}$, have the same meaning as in theorem 2, denote $\mathcal{R}M = \pi^{-1}\pi M$ for $M \subset Q$. Let t be an F-topology on Q such that P is t-closed, $t/(Q - P)$ is discrete and $\pi \mid P$ is a homeomorphic mapping of (P, u) onto the fine quotient space. Denote $v = t/P$. We must prove that $v \leqq \tilde{u}$. The system $\{uX; X \subset P\}$ is a base for $\tilde{u}$-closed sets and therefore it is sufficient to prove that every set uX is v-closed. Evidently the set $t\mathcal{R}X$ is t-closed and therefore $uX = P \cap t\mathcal{R}X$ is v-closed. Conversely, let $v \in \mathbf{P}$ and let w be an F-topology on P such that $v \leqq w \leqq \tilde{u}$. Let t_0 be an F-topology on Q such that $t_0/P = v$, $t_0/(Q - P)$ is discrete, P is t_0-closed and $uX = P \cap t_0\mathcal{R}X$ for $X \subset P$. We define a topology t on Q in the following way: for $M \subset Q$ we put $tM = w(M \cap P) \cup t_0(M - P)$. Evidently, $tP = P$, $t/P = w$, $t/(Q - P) = t_0/(Q - P)$ is discrete and for $X \subset P$, $uX = P \cap t_0\mathcal{R}X = P \cap t_0[(\mathcal{R}X \cap P) \cup (\mathcal{R}X - P)] = vX \cup [P \cap t_0(\mathcal{R}X - P)] \subset wX \cup [P \cap t_0(\mathcal{R}X - P)] = P \cap t\mathcal{R}X \subset \tilde{u}X \cup [P \cap t_0(\mathcal{R}X - P)] \subset uX$. Hence $uX = P \cap t\mathcal{R}X$ and consequently, using lemma 1, $\pi \mid P$ is a homeomorphic mapping of (P, u) onto the fine quotient-space of the space (Q, t). Consequently $t/P = w \in \mathbf{P}$.

Moreover, let every element of the decomposition $\mathscr{P}$ contain at least two points of Q. For $M \subset Q$ put $M' = P \cap \mathscr{R}(M - P)$. Evidently for $X \subset P, X' = \emptyset, (\mathscr{R}X)' = X$. If we put, for $M \subset Q, tM = M \cup \operatorname{der} M'$, then $t/P \in \mathbf{P}$ (the proof is analogical to that of Theorem 1). But t/P is discrete, hence every F-topology on P finer than $\tilde{u}$ belongs to $\mathbf{P}$.

III

Definition 1. Let (P, v) be an F-space. We call (P, v) a strong F-space (and v a strong F-topology) if the following condition holds: if v is the F-modification of some topology u, then $u = v$.

Note 1. As noted previously, the quotient-topology is the F-modification of the fine quotient-topology. Hence if the quotient-topology is a strong F-topology, then it is necessarily identical with the fine quotient-topology. Consequently, the following proposition holds (the same notation is used as in lemma 1): Let (Q, t) be an F-space, let $\mathscr{P}$ be a decomposition of (Q, t) such that the quotient-space is a strong F-space. Then for every $M \subset Q$ the set $\mathscr{R}t\mathscr{R}M$ is closed. In this section a necessary and sufficient condition for a space to be a strong F-space is given. From it there follows immediately that, for example, every metric space is a strong F-space, but the cube with uncountably many dimensions is not a strong F-space.

In [1] is given[2]) the following definition 2.

Definition 2. A point x of a space (P, u) is called an F-point if each of its neighbourhoods contains some open neighbourhood of x. Clearly, a space is an F-space if and only if each point is an F-point (cf. [1], p. 66).

Definition 3. A point x of an F-space (P, v) is called a strong F-point if for every $A \subset P$ with $x \in vA - A$, there exists a set B such that $B \subset A$, $x \in vB$, $x \notin v(vB - A - \{x\})$.

Theorem 3. *A point x of a regular F-space (P, v) is a strong F-point if and only if for every $A \subset P$ with $x \in vA - A$ there exists a set B such that $B \subset A$, $vB - B = \{x\}$.*

Proof. Evidently, if $vB - B = \{x\}$ for some $x \in P$ and some $B \subset A \subset P$, then $x \in vB$, $x \notin v(vB - A - \{x\})$. Consequently, every point which satisfies the condition from the theorem 3 is a strong F-point. Let x be a strong F-point of a regular F-space (P, v). Let $A \subset P$, $x \in vA - A$. Then there exists a set C such that $C \subset A$, $x \in vC$, $x \notin v(vC - A - \{x\})$. Let U be a closed neighbourhood of x, disjoint with $vC - A - \{x\}$. Denote $A^* = A \cup \{x\}$. Evidently, $U \cap vC = U \cap [(vC \cap A^*) \cup (vC - A^*)] = U \cap (vC \cap A^*) \subset A^*$. Consequently the set $B = U \cap vC - \{x\}$ satisfies $B \subset A$, $vB \subset U \cap vC = B \cup \{x\}$ and $x \in vB$.

[2]) In [1] an F-point is called a strong F-point. The notion of strong F-points from definition 3 does not occur in [1].

Lemma 2. *Let x be a strong F-point of an F-space (P, v). If v is an F-modification of some topology u on P, then x is an F-point of the space (P, u).*

Proof. Let u be a topology on a set P, let v be an F-modification of u. Let x be a strong F-point of (P, v) but not an F-point of (P, u). Then there exists an u-neighbourhood U which contains no u-open (and consequently no v-open) neighbourhood of x. Consequently U is not v-neighbourhood of x; hence $x \in vA - A$ for $A = P - U$. Let B be a set such that $B \subset A$, $x \in vB$, $x \notin v(vB - A - \{x\})$. Let V be a v-open neighbourbood of x, disjoint with $vB - A - \{x\}$. Put $G = V - vB$. Evidently G is v-open. It is easy to see that $V \cap vB = V \cap [vB \cap (A \cup \{x\})]$. By virtue of $U = P - A$ there is $U \cap V \cap vB = \{x\}$. Consequently the u-neighbourhood $U \cap V$ of x is contained in $G^* = G \cup \{x\}$. Thus the set G^* is a u-neighbourhood of each of its points and therefore it is u-open. Hence G^* is also v-open. But this is imposible, because $x \notin B$, $G^* \cap vB = \{x\}$.

Lemma 3. *Assume $x \in P$ is not a strong F-point of an F-space (P, v). Then there exists a topology u on P such that v is the F-modification of u and x is not an F-point of the space (P, u).*

Proof. If $x \in P$ is not a strong F-point of an F-space (P, v), there exists a set $A \subset P$ such that $x \in vA - A$ and that for every $B \subset A$ with $x \in vB$, there is $x \in v(vB - A - \{x\})$. Now we define a topology u on P with the required properties: for $y \in P$, $y \neq x$ let $u_y = v_y$; the system of all sets $V - A$, where V runs ever all v-neighbourhoods of x, is a complete collection of u-neighbourhood of x. Now we prove that $u^2M = vM$ for every $M \subset P$ (hence v is the F-modification of u). If $M \subset P$, $M \cap A = \emptyset$, then evidently $uM = vM$. If $M \subset A$, then $uM = vM - \{x\}$, $u^2M \subset vM$; but if $x \in vM$, then $x \in v(vM - A - \{x\})$ and therefore $u^2M = vM$. Consequently, $u^2M = u^2(M - A) \cup u^2(M \cap A) = v(M - A) \cup v(M \cap A) = vM$ for every $M \subset P$. Evidently, x is not an F-point of (P, u) because $x \in u^2A - uA$ and its u-neighbourhood $P - A$ contains no u-open neighbourhood of x.

Theorem 4. *An F-space (P, v) is a strong F-space if and only if each of its point is a strong F-point.*

The proof follows from lemmas 2 and 3.

Theorem 5. *Every subspace of a strong F-space is a strong F-space.*

Proof. If (Q, w) is a subspace of an F-space (P, v) and if for some $x \in Q$ and some $A \subset Q$ there holds: $x \in wA - A$ and for every $B \subset A$ with $x \in wB$ there is $x \in w(wB - A - \{x\})$, then evidently $x \in vA - A$ and for every $B \subset A$ with $x \in vB$, there is $x \in v(vB - A - (x))$.

Note 2. Every metric space, and more generally every $\mathscr{L}F$-space (cf. definition in [1]), is a strong F-space; this follows immediatelly from theorems 3 and 4. Every topological product of uncountably many spaces, each of which contains at least two points, is not a strong F-space. It is sufficient to show this for the topological product of uncountably many two-point spaces only; we shall use the theorem 5. Let P be the set of all characteristic functions on some uncountable set Λ, let v be the product-topology on P. Denote by 0 the function zero. Let A be the set of all functions $\chi \in P$ such that $\chi(x) = 0$ only for a finite number of $x \in \Lambda$. Evidently, $0 \in vA - A$. Let $B \subset A$, $0 \in vB$; we must prove that $0 \in v(vB - A - \{0\})$. Let $l \subset \Lambda$ be finite; it is sufficient to find $\varphi \in vB - A - \{0\}$ such that $\varphi(x) = 0$ for $x \in l$. By virtue of $0 \in vB$, there exists a $\varphi_1 \in B$ such that $\varphi_1(x) = 0$ for $x \in l$. Let $l_1 = \{x \in \Lambda; \varphi_1(x) = 0\}$; l_1 is finite and therefore we can choose a finite $l_1^* \subset \Lambda$ such that $l_1 \subsetneqq l_1^*$. Then we can find $\varphi_2 \in B$ such that $\varphi_2(x) = 0$ for $x \in l_1^*$. Let $l_2 = \{x \in \Lambda; \varphi_2(x) = 0\}$. Choose a finite set $l_2^* \subset \Lambda$ such that $l_2 \subsetneqq l_2^*$ and find $\varphi_3 \in B$ with $\varphi_3(x) = 0$ for $x \in l_2^*$ and so on. If we denote $L = \bigcup_{n=1}^{\infty} l_n$ and if φ is the characteristic function of $\Lambda - L$, then, evidently, φ satisfies our requirements.

IV

In this section the following question is solved: If (P, v) is an F-space, $\emptyset \neq D \subset P$, and no $x \in D$ is a strong F-point of (P, v), does there exists a topology on P, the F-modification of which is v and in which no $x \in D$ is an F-point. Such as topology does not always exist, as shown in the following example:

Example 1. In this example a T_1 space (P, v) is construct such that there exists a set $D \subset P$, no point of which is a strong F-point of (P, v), but if u is a topology on P with F-modification v, then some $x \in D$ is an F-point of (P, u). The following trivial results are used:

a) Let (P, u) be a space, $D \subset P$ be u-closed. Let v be the F-modification of u, let v_D be the F-modification of u/D. Then $v_D = v/D$.

b) Let A be an infinite discrete T_1 space, let $(\beta A, t)$ be its Čech-Stone compactification. Let $x_0 \in \beta A - A$. If u is a topology on $A \cup \{x_0\}$ with $u < t/A \cup \{x_0\}$, then u is discrete.

Now we construct the space (P, v): Let (D, w) be a dense-in-itself strong T_1F-space, $D \neq \emptyset$. Let A be an infinite discrete space, $A \cap D = \emptyset$, let $(\beta A, t)$ be its Čech-Stone compactification. We choose a point $x_0 \in \beta A - A$. We put $P = D \cup A$ and define the following F-topology v on P: for $Q \subset P$ let $vQ = Q \cup D$ if $x_0 \in t(Q \cap A)$, and $vQ = Q \cup w(Q \cap D)$ if $x_0 \notin t(Q \cap A)$. Evidently, (P, v) is a T_1F-space. All points from A are isolated in (P, v), hence D is v-closed. If $x \in D$, then $x \in vA - A$; if $B \subset A$, $x \in vB \cap D$, then $vB = B \cup D$, $vB - A - \{x\} = D - \{x\}$, $v(vB - A - \{x\}) = w(D - \{x\}) = D$. Consequently, no point from D is a strong F-point of (P, v). Let u be a topology on P, the F-modification of which is v. We show that some $x \in D$ necessarily must be an F-point of (P, u). D is v-closed, $v/D = w$ is a strong F-topology and therefore (using a)) $u/D = v/D$. If some $x \in D$ is not na F-point of (P, u), then

$u_x < v_x$; by virtue of b) and of the equality $u/D = v/D$, we have that $x \notin uA$. Consequently, if no point from D is an F-point of (P, u), then $uA = A$ and therefore v is not the F-modification of u, because A is not v-closed.

Now we prove a sufficient condition for the existence of a topology with a given set of non F-points and with a given F-modification (theorem 6). We shall aply this condition to the topological product of uncountably many F-spaces (theorems 7 and 8).

Theorem 6. *Let D be a subset of an F-space (P, v). For every $x \in D$ let there exist $A_x \subset P$ such that*

1) $x \in vA_x - A_x$ *and if* $B \subset A_x$, $x \in vB$ *then* $x \in v(vB - A_x - \{x\})$;

2) *the system* $\{A_x;\ x \in D\}$ *is disjoint.*

Then there exists a topology u on P such that no point from D is its F-point, and that v is its F-modification. Moreover, if $D = P$ and $vA_x = P$ for all x, then $\tilde{u}$ is discrete.

Proof. We define the topology u on P with the required properties: denote by $\mathscr{V}_x$ a complete collection of v-neighbourhoods of x. If $x \notin D$, then $\mathscr{V}_x$ is also a complete collection of u-neighbourhoods of x. If $x \in D$, then $\{V - A_x;\ V \in \mathscr{V}_x\}$ is a complete collection of u-neighbourhoods of x. Evidently $x \in u^2A_x - uA_x$ for $x \in D$, hence no point from D is an F-point of u. Denote by u^* the F-modification of u. Evidently $u \leqq v$, hence $u^* \leqq v$. Suppose that for some $x \in P$ there is $u^*_x < v_x$, and obtain a contradiction. Clearly $x \in D$. Let U be an u^*-open neighbourhood of x, which is not its v-neighbourhood. Clearly U is a u-neighbourhood of all of its point. By definition of the topology u, there exists a v-open neighbourhood V of x such that $V - A_x \subset$ $\subset U$. We put $B = V - U$. Evidently $B \subset A_x$, $x \in vB$. But $x \notin v(vB - A_x - \{x\})$, because $V \cap (vB - A_x - \{x\}) = \emptyset$. Indeed, let $y \in V \cap (vB - A_x - \{x\})$. Clearly $y \in V - A_x \subset U$. If $y \notin D$, then U contains a v-neighbourhood of y, and consequently $y \notin vB$. If $y \in D$, $y \neq x$, then there exists a v-neighbourhood V_y of y such that $V_y -$ $- A_y \subset U$. Put $W_y = V \cap V_y$; clearly, $W_y = (W_y - A_x) \cup (W_y \cap A_x)$, $W_y - A_x \subset$ $\subset V - A_x \subset U$, $W_y \cap A_x \subset V_y - A_y \subset U$. Consequently $W_y \subset U$, hence $y \notin vB$. Evidently, if $D = P$ and if $vA_x = P$ for every x, then $uA_x = P - \{x\}$, consequently $\{x\}$ is $\tilde{u}$-open, hence $\tilde{u}$ is discrete.

Note 3. May be noted from the preceding proof that:

a) Condition 2 from theorem 6 may be replaced by the weaker condition: If $x \in D$, $y \in D \cap (vA_x - A_x - \{x\})$ then $y \notin v(A_x \cap A_y)$.

b) It is easy to show that the order of the topology u, constructed in the preceding proof is 2.

c) Evidently, if $\{A_x;\ x \in D\}$ and $\{B_x;\ x \in D\}$ are systems satisfying conditions 1) and 2) from theorem 6, $x \in v(A_x - B_x)$ for some $x \in D$, then the topologies constructed from these systems are different.

d) If u denotes the topology constructed on the set P in the proof of theorem 6, then for every $x \in D$ there exists a set M such that $x \in u^2M - uM$ (put $M = A_x$). This property need not be satisfied in general for all not F-points. The following trivial example is given: Let $P = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\}$. On P we consider the following topology u: for $M \subset P$ let $0 \in uM$ if either $0 \in M$ or M is infinite; $1/n \in uM$ if either $1/n \in M$ or $1/(n-1) \in M$. Evidently 0 is not an F-point, because the only u-open set containing 0 is P. But $0 \in u^2M - uM$ for no set $M \subset P$. A point x for which $x \in u^2M$ implies $x \in uM$, was called a weak F-point in [1]. In [1] p. 136 there is also given an example of a T_1-space with a non F weak F-point.

Lemma 4. *Let A be an infinite set, let $[A]$ denote the set of all its finite subsets. There exists a disjoint system $\{M_{k,l};\ k \in [A],\ l \in [A]\}$ of subsets of $[A]$ such that*

1) *if $[k, l] \in [A] \times [A]$, $q \in [A]$, $p \in [A]$ are given, $q \cap p = \emptyset$, then there exist $m \in M_{k,l}$ such that $q \subset m$, $m \cap p = \emptyset$;*

2) *if $[k, l] \neq [k', l']$ then for every $m \in M_{k,l}$, $m' \in M_{k',l'}$ the set $\Delta(m, m') - k \cup k'$ is not empty (where by $\Delta(m, m')$ is denoted the symmetric difference of m and m').*

Proof. Let A be an infinite set, let $[A]$ be the set of all its finite subsets. Let π_1, π_2 be the projections of $[A] \times [A]$ onto $[A]$, $\pi_1(k, l) = k$, $\pi_2(k, l) = l$. Let Γ be the set of all couples $[k, l]$, $k, l \in [A]$ for which $k \cap l = \emptyset$. Let A be a set, card A = card A, let $\prec$ be a well ordering of A such that the power of $A_\alpha = \{\beta \in A;\ \beta \prec \alpha\}$ is smaller than the power of A for all $\alpha \in A$. Let ψ be a one-to-one mapping of A onto Γ. Set $\psi_1 = \pi_1\psi$, $\psi_2 = \pi_2\psi$. Let φ be a mapping of A onto $[A] \times [A]$ such that for every $[k, l] \in [A] \times [A]$, and for every $\alpha \in A$ there exists a $\beta \in A$ such that $\beta \geqq \alpha$, $\varphi(\beta) = [k, l]$ (it may be easily proved that such a mapping exists). Set $\varphi_1 = \pi_1\varphi$, $\varphi_2 = \pi_2\varphi$. Now we shall construct a system $\{P_\alpha;\ \alpha \in A\}$ of subsets of $[A]$ by transfinite induction. If α_0 is the first element of A, put $P_{\alpha_0} = \{l\}$, where $l \in [A]$, $\psi_1(\alpha_0) \subset l$, $l \cap \psi_2(\alpha_0) = \emptyset$. Let $\alpha \in A$ and let there be defined sets P_β ($\beta \in A$, $\beta \prec \alpha$) such that:

1) card $P_\beta \leqq$ card A_β and if $\gamma \in A$, $\gamma \preceq \beta$, then there exists an $l \in P_\beta$ with $\psi_1(\gamma) \subset l$, $l \cap \psi_2(\gamma) = \emptyset$;

2) if $\beta \prec \alpha$, $\gamma \prec \alpha$, $m \in P_\beta$, $q \in P_\gamma$, then $\Delta(m, q) - \varphi_1(\beta) \cup \varphi_1(\gamma) \neq \emptyset$.

Set $\bar{P}_\beta = \bigcup_{l \in P_\beta} l$, $P = \bigcup_{\beta \preceq \alpha} \bar{P}_\beta$, $F = \bigcup_{\beta \preceq \alpha} \varphi_1(\beta)$. Evidently card $(A - P \cup F)$ = card A and therefore for every $\beta \preceq \alpha$ we can choose an element $l_\beta \in [A]$ such that $l_\beta \cap (A - P \cup F) \neq \emptyset$, $\psi_1(\beta) \subset l_\beta$, $l_\beta \cap \psi_2(\beta) = \emptyset$. Let P_α be the set of all $l_\beta(\beta \preceq \alpha)$. Evidently card $P_\alpha \leqq$ card A_α and for every $\beta \preceq \alpha$ there exists an $m \in P_\alpha$, for which $\psi_1(\beta) \subset m$, $m \cap \psi_2(\beta) = \emptyset$ and if $\beta \prec \alpha$, $k \in P_\beta$, then $l - k \cup \varphi_1(\alpha) \cup \varphi_1(\beta) \neq \emptyset$ for every $l \in P_\alpha$. Now for $[k, l] \in [A] \times [A]$ we put $M_{k,l} = \bigcup_{\alpha \in A,\ \varphi(\alpha) = [k,l]} P_\alpha$. Next show that the system $\{M_{k,l};\ k, l \in [A]\}$ has properties 1) and 2) from the lemma: Let $[k, l] \in [A] \times [A]$, $q \in [A]$, $p \in [A]$, be given, $q \cap p = \emptyset$. Consequently $[q, p] \in \Gamma$

and therefore $\psi(\alpha) = [q, p]$ for some $\alpha \in A$. Let $\beta \in A$, $\beta \geqq \alpha$, be such that $\varphi(\beta) = [k, l]$. Consequently $P_\beta \subset M_{k,l}$ and P_β contains an m with $q = \psi_1(\alpha) \subset m$, $m \cap \psi_2(\alpha) = \emptyset$. Let $[k, l] \neq [k', l']$, $m \in M_{k,l}$, $m' \in M_{k',l'}$. Let α (or β) be the element of A for which $m \in P_\alpha$ (or $m' \in P_\beta$ respectively). Evidently $\alpha \neq \beta$. If $\alpha \prec \beta$, then $m' - m \cup k \cup k' \neq \emptyset$, if $\beta \prec \alpha$, then $m - m' \cup k \cup k' \neq \emptyset$.

Lemma 5. *Let Λ be an infinite set. For every $\lambda \in \Lambda$ let (P_λ, v_λ) be a T_1F-space such that $2 \leqq \operatorname{card} P_\lambda \leqq \operatorname{card} \Lambda$; let (P, v) be the topological product of all spaces (P_λ, v_λ).*

Then for every $x \in P$ there exists an $A_x \subset P$ such that $vA_x = P$ and that the system $\{A_x;\ x \in P\}$ is disjoint.

Moreover, if Λ is uncountable and for every $\lambda \in \Lambda$ the space (P_λ, v_λ) either satisfies the first axiom of countability or it is regular and sequentionally compact, then for every $x \in P$ is satisfied: $x \in vA_x - A_x$ and if $B \subset A_x$, $x \in vB$ then $x \in v(vB - A_x - \{x\})$.

Proof. Let Λ be an infinite set, denote by $[\Lambda]$ the set of all its finite subsets. For $\lambda \in \Lambda$ let (P_λ, v_λ) be a T_1F-space such that $2 \leqq \operatorname{card} P_\lambda \leqq \operatorname{card} \Lambda$, let (P, v) be their topological product. For $x, y, z \in P$ we denote by $x_\lambda, y_\lambda, z_\lambda$ the λ^{th} coordinate of the points x, y, z respectively.

For $x, y \in P$ let x be equivalent to y if and only if $x_\lambda \neq y_\lambda$ for at most a finite number of elements of Λ. This equivalence defines a decomposition of P. Let $S \subset P$ be a system of representants of this decomposition. For $y \in S$ denote by C_y the set of all $x \in P$ equivalent to y. For $x \in C_y$ denote by $k(x)$ the set of all $\lambda \in \Lambda$ for which $x_\lambda \neq y_\lambda$. Evidently $k(x) \in [\Lambda]$. For $y \in S$, $l \in [\Lambda]$ let $B_{y,l}$ be the set of all $x \in C_y$ for which $k(x) = l$. Evidently $\{B_{y,l};\ y \in S,\ l \in [\Lambda]\}$ is the decomposition of P and $\operatorname{card} B_{y,l} \leqq \operatorname{card} [\Lambda]$. Now we choose a mapping φ from P into $[\Lambda]$ such that φ is one-to-one on every set $B_{y,l}$. Next for every $\lambda \in \Lambda$ we choose a one-to-one mapping ψ_λ from P_λ into P_λ such that $\psi_\lambda(c) \neq c$ for every $c \in P_\lambda$. Let $\mathscr{M} = \{M_{k,l};\ k \in [\Lambda],\ l \in [\Lambda]\}$ be a disjoint system of subsets of Λ with properties 1) and 2) from lemma 4. For $x \in P$ consider the set $M_{k(x),\varphi(x)}$. For $x \in P$ let A_x be the set of all $z \in P$ for which there exist $l \in M_{k(x),\varphi(x)}$ and $p \in [\Lambda]$, $p \cap l = \emptyset$ such that $z_\lambda = \psi_\lambda(x_\lambda)$ for $\lambda \in \Lambda - l \cup p$, $z_\lambda = x_\lambda$ for $\lambda \in l$ and $z_\lambda \neq x_\lambda$ for $\lambda \in p$. Now we show, that the system $\mathscr{A} = \{A_x;\ x \in P\}$ has the required properties:

I. Prove that the system $\mathscr{A}$ is disjoint: a) Let $x, x' \in P$ be not equivalent. Consequently for some infinite $\Gamma \subset \Lambda$ there is $x_\lambda \neq x'_\lambda$ for every $\lambda \in \Gamma$. Let $z \in A_x$, $z' \in A_{x'}$. We have to prove that $z \neq z'$. By definition of the sets A_x, $A_{x'}$ there exist sets $l, l' \in [\Lambda]$ such that $z_\lambda = \psi_\lambda(x_\lambda)$ for $\lambda \notin l$, $z'_\lambda = \psi(x'_\lambda)$ for $\lambda \notin l'$. All the mappings ψ_λ are one-to-one, consequently $z_\lambda \neq z'_\lambda$ for $\lambda \in \Gamma - l \cup l'$.

b) Let $x \neq x'$, but $x \in C_y$, $x' \in C_y$ for some $y \in S$. We choose $z \in A_x$, $z' \in A_{x'}$, and show that $z \neq z'$. Take $l \in [\Lambda]$, $l' \in [\Lambda]$ such that $z_\lambda = x_\lambda$ if and only if $\lambda \in l$, $z'_\lambda = x'_\lambda$ if and only if $\lambda \in l'$. By definition of the sets A_x and $A_{x'}$ there is $l \in M_{k(x),\varphi(x)}$, $l' \in$

$\in M_{k(x'),\varphi(x')}$. But then (from property 2, lemma 4) there is $m = \Delta(l, l') - k(x) \cup \cup k(x') \neq \emptyset$. Prove that $z_\mu \neq z'_\mu$ for $\mu \in m$. Let $\mu \in m$, let, for example, $\mu \in l - l' \cup \cup k(x) \cup k(x')$. Then $x_\lambda = y_\lambda$ for $\lambda \notin k(x)$, $x'_\lambda = y_\lambda$ for $\lambda \notin k(x')$ $z'_\lambda \neq x'_\lambda$ for $\lambda \notin l'$ and $z_\lambda = x_\lambda$ for $\lambda \in l$. Consequently $z_\mu = x_\mu = y_\mu = x'_\mu \neq z'_\mu$.

II. Now we prove that $vA_x = P$: Let $t \in P$, $m \in [\Lambda]$ be given, we have to find $z \in A_x$ such that $z_\lambda = t_\lambda$ for $\lambda \in m$. Let $q = \{\lambda \in m;\ t_\lambda = x_\lambda\}$, $p = m - q$. Clearly $q \cap p = \emptyset$, hence there exists a set $l \in M_{k(x),\varphi(x)}$ such that $q \subset l$, $l \cap p = \emptyset$. Take $z \in P$ such that $z_\lambda = x_\lambda$ for $\lambda \in l$, $z_\lambda = t_\lambda$ for $\lambda \in p$, $z_\lambda = \psi_\lambda(x_\lambda)$ for $\lambda \in \Lambda - l \cup p$. Then $z \in A_x$.

III. Now suppose that the set Λ is uncountable and that for every $\lambda \in \Lambda$ and every $c \in P_\lambda$ the following condition holds:

If V_c is a v_λ-neighbourhood of c, then there exists a sequence $\{V_c^n;\ n = 1, 2, \ldots\}$ of v_λ-neighbourhoods of c such that $\psi_\lambda(c) \notin V_c^n$ for every n, and that every sequence $\{d_n\}$ of points of P_λ where $d_n \in V_c^n$ contains a subsequence converging to some point of V_c different from $\psi_\lambda(c)$. (*)

The condition (*) is clearly satisfied if c has countable character or if (P_λ, v_λ) is regular and sequentionally compact.

For $x \in P$ evidently $x \in vA_x - A_x$. Let $B \subset A_x$, $x \in vB$. Prove that $x \in v(vB - A_x - \{x\})$. Let $p \in [\Lambda]$, for $\lambda \in p$ let a v_λ-neighbourhood V_λ of x_λ be given. Set $Q_\lambda = P_\lambda$ for $\lambda \in \Lambda - p$, $Q_\lambda = V_\lambda$ for $\lambda \in p$ and let U be the cartezian product of all sets $Q_\lambda (\lambda \in \Lambda)$. We have to find a point $z \in U \cap (vB - A_x - \{x\})$. For every $\lambda \in \Lambda$ let $\{V_{x_\lambda}^n\}$ be the sequence of v_λ-neighbourhoods of x_λ with the properties from condition (*) Q_λ written instead of V_c. By virtue of $x \in vB$ there exists a $z^1 \in B$ such that $z_\lambda^1 \in V_{x_\lambda}^1$ for $\lambda \in p$. Let $p_1 = \{\lambda \in \Lambda;\ z_\lambda^1 \neq \psi_\lambda(x_\lambda)\}$. p_1 is finite because $B \subset A_x$. Choose $p_1^* \in [\Lambda]$, $p_1^* \supsetneqq p_1$. Find $z^2 \in B$ such that $z_\lambda^2 \in V_{x_\lambda}^2$ for $\lambda \in p_1^*$. Set $p_2 = \{\lambda \in \Lambda;\ z_x^2 \neq \psi_\lambda(x_\lambda)\}$ and choose $p_2^* \in [\Lambda]$, $p_2^* \supsetneqq p_2$ and so on. In this manner one defines an infinite countable set $L = \bigcup_{n=1}^{\infty} p_n$ and a sequence $\{z^n;\ n = 1, 2, \ldots\}$ of points of B. For every $\lambda \in L$ the sequence $\{z_\lambda^n;\ n = 1, 2, \ldots\}$ contains subsequence converging to a point of Q_λ. If we use the well-known "diagonal method", we obtain an increasing sequence $\{k_n;\ n = 1, 2, \ldots\}$ of positive integers such that $\lim\limits_{n\to\infty} z_\lambda^{k_n}$ exist for every $\lambda \in L$. If we take $z \in P$ with $z_\lambda = \psi_\lambda(x_\lambda)$ for $\lambda \in \Lambda - L$ and $z_\lambda = \lim\limits_{n\to\infty} z_\lambda^{k_n}$ for $\lambda \in L$, then it is possible to show that $z \in U \cap (vB - A_x - \{x\})$. Indeed $\Lambda - L \neq \emptyset$, $\psi_\lambda(x_\lambda) \neq x_\lambda$ consequently $z \neq x$. $z \notin A_x$ because $z_\lambda \neq \psi_\lambda(x_\lambda)$ for infinitely many λ. Evidently, the sequence $\{z^{k_n};\ n = 1, 2, \ldots\}$ converges to z, consequently $z \in vB$. By virtue of (*), z is an element of U.

Note 4. The system S of representants in the proof of lemma 5 may be chosen in $2^{2^{\operatorname{card}\Lambda}}$ ways. If S and S' are different system of representants, $x \in S$, $x \notin S'$, it is possible to show that the set A_x constructed using S, and the set A'_x constructed using S', are disjoint. Consequently $x \in v(A_x - A'_x)$.

Theorem 7. *Let (P, v) be a topological product of T_1F-spaces (P_λ, v_λ), $\lambda \in \Lambda$, let $2 \leqq \operatorname{card} P_\lambda \leqq \operatorname{card} \Lambda > \aleph_0$. Let each space (P_λ, v_λ) either satisfy the first axiom of countability or be regular and sequentionally compact. Then there exists at least $2^{\operatorname{card} P}$ different topologies u on P such that $\tilde{u}$ is discrete, the order of u is 2 and the F-modification of u is v.*

Proof. Follows immediately from theorem 6, lemma 5, note 3. b) and note 4.

Theorem 8. *Let $\aleph$ be an arbitrary uncountable cardinal number, let (P, v) be a topological product of $\aleph$-many T_1F-spaces of cardinality $\geqq 2$. Then there exist at least $2^{2^{\aleph}}$ different topologies u on P such that the F-modification of u is v, the order of u is 2 and every $x \in P$ is an element of $u^2A - uA$ for some $A \subset P$.*

Proof. Let (P, v) be a topological product of T_1F-spaces (P_λ, v_λ), $\lambda \in \Lambda$, $2 \leqq \operatorname{card} P_\lambda$, $\operatorname{card} \Lambda = \aleph > \aleph_0$. Let $\mathscr{G}_\lambda$ be a closed decomposition of the space (P_λ, v_λ) such that $2 \leqq \operatorname{card} G_\lambda \leqq \aleph$ for every $G_\lambda \in \mathscr{G}_\lambda$ and that the space $(G_\lambda, v/G_\lambda)$ either satisfies the first axiom of countability or is regular and sequentially compact (for example G_λ finite). Let $\mathscr{G}$ be a closed decomposition (P, v) such that $G \in \mathscr{G}$ if and only if G is the product of $G_\lambda \in \mathscr{G}_\lambda$, $\lambda \in \Lambda$. For every $G \in \mathscr{G}$ construct the system $\{A_x;\ x \in G\}$ of subsets of G with the properties described in lemma 5 and then apply theorem 6, using the system $\bigcup_{G \in \mathscr{G}} \{A_x;\ x \in G\}$.

V

Now we generalize the methods of section 3 in the following direction: the order of all topologies, constructed in sections 2 and 3 was $\leqq 2$. Now we shall study topologies with a given F-modification but with an order as great as possible. We define the F-order of an F-space and show that among generalized Cantor's discontinua there exist spaces with an arbitrarily great F-order.

Definition 4. Let (P, v) be an F-space. We call the F-order of (P, v) (and denote it by F-ord (P, v)) the supremun of the orders of all topologies on P whose F-modification is v.

This definition generalizes the definition of a strong F-space. An F-space (P, v) is a strong F-space if and only if the F-order of (P, v) is 1.

We have proved that every subspace of a strong F-space is a strong F-space. But if F-ord $(P, v) = \alpha > 1$, then it is in general not true that every subspace has F-order $\leqq \alpha$. We give a trivial example:

Example 2. The following trivial result is used: Let v be the F-modification of both topologies u_1 and u_2 on P. If $u_1 \leqq u_2$, then the order of u_1 is greater than or equal to the order of u_2.

Let $P = \{1, 2, 3, 4, 5\}$; on P the following F-topology v is given: $v\{1\} = \{1, 2, 3, 4, 5\}$, $v\{2\} = \{2, 3, 4\}$, $v\{3\} = \{3, 4\}$, $v\{4\} = \{4\}$, $v\{5\} = \{4, 5\}$, for $A \subset P$ is $vA = \bigcup_{x \in A} v\{x\}$.

Evidently if u is some topology on P with F-modification v, then necessarily $u\{4\} = \{4\}$, $u\{3\} = \{3, 4\}$, $u\{5\} = \{4, 5\}$. Moreover if we put $u\{2\} = \{2, 3\}$, $u\{1\} = \{1, 2, 5\}$ then evidently u is the finest topology on P with F-modification v. Consequently F-ord $(P, v) = 2$. But if we consider $Q = \{1, 2, 3, 4\}$ it may be easily shown that F-ord $(Q, v/Q) = 3$. A regular $T_1 F$-space with an analogous property can be constructed (of course this is more complicated).

Lemma 6. *Let Q be an open subset of a space (P, u), let $M \subset Q$. Then for every ordinal α there is $Q \cap u^\alpha M = (u/Q)^\alpha M$.*

Proof. If $\alpha = 1$, then the equality holds. Let it is true for every $\beta < \alpha$. Then $(u/Q)^\alpha M = u/Q(\bigcup_{\beta<\alpha} (u/Q)^\beta M) = Q \cap u(\bigcup_{\beta<\alpha} (Q \cap u^\beta M)) = Q \cap u(Q \cap \bigcup_{\beta<\alpha} u^\beta M)$. But, Q being open, $Q \cap u(Q \cap \bigcup_{\beta<\alpha} u^\beta M) = Q \cap u(\bigcup_{\beta<\alpha} u^\beta M) = Q \cap u^\alpha M$.

Lemma 7. *Let Q be the intersection of an open subset and a closed subset of a space (P, u). Then the order of u is not less than the order of u/Q. If the F-modifications of u/Q and u are denoted by w or v respectively then $w = v/Q$.*

Proof. Let μ, η be the orders of u or u/Q respectively. The inequality $v \leqq \mu$ can be easily proved using lemma 6. The proposition concerning the F-modifications of u and u/Q is proved in [1] (p. 75, theorem 4.6.17).

Lemma 8. *Let (P, v) be an F-space, $Q \subset P$ be the intersection of a v-closed and a v-open set. Every topology on Q with F-modification v/Q can be extended to the whole P so that the F-modification of this extension is v.*

Proof. Let (P, v) be an F-space, $Q \subset P$, let t be a topology on Q with F-modification v/Q. We construct a topology u on P such that $u/Q = t$ and the F-modification of u is v in the following cases:

a) Q is v-closed. For $A \subset P$ put $uA = v(A - Q) \cup t(A \cap Q)$. Evidently u is a topology on P, $u/Q = t$, $u \leqq v$. Consequently it is sufficient to prove that, if $uA = A$ for some $A \subset P$, then $vA = A$. Let $A = v(A - Q) \cup t(A \cap Q)$. Then $t(A \cap Q) \subset A \cap Q$, hence $A \cap Q$ is t-closed and therefore it is (v/Q)-closed. Consequently $Q \cap v(A \cap Q) = A \cap Q$. But Q is v-closed so that $Q \cap v(A \cap Q) = v(A \cap Q)$. Consequently $A \supset v(A - Q) \cup v(A \cap Q)$.

b) Q is v-open. For $A \subset P$ put $uA = (vA - Q) \cup t(A \cap Q)$. Evidently u is a topology on P, $u/Q = t$, $u \leqq v$. Let $A \subset P$ be u-closed, and prove that it is v-closed: $A = (vA - Q) \cup t(A \cap Q)$. $A \cap Q$ is t-closed, consequently $A \cap Q = Q \cap v(A \cap Q)$. But Q is v-open, so that $Q \cap v(A \cap Q) = Q \cap vA$. Consequently $A \supset (vA - Q) \cup (Q \cap vA)$.

c) Now, if $Q = G \cap H$, G is v-open and H is v-closed, then we extend t from Q to G by a) and then from G to P by b).

Theorem 9. *Let Q be the intersection of a closed set and an open set of an F-space (P, v). Then F-ord $(Q, v/Q) \leqq F$-ord (P, v).*

Proof. This follows immediately from lemmas 8 and 7.

Theorem 10. *Let (P, v) be a topological product of T_1F-spaces (P_λ, v_λ) $(\lambda \in \Lambda)$. Then F-ord $(P, v) \geqq \sup_{\lambda \in \Lambda} F$-ord (P_λ, v_λ).*

Proof. This follows immediately from theorem 9.

Note 5. The F-order of the topological product of T_1F-spaces may exceed the supremum of the F-orders of these spaces. This follows trivially from theorem 8 or 7. But even the topological product of two strong FT_1-spaces is not necessar ily a strong F-space. An example of such spaces is given.

Example 3. Let (P_1, v_1) be the countable bicompact T_1-space with only one non-isolated point a. Let (P_2, v_2) be an uncountable, non-bicompact, Lindelöff T_1-space with only one non-isolated point b. Clearly (P_1, v_1) and (P_2, v_2) are strong F-spaces, but the F-order of their topological product (P, v) is 2. For, the F-modification of the following topology u is the product-topology and the order of u is 2: if $x \in P$, $x \neq [a, b]$, then $u_x = v_x$; the system of all sets $[(a) \times V_2] \cup [V_1 \times (b)]$, where V_1 is a v_1-neighbourhood of a, V_2 is v_2-neighbourhood of b, forms a complete collection of u-neighbourhood of $[a, b]$.

Problems. 1. Does there exist two strong F-spaces, satisfying axiom T_1, such that the F-order of their product is > 2?

2. Let (P, v) be a topological product of T_1F-spaces (P_λ, v_λ), $\lambda \in \Lambda$. Does there exist some upper bound for the F-order of (P, v) by means of the F-orders of (P_λ, v_λ) and card Λ?

Convention. Let μ be the order of a space (P, u). We shall say that the order of (P, u) is μ_+ if there exists a set $A \subset P$ such that $u^\alpha A \neq u^{\alpha+1} A$ for all $\alpha < \mu$; we shall say that the order of (P, u) is μ_- if such a set does not exist.

Lemma 9. *Let (P, u) be a space, let v be the F-modification of u, $M \subset P$, α an ordinal. Then*

$$vM - \bigcup_{\gamma<\alpha} u^\gamma M \subset v(u^\alpha M - \bigcup_{\gamma<\alpha} u^\gamma M), \quad vM - M \subset v(uM - M).$$

Proof. Set $C_\alpha = \bigcup_{\gamma<\alpha} u^\gamma M$. By definition, $u^\alpha M = uC_\alpha$. Let $x \in vM - C_\alpha$, $x \notin v(uC_\alpha - C_\alpha)$, we obtain a contradiction. Let V be v-open neighbourhood of x, which has an empty intersection with $uC_\alpha - C_\alpha$. Then $x \notin uC_\alpha$, consequently there exists an u-neighbourhood U_1 of x such that $U_1 \subset V$, $U_1 \cap C_\alpha = \emptyset$. Thus $U_1 \cap uC_\alpha = \emptyset$ and therefore for every $y \in U_1$ there exists an u-neighbourhood W_y^1 of y

such that $W_y^1 \subset V$, $W_y^1 \cap C_\alpha = \emptyset$. Set $U_2 = \bigcup_{y \in U_1} W_y^1$. Evidently $U_2 \subset V$, $U_2 \cap C_\alpha = \emptyset$, consequently $U_2 \cap uC_\alpha = \emptyset$. For every $y \in U_2$ one can find an u-neighbourhood W_y^2 of y such that $W_y^2 \subset V$, $W_y^2 \cap C_\alpha = \emptyset$. Set $W_3 = \bigcup_{y \in U_2} W_y^2$. The subsequent construction is evident. If we put $U = \bigcup_{n=1}^{\infty} U_n$, then evidently $x \in U$, U is u-open and consequently v-open, $U \cap C_\alpha = \emptyset$ and therefore $x \notin vC_\alpha$. But $vM \subset vC_\alpha$ and thus $x \notin vM$, which is a contradiction.

Theorem 11. *Let (P, v) be an F-space, μ an ordinal. Then there exists a topology on P with F-modification v and order μ_+ if and only if for every ordinal $\alpha \leqq \mu$ there exists an $A_\alpha \subset P$ such that*

1) *the system $\mathscr{A} = \{A_\alpha; \alpha \leqq \mu\}$ is disjoint, $A_\alpha \neq \emptyset$ for $\alpha < \mu$ and if μ is an isolated ordinal, then $A_\mu \neq \emptyset$;*

2) $vA_0 = \bigcup \mathscr{A}$, $\bigcup_{\beta > \alpha} A_\beta \subset vA_\alpha$;

3) $\alpha < \beta \leqq \mu$, $B \subset \bigcup_{\gamma < \alpha} A\gamma \Rightarrow A_\beta \cap vB \subset v(A_\alpha \cap vB)$.

Proof. Let u be a topology on P with F-modification v and order μ_+. We shall obtain system $\mathscr{A}$ with the properties 1) 2) 3) from the theorem. Let M be a subset of P such that $u^\alpha M \neq u^{\alpha+1} M$ for all $\alpha < \mu$. Set $A_0 = M$, $A_1 = uM - M$ and for $1 < \alpha \leqq \mu$ let $A_\alpha = u^\alpha M - \bigcup_{\gamma < \alpha} u^\gamma M$. The system $\{A_\alpha;\ \alpha \leqq \mu\}$ satisfies 1) trivially. 2) can be easily proved using lemma 9. Now we prove 3): set $C_\alpha = \bigcup_{\gamma < \alpha} A_\gamma$; clearly $uC_\alpha = C_\alpha \cup A_\alpha$. Let $\alpha < \beta \leqq \mu$, $B \subset C_\alpha$. Set $B^* = C_\alpha \cap vB$. Evidently $B \subset B^* \subset vB$, consequently $vB^* = vB$. B^* is a subset of C_α, hence $uB^* \subset uC_\alpha = C_\alpha \cup A_\alpha$ and therefore $uB^* = (uB^* \cap C_\alpha) \cup (uB^* \cap A_\alpha)$. But $uB^* \cap C_\alpha \subset vB^* \cap C_\alpha = vB \cap C_\alpha = B^*$ and therefore $uB^* - B^* \subset uB^* \cap A_\alpha \subset vB^* \cap A_\alpha$. Let $x \in A_\beta \cap vB$, $x \notin v(A_\alpha \cap vB)$. Let V be a v-open neighbourhood of x, which has an empty intersection with $A_\alpha \cap vB = A_\alpha \cap vB^*$. Then $v(uB^* - B^*) \subset P - V$ and therefore, using lemma 9 there is $vB^* - B^* \subset P - V$. Consequently $x \notin vB^* = vB$, which is a contradiction.

Now let μ be an ordinal, let $\mathscr{A} = \{A_\alpha; \alpha \leqq \mu\}$ be a system of subsets of an F-space (P, v) satisfying conditions 1) 2) 3) from the theorem. We shall construct a topology u on P with F-modification v and order μ_+. Consider the following topology u: for $x \in P - \bigcup \mathscr{A}$ let $u_x = v_x$; for $x \in A_\beta$ $(0 \leqq \beta \leqq \mu)$ all sets $V - \bigcup_{\gamma < \alpha} A_\gamma$, where $\alpha < \beta$, V is a v-neighbourhood of x, form a complete collection of u-neighbourhoods of x (consequently $u_x = v_x$ for $x \in A_0 \cup A_1$). Clearly $u^\alpha A_0 = \bigcup_{\gamma \leqq \alpha} A_\gamma$ for $\alpha \leqq \mu$, hence the order of u is μ_+. We shall prove that v is the F-modification of u. Denote by u^* the F-modification of u. Clearly $u^* \leqq v$ and for $x \in A_0 \cup A_1 \cup (P - \bigcup \mathscr{A})$ there is

$u_x^* = v_x$. Let $1 < \beta \leqq \mu$ and for all $y \in \bigcup_{\gamma<\beta} A_\gamma$ let $u_y^* = v_y$. Let $x \in A_\beta$, we are to prove that then also $u_x^* = v_x$. Suppose that $u_x^* < v_x$; let U be an u^*-open neighbourhood of x, but not an v-neighbourhood of x. Then there exist a v-open neighbourhood V of x and $\alpha < \beta$ such that $V - \bigcup_{\gamma<\alpha} A_\gamma \subset U$. Set $B = V - U$. Then $B \subset \bigcup_{\gamma<\alpha} A_\gamma$, $x \in \in A_\beta \cap vB$, consequently $x \in v(A_\alpha \cap vB)$. But now we can show that $V \cap A_\alpha \cap vB = \emptyset$: let $y \in V \cap A_\alpha$, hence $y \in U$ because $y \in V$, $y \notin \bigcup_{\gamma<\alpha} A_\gamma$. U, being u^*-open, is its u^* neighbourhood, consequently, using $u_y^* = v_y$, its v-neighbourhood also. Clearly $U \cap B = \emptyset$, consequently $y \notin vB$.

Convention. We will say that the cardinality of a set is $\overline{\aleph}_0$ if the set is finite. If α is a finite positive ordinal we denote $\overline{\aleph}_\alpha = \aleph_{\alpha-1}$. If α is an infinite ordinal, we put $\overline{\aleph}_\alpha = \aleph_\alpha$.

Theorem 12. *Let (P, v) be a topological product of $\overline{\aleph}_\mu$ of two-point T_1F-spaces. Then F-ord $(P, v) \geqq \mu$.*

Proof. If $\mu \leqq 1$ theorem 12 is trivial. Let $\mu > 1$. We shall find a system $\mathscr{A} = \{A_\alpha; \alpha \leqq \mu\}$ of subsets of P with properties 1) 2) 3) from theorem 11. Consider the set P as the set of all characteristic functions on some set Λ, card $\Lambda = \overline{\aleph}_\mu$. For $\alpha \leqq \mu$ we define A_α as the set of all points $y \in P$ such that $y_\lambda = 0$ precisely on a set of cardinality $\overline{\aleph}_\alpha$. Clearly $vA_\alpha = P$ for every $\alpha \leqq \mu$. Evidently the system $\{A_\alpha; \alpha \leqq \mu\}$ has properties 1) and 2) from theorem 11. We prove that 3) is also satisfied: let $\alpha < \beta \leqq \mu$, $B \subset \bigcup_{\gamma<\alpha} A_\gamma$, $x \in A_\beta \cap vB$; we proceed to prove that $x \in v(A_\alpha \cap vB)$. Let $p \subset \Lambda$ be finite, take $z \in A_\alpha \cap vB$ such that $z_\lambda = x_\lambda$ for $\lambda \in p$. Let Λ_0 be the subset of Λ such that $x_\lambda = 0$ it and only if $\lambda \in \Lambda_0$; clearly card $\Lambda_0 = \overline{\aleph}_\beta$. Choose $\Lambda_1 \subset \Lambda_0$ card $\Lambda_1 = \overline{\aleph}_\alpha$. For every finite $l \subset \Lambda_1$ there exists an ${}^1y^l \in B$ such that ${}^1y^l_\lambda = x_\lambda$ for $\lambda \in l \cup p$. Denote by $\langle \Lambda_1 \rangle$ the set of all these ${}^1y^l$. Let Λ_2 be the set of all $\lambda \in \Lambda$ such that for some ${}^1y^l \in \langle \Lambda_1 \rangle$ there is ${}^1y^l_\lambda = 0$; then card $\Lambda_2 = \overline{\aleph}_\alpha$, because card $\langle \Lambda_1 \rangle \leqq \leqq \overline{\aleph}_\alpha$ and $B \subset \bigcup_{\gamma<\alpha} A_\gamma$. For every finite $l \subset \Lambda_2$ let ${}^2y^l$ be a point of B such that ${}^2y^l_\lambda = x_\lambda$ for $\lambda \in l \cup p$, let $\langle \Lambda_2 \rangle$ be the set of all these ${}^2y^l$. Denote by Λ_3' the set of all $\lambda \in \Lambda$ such that for some ${}^2y^l \in \langle \Lambda_2 \rangle$ there is ${}^2y^l_\lambda = 0$, let $\Lambda_3 = \Lambda_3' \cup \Lambda_2$. Now the construction of Λ_4' is clear, $\Lambda_4 = \Lambda_4' \cup \Lambda_3$ and so on. Put $\Gamma = \bigcup_{n=1}^{\infty} \Lambda_n$, $\Gamma_0 = \Lambda_0 \cap \Gamma$. Evidently card Γ = card $\Gamma_0 = \overline{\aleph}_\alpha$. Let z be the point of P such that $z_\lambda = 0$ if and only if $\lambda \in \Gamma_0$. Evidently $z \in A_\alpha$, $z_\lambda = x_\lambda$ for $\lambda \in p$; we shall prove that $z \in vB$. Let a finite $m \subset \Lambda$ be given; we may take $y \in B$ such that $y_\lambda = z_\lambda$ for $\lambda \in m$. Set $k = m \cap \Gamma$. Then there exists a positive integer n with $k \subset \Lambda_n$. The point ${}^ny^k \in \langle \Lambda_n \rangle$ has the required properties. Evidently ${}^ny^k \in B$. For $\lambda \in \Lambda - \Gamma$ there is ${}^ny^k_\lambda = 1 = z_\lambda$, for $\lambda \in m \cap \Gamma_0 = = k \cap \Lambda_0$ there is ${}^ny^k_\lambda = x_\lambda = 0 = z_\lambda$ and for $\lambda \in m \cap (\Gamma - \Gamma_0) = k - \Gamma_0$ there is ${}^ny^k_\lambda = x_\lambda = 1 = z_\lambda$.

VI

Now we show that if, in the definition of a strong F-space (definition 1), we replace the notion of F-modification by the notion of F-reduction, the situation is quite different.

First only T_1-spaces with be considered. We recall that a T_1-topology v on P is called maximal if $vA = P$ for every infinite $A \subset P$.

Theorem 13. *Let (P, v) be a T_1F-space. Then there exists no T_1-topology u on P such that $u \neq v$, $\tilde{u} = v$ if and only if v is maximal.*

Proof. Let a T_1F-space (P, v) be given. If v is maximal and u is a T_1-topology on P for which $\tilde{u} = v$, then $v \leqq u$ and hence $v = u$. Suppose that v is not maximal. We have to find a T_1-topology u on P such that $u \neq v$, $\tilde{u} = v$. First we construct such a topology in four special cases:

a) Let there exist an infinite v-closed set $T \subset P$ such that $P - T \neq \emptyset$ and v/T is maximal: We choose $a \in P - T$ and infinite sets M, N with $M \cap N = \emptyset$, $M \cup N = T$. Consider the following topology u: for $x \in P$, $x \neq a$ let $u_x = v_x$, a complete collection of u-neighbourhoods of a is the system of all sets $U \cup (M - K)$, where U is a v-neighbourhood of a and K is finite. Evidently $u \neq v$, $\tilde{u} = v$.

b) Let there exist an infinite v-closed discrete set $C \subset P$: We choose a T_1-topology u_1 on C such that $u_1 \neq \tilde{u}_1$, $\tilde{u}_1$ is discrete. Consider the following topology u on P: for $x \notin C$ let $u_x = v_x$, for $x \in C$ a complete collection of u-neighbourhoods is the system of all sets $V \cup U$, where V is a v-neighbourhood of x and U is an u_1-neighbourhood of x.

c) Let there exist an infinite discrete set $D \subset P$ such that if $x \in vD - D$, then every v-neighbourhood of x contains D, except for a finite number of points. A T_1-topology u, for which $u \neq v$, $\tilde{u} = v$, may be constructed in the same way as in b) (replacing always C by D).

d) Let there exist an infinite discrete set $E \subset P$ such that the set $vE - E$ is infinite. Choose $a \in E$. Consider the following topology u on P: for $x \in P$, $x \neq a$ let a complete system of u-neighbourhoods consists of all sets $V \cup (vE - E - K)$, where V is a v-neighbourhood of a and K is finite. Clearly $u \neq v$, $\tilde{u} = v$.

Now we show that for every non-maximal T_1F-topology one of the four cases considered obtains. v is not maximal, hence there exists an infinite v-closed set T_1 such that $P - T_1 \neq \emptyset$. Choose $a_1 \in P - T_1$. If v/T_1 is maximal then we have case a). If v/T_1 is not maximal, then there is an infinite v-closed set $T_2 \subset T_1$ such that $T_1 - T_2 \neq \emptyset$. Choose $a_2 \in T_1 - T_2$. If v/T_2 is maximal we have case a) again. If v/T_2 is not maximal, then there exists an infinite v-closed set $T_3 \subset T_2$, $T_2 - T_3 \neq \emptyset$ and so on. It this process does not terminate after a finite number of steps, we obtain an infinite discrete set $A = \{a_1, a_2, \ldots\}$. If $vA - A = \emptyset$, then we have case b). If $vA - A$ is infinite, then we have case d). If $vA - A$ is finite, then either there exists an infinite $C \subset A$ such that $vC - C = \emptyset$ and then b) holds or there is an infinite

$D \subset A$ such that if $x \in vD - D$, then every v-neighbourhood of x contains D, except for a finite set of points. Thus c) holds.

Note. If we do not suppose axiom T_1 the situation is more complicated. For brevity we shall say that an F-topology v on P has the F-reduction property if the following holds: if u is a topology on P, $\tilde{u} = v$ then $u = v$. The following proposition may be proved, using the proof of theorem 13 without any changes:

Proposition. *Let a topology v on P have the F-reduction property. Then $vA = P$ for every infinite $A \subset P$.*

Conversely it may be easily shown using the space shown in the part I that every space with at least three closed points, fails to have the F-reduction property. Consequently only very special topologies have the F-reduction property. In any case, the following proposition holds:

Proposition. *One very infinite set there exist infinitely many non-homeomorphic topologies with the F-reduction property.*

Proof. Let P be an infinite set, let n be a positive integer. We choose different points $a_1, a_2, \ldots, a_n \in P$. Set $A = \{a_1, \ldots, a_n\}$, $Q = P - A$, and consider the following topology v on P: for $x \in Q$ let $v\{x\} = P$, for $1 \leqq l \leqq n$ let $v\{a_l\} = \bigcup_{k=1}^{l} \{a_k\}$, and for $M \subset P$ let $vM = \bigcup_{x \in M} v\{x\}$. We prove that v has the F-reduction property. Let there exist a topology u on P such that $u \neq v$, $\tilde{u} = v$. We are to obtain a contradiction. If $u_x > v_x$ for some x, then, since the complement of every v-neighbourhood of x is finite, there exists an $y \in P$ such that $x \in u\{y\} - v\{y\}$. Clearly $y \in A$. Let j be the first integer such that $1 \leqq j \leqq n$ and that there exists an $x \in P$ with $x \in u\{a_j\} - v\{a_j\}$. If $x \in Q$, then $a_l \in u\{a_j\} - v\{a_j\}$ for every $l \leqq n$, $l > j$. For, if $a_l \notin u\{a_j\}$, then $x \notin \operatorname{Int} U$ for some u-neighbourhood U of a_l. $\operatorname{Int} U$ is a $\tilde{u}$-neighbourhood of a_l; but every $\tilde{u}$-neighbourhood of a_l necessarily contains x. If $x = a_l$ for some $l > j$, then it may be proved in the same manner that $a_m \in u\{a_j\}$ for all $m \geqq j$, $m \leqq l$. Consequently we may suppose that $x \in Q$ if $j = n$, and $x = a_{j+1}$ if $j < n$. Let V be the smallest v-neighbourhood of x, let U be a u-neighbourhood of x such that $\operatorname{Int} U \subset V$. Clearly $a_j \notin V$, $a_j \in U$. Hence $a_j \in u(P - U) \subset u(\bigcup_{k=1}^{j-1} \{a_k\})$. Consequently for some $i < j$ there is $a_j \in u\{a_i\} - v\{a_i\}$, which is the contradiction.

On an infinite countable set there may be constructed $\aleph_0$ non-homeomorphic topologies with the F-reduction property and such that the closure of no point is the whole space.

References

[1] *E. Čech:* Topologické prostory. Praha, 1959.

[2] *V. Šedivá-Trnková:* Non-F-spaces. General topology and its Relations to Modern Analysis and Algebra. Proceedings of the Symposium, held in Prague in September 1961, 311–315.

Резюме

ТОПОЛОГИИ НА ПРОИЗВЕДЕНИЯХ И РАЗЛОЖЕНИЯХ ТОПОЛОГИЧЕСКИХ ПРОСТРАНСТВ

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Пусть $\mathscr{P}$ – разложение топологического пространства P, а π – проекция P на $\mathscr{P}$. На $\mathscr{P}$ мы определяем операцию u замыкания следующим образом: для $\mathscr{M} \subset \mathscr{P}$ имеет место $u\mathscr{M} = \pi(\overline{\pi^{-1}\mathscr{M}})$ (где $\bar{A}$ означает замыкание множества A в пространстве P). Эта операция удовлетворяет следующим аксиомам: $u\emptyset = \emptyset$, $\mathscr{M} \subset u\mathscr{M}$, $u(\mathscr{M}_1 \cup \mathscr{M}_2) = u\mathscr{M}_1 \cup u\mathscr{M}_2$. Вообще говоря, не обязательно должна быть выполнена аксиома $u(u\mathscr{M}) = u\mathscr{M}$.

В статье исследуются свойства операции замыкания, выполняющей эти три аксомы (такая операция замыкания называется топологией; операция же, выполняющая все четыре аксиомы, называется F-топологией; этому соответствуют и понятия топологическое пространство и топологическое F-пространство).

Разложение $\mathscr{P}$, снабженное топологией u, названо *плотным квоциент-пространством* в отличие от квоциент-пространства, определенного обычным образом.

Если u – топология на множестве P, то мы определим для $M \subset P$ $u^\alpha M = u(\bigcup_{\beta<\alpha} u^\beta M)$ (α, β означают порядковые числа). Если φ – порядковое число такое, что $u^\varphi M = u^{\varphi+1}M$ для всех M, то F-топологию u^φ мы называем F-модификацией топологии u. Наименьшее порядковое число φ такое, что u^φ является F-модификацией u, мы называем порядком u. Очевидно, квоциент-топология является F-модификацией плотной квоциент-топологии.

Теорема 1. *Каждое топологическое пространство является плотным квоциент-пространством F-пространства.*

Определение 1. F-пространство (P, v) мы назовем сильным F-пространством, если справедливо утверждение: если v является F-модификацией какой-либо топологии u, то $v = u$.

Дается необходимое и достаточное условие для того (теорема 4), чтобы F-пространство было сильным F-пространством. Это условие имеет простой вид для регулярных F-пространств: Регулярное F-пространство (P, v) будет сильным F-пространством, если и только если для каждого $A \subset P$, $x \in vA - A$ существует $B \subset A$ так, что $vB = B \cup \{x\}$.

В дальнейшем понятие T_1-пространство означает топологическое пространство, в котором конечные множества замкнуты.

Теорема 8. *Пусть (P, v) – топологическое произведение несчетного количества хотя бы двухточечных T_1- и F-пространств. Тогда существует $2^{\operatorname{card} P}$ различных топологий и на P порядка 2, F-модификацией которых является v, и для каждого $x \in P$ существует $H \subset P$ так, что $x \in u^2 H - uH$.*

Назовем *обобщенным дисконтинуумом Кантора* каждое топологическое произведение хотя бы $\aleph_0$ двухточечных T_1-пространств.

Теорема 12. *Для каждого порядкового числа μ существует обобщенный дисконтинуум Кантора (P, v) так, что v является F-модификацией топологии порядка μ.*

В статье исследу тся некоторые дальнейшие соотношения и понятия (F-редукция, F-порядок), связанные с определенными таким общим образом топологиями.