Czechoslovak Mathematical Journal

Josef Král

Non-tangential limits of the logarithmic potential

Czechoslovak Mathematical Journal, Vol. 14 (1964), No. 3, 455–482

Persistent URL: http://dml.cz/dmlcz/100633

Чехословацкий математический журнал, т. 14 (89) 1964, Прага

NON-TANGENTIAL LIMITS OF THE LOGARITHMIC POTENTIAL

JOSEF KRÁL, Praha

(Received July 26, 1963)

Necessary and sufficient conditions are established securing the existence of non-tangential limits of the logarithmic potential of the double distribution with any continuous density.

Introductory remark. In the present paper we continue the investigation of the behaviour of the logarithmic potential of the double distribution

$$W_F(z) = \operatorname{Im} \int_K \frac{F(\zeta)}{\zeta - z} \, \mathrm{d}\zeta$$

with any continuous (real-valued) density F for z approaching the curve K. We have showed in [5] (cf. also [2]) that for a simple closed curve K a necessary and sufficient condition securing the uniform continuity of $W_F(z)$ on every complementary domain of K can be formulated in terms of the quantity $v^K(\zeta)$ defined as follows:

$$v^K(\zeta) = \int_0^{2\pi} \mu^K(\zeta, \alpha) \, \mathrm{d}\alpha \,,$$

where $\mu^K(\zeta, \alpha)$ denotes the number of points at which K meets the half-line $\{z; z = \zeta + r \exp i\alpha, r > 0\}$. Now we show that $v^K(\zeta)$ together with an analogous quantity $u_r^K(\zeta)$ to be defined below is useful also for investigation of non-tangential limits of $W_F(z)$.

Suppose that K is a simple oriented rectifiable curve (which need not be closed) in the plane and fix an $\eta \in K$. Let $S = \{z; z = \eta + r \exp i\beta, 0 < r < R\}$ be a "non-tangential" segment which means that, for sufficiently small $\delta > 0$,

$$\eta \pm r \exp i\gamma \notin K \text{ whenever } |\gamma - \beta| < \delta \quad \text{and} \quad 0 < r < R \,.$$

We are concerned with the following problem:

What necessary and sufficient restrictions are to be imposed on K to secure the existence of

$$\lim_{z \to \eta} W_F(z) \,, \quad z \in S \tag{1}$$

for every continuous function F on K?

Defining

$$u_r^K(\zeta) = \int_0^r v^K(\zeta, \varrho)\, d\varrho\,,$$

where $v^K(\zeta, \varrho)$ denotes the number of points at which K meets the circle $\{z; |z - \zeta| = \varrho\}$, we obtain the following answer (which was announced without proof in [2]):

In order that the limit (1) *exist for every continuous F on K it is necessary and sufficient that*

$$v^K(\eta) < +\infty\,, \tag{2}$$

$$\sup_{r>0} r^{-1}\, u_r^K(\eta) < +\infty\,. \tag{3}$$

Note that (2), (3) do not depend on S. If (1) exists for at least one non-tangential segment (and every continuous F) then (2) and (3) hold and, consequently, (1) exists for every non-tangential segment S. If (2), (3) take place then more can be said about (1) and its value can be determined. This is done in § 2 where also some related results for a more general class of (non-simple) curves are presented. Proofs depend on some results concerning v^K and u_r^K established in [4] (cf. also [3]) which are completed in § 1 of the present paper.

It is interesting to observe that (2) and (3) imply

$$\sup_{r>0} r^{-1}\, \lambda\{\zeta;\ \zeta \in K,\ |\zeta - \eta| < r\} < +\infty\,, \tag{4}$$

where λ stands for the Hausdorff linear measure (= length) on K. The converse is not generally true; only the implication

$$(4) \Rightarrow (3)$$

is easily verified. While (4) holds almost everywhere (λ) on K provided K is rectifiable the length of $\{\eta;\ \eta \in K,\ v^K(\eta) = +\infty\}$ may be positive as shown by an example in § 3.

1

Let us first recall some definitions introduced in [4] (cf. also [2]). Let ψ be a path (= continuous complex-valued function) on $\langle a, b\rangle = \{t;\ a \leqq t \leqq b\}$. If $z \in E_2$ (= the Euclidean plane, which is identified with the set of finite complex numbers), $0 < r \leqq \infty$ and $\alpha \in E_1$ (= the set of finite real numbers) we denote by $\mu_r^\psi(\alpha; z)$ the number (possibly zero or infinite) of points in $\{t;\ t \in \langle a, b\rangle,\ 0 < |\psi(t) - z| < r,\ \psi(t) - z = |\psi(t) - z| \exp i\alpha\}$. Since $\mu_r^\psi(\alpha; z)$ is Lebesgue measurable with respect to α we may define

$$v_r^\psi(z) = \int_0^{2\pi} \mu_r^\psi(\alpha; z)\, d\alpha\,.$$

Similarly, let $v^{\psi}(\varrho; z)$ stand for the number $(0 \leqq v^{\psi}(\varrho; z) \leqq +\infty)$ of points in

$$\{t;\, t \in \langle a, b\rangle,\, |\psi(t) - z| = \varrho\}\,.$$

$v^{\psi}(\varrho; z)$ being Lebesgue measurable with respect to ϱ we may put

$$u_r^{\psi}(z) = \int_0^r v^{\psi}(\varrho; z)\, \mathrm{d}\varrho\,.$$

The following result derived in [4] will be useful below:

1·1. Lemma. *Let $\eta \in \psi(\langle a, b\rangle)$, $R > 0$, $\beta \in E_1$ and suppose that there exists a $\delta > 0$ such that $\eta \pm r \exp i\gamma \notin \psi(\langle a, b\rangle)$ whenever $|\gamma - \beta| < \delta$, $0 < r < 2R$.*

Then

$$\sup_{0<r<R} r^{-1}\, u_r^{\psi}(\eta) \leqq k\,.\,\bigl(v_R^{\psi}(\eta) + \sup_{0<r<2R} v_{2r}^{\psi}(\eta + r \exp i\beta)\bigr)\,,$$

$$\sup_{0<r<R} v_R^{\psi}(\eta + r \exp i\beta) \leqq m\,.\,\bigl(v_{2R}^{\psi}(\eta) + \sup_{0<r<2R} r^{-1} u_r^{\psi}(\eta)\bigr)$$

with constants k, m depending on δ only.

1·2. Notation. $N_{\psi}(z)$ will stand for the number (possibly zero or infinite) of points in $\psi^{-1}(z) = \{t;\, t \in \langle a, b\rangle,\, \psi(t) = z\}$. We shall write simply $v^{\psi}(z)$ instead of $v_{\infty}^{\psi}(z)$. If f is a (complex- or real-valued) function on the interval J we write var $[f; J]$ for the variation of f on J which is defined as the least upper bound of all the sums

$$\sum_{j=1}^{n} |f(b_j) - f(a_j)|$$

$\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle$ ranging over all finite systems of non-overlapping compact intervals contained in J.

Suppose now that φ is a continuous complex-valued function $\neq 0$ on J and denote by $\vartheta(t)$ a continuous single-valued argument of $\varphi(t)$ on J. If $a \leqq b$ are the end-points of J and if there exist finite limits

$$\vartheta(a+) = \lim_{t \to a+} \vartheta(t)\,, \quad \vartheta(b-) = \lim_{t \to b-} \vartheta(t)$$

we put

$$\Delta \arg [\varphi; J] = \vartheta(b-) - \vartheta(a+)\,.$$

(Clearly, $\Delta \arg [\varphi; J]$ is independent of the particular choice of ϑ.)

1·3. Remark. Let ψ be a path on $\langle a, b\rangle$, $R > 0$, $z \in E_2$ and denote by $\mathfrak{S}$ the system of all components of

$$\{t;\, t \in \langle a, b\rangle,\, 0 < |\psi(t) - z| < R\}\,.$$

Using the Banach theorem on variation of a continuous function one easily proves that the quantities $u_R^\psi(z)$, $v_R^\psi(z)$ defined above have the following meaning:

$$u_R^\psi(z) = \sum_I \operatorname{var}_t [|\psi(t) - z|; I], \quad I \in \mathfrak{S}, \tag{5}$$

$$v_R^\psi(z) = \sum_I \operatorname{var} [\vartheta_I; I], \quad I \in \mathfrak{S}, \tag{6}$$

where $\vartheta_I(t)$ denotes a continuous single-valued argument of $\psi(t) - z$ on I (cf. 2·2 and 2·5 in [4]). Hence it follows easily that, for any $R > 0$ and $z \in E_2$,

$$u_R^\psi(z) \leqq \operatorname{var}_t [|\psi(t) - z|; \langle a, b\rangle] \leqq \operatorname{var} [\psi; \langle a, b\rangle]. \tag{7}$$

We have also

$$v^\psi(z) = \sup \sum_{j=1}^{n} |\Delta_t \arg [\psi(t) - z; \langle a_j, b_j\rangle]|, \tag{8}$$

$\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle$ ranging over all finite systems of non-overlapping compact subintervals in $\langle a, b\rangle - \psi^{-1}(z) = \{t;\, t \in \langle a, b\rangle, \psi(t) \neq z\}$. If $\operatorname{var} [\psi; \langle a, b\rangle] < +\infty$ and $z \notin \psi(\langle a, b\rangle)$ then

$$v^\psi(z) \leqq \varrho^{-1}(z) \,.\, \operatorname{var} [\psi; \langle a, b\rangle], \tag{9}$$

where $\varrho(z) = \inf \{|z - \psi(t)|;\, a \leqq t \leqq b\}$ (cf. 1·12 in [5]).

Using the notation introduced in 1·2 we can formulate the following lemma:

1·4. Lemma. *Let ψ be a path on $\langle a, b\rangle$, $\zeta \in E_2$, $z \in E_2 - \psi(\langle a, b\rangle)$ and suppose that the segment with end-points z, ζ meets $\psi(\langle a, b\rangle)$ at most at ζ. Then*

$$|\Delta_t \arg [\psi(t) - z; \langle a, b\rangle]| \leqq v^\psi(\zeta) + 2\pi(N_\psi(\zeta) + 1).$$

Proof. If the segment with end-points z, ζ does not meet $\psi(\langle a, b\rangle)$ then, by lemma 1·6 in [5],

$$|\Delta_t \arg [\psi(t) - z; \langle a, b\rangle]| \leqq |\Delta_t \arg [\psi(t) - \zeta; \langle a, b\rangle]| + 2\pi \leqq v^\psi(\zeta) + 2\pi.$$

In case $\zeta \in \psi(\langle a, b\rangle)$ our lemma reduces to lemma 1·8 proved in [5].

On account of 1·4 we shall prove the following result to be applied in § 2:

1·5. Proposition. *Let ψ be a path on $\langle a, b\rangle$, $\psi(\langle a, b\rangle) = C$ and let $G \subset E_2 - C$ be an open set with boundary $B \neq \emptyset$. Then*

$$\sup_{z \in G} v^\psi(z) \leqq 2 \sup_{\eta \in B} v^\psi(\eta) + 8\pi \sup_{\eta \in B} N_\psi(\eta). \tag{10}$$

Proof is similar to that of theorem 1·11 in [5]. Suppose that

$$c_1 = \sup_{\eta \in B} v^\psi(\eta) < +\infty, \quad c_2 = \sup_{\eta \in B} N_\psi(\eta) < +\infty$$

and put $c = 2c_1 + 8\pi c_2$. Further fix a $z_0 \in G$ and an arbitrary d with $d < v^{\psi}(z_0)$. In order to prove (10) it is sufficient to show that $d \leqq c$. Denote by $\vartheta(t)$ a continuous single-valued argument of $\psi(t) - z_0$ on $\langle a, b \rangle$ (so that, by 1·3, $v^{\psi}(z_0) = \operatorname{var}[\vartheta; \langle a, b \rangle]$). There is a subdivision $\{a = t_0 < \ldots < t_p = b\}$ of $\langle a, b \rangle$ such that

$$d < \sum_{j=1}^{p} |\vartheta(t_j) - \vartheta(t_{j-1})| .$$

Put $s_j = \operatorname{sign}(\vartheta(t_j) - \vartheta(t_{j-1}))$ and define

$$h(z) = \sum_{j=1}^{p} s_j \Delta_t \arg[\psi(t) - z; \langle t_{j-1}, t_j \rangle], \quad z \in E_2 - C .$$

Thus

$$h(z_0) = \sum_{j=1}^{p} |\vartheta(t_j) - \vartheta(t_{j-1})| > d$$

and h is a harmonic function on $E_2 - C \supset G$ with $\lim_{|z| \to \infty} h(z) = 0$ (cf. 1·10 in [5]). We shall prove that, for every $\eta \in B$,

(11) $$\limsup_{\substack{z \to \eta \\ z \in G}} h(z) \leqq c .$$

This will imply, by the well-known property of harmonic functions (cf. 1·9 in [5]), that $h \leqq c$ on G; in particular, $c \geqq h(z_0) > d$ and the proof will be completed.

If $\eta \in B - C$ then h is continuous at η,

$$\lim_{z \to \eta} h(z) = h(\eta) \leqq \sum_{j=1}^{p} |\Delta \arg[\psi - \eta; \langle t_{j-1}, t_j \rangle]| \leqq v^{\psi}(\eta)$$

(cf. remark 1·3) and (11) is true.

Suppose now that $\eta \in B \cap C$, denote by J_1 the set of all $j \in \{1, \ldots, p\}$ with $\langle t_{j-1}, t_j \rangle \cap \psi^{-1}(\eta) = \emptyset$ and put $J_2 = \{1, \ldots, p\} - J_1$,

$$C_k = \bigcup_j \psi(\langle t_{j-1}, t_j \rangle), \ j \in J_k \quad (k = 1, 2) .$$

Further define the function h_k on $E_2 - C_k = G_k$ by

$$h_k(z) = \sum_j s_j \Delta \arg[\psi - z; \langle t_{j-1}, t_j \rangle], \ j \in J_k \quad (k = 1, 2) .$$

Then h_1 is continuous at $\eta \in G_1$ and

(12) $$\lim_{z \to \eta} h_1(z) = h_1(\eta) \leqq \sum_{j \in J_1} |\Delta \arg[\psi - \eta; \langle t_{j-1}, t_j \rangle]| \leqq v^{\psi}(\eta) \leqq c_1 .$$

Let us associate with any $z \in G$ a point $\zeta_z \in B$ such that the segment with end-points z,

ζ_z meets B at ζ_z only. Writing $\psi_j = \psi|_{\langle t_{j-1}, t_j \rangle}$ and using 1·4 we obtain for every $z \in G$ and $j \in J_2$

$$(13) \qquad |\Delta \arg [\psi - z; \langle t_{j-1}, t_j \rangle]| \leqq v^{\psi_j}(\zeta_z) + 2\pi(N_{\psi_j}(\zeta_z) + 1) .$$

We have

$$(14) \qquad v^{\psi}(\zeta_z) = \sum_{j=1}^{p} v^{\psi_j}(\zeta_z) \geqq \sum_j v^{\psi_j}(\zeta_z) , \quad j \in J_2$$

(cf. the equality (8) in [5]). The number of elements in J_2 does not exceed $2N_\psi(\eta)$ and, consequently,

$$\sum_{j \in J_2} (N_{\psi_j}(\zeta_z) + 1) \leqq 2N_\psi(\zeta_z) + 2N_\psi(\eta) \leqq 4c_2 .$$

Hence we conclude on account of (13), (14)

$$h_2(z) \leqq \sum_{j \in J_2} |\Delta \arg [\psi - z; \langle t_{j-1}, t_j \rangle]| \leqq v^{\psi}(\zeta_z) + 8\pi c_2 \leqq c_1 + 8\pi c_2 .$$

This together with (12) implies

$$\limsup_{\substack{z \to \eta \\ z \in G}} h(z) = \limsup (h_1(z) + h_2(z)) \leqq \lim_{z \to \eta} h_1(z) + \sup_{z \in G} h_2(z) \leqq 2c_1 + 8\pi c_2 .$$

We see that (11) holds again.

1·6. Remark. The inequality (10) could be further improved. This will not be discussed here because the only fact relevant for our purposes is that the boundedness of the right-hand side in (10) implies the boundedness of v^ψ on G (cf. also [3]).

2

We shall assume throughout this paragraph that $\psi = \psi_1 + i\psi_2$ (ψ_1, ψ_2 real-valued) is a path on $\langle a, b \rangle$ with

$$\operatorname{var} [\psi; \langle a, b \rangle] < +\infty , \quad \psi(\langle a, b \rangle) = C .$$

2·1. Definition. Let $z \in E_2$ and denote by $\mathfrak{S}$ the system of all components of $\langle a, b \rangle - \psi^{-1}(z)$. With every $I \in \mathfrak{S}$ we associate a continuous single-valued argument $\vartheta_z^I(t)$ of $\psi(t) - z$ on I. Given a (real-valued) function f on $\langle a, b \rangle$ we define

$$w^\psi(z; f) = \sum_{I \in \mathfrak{S}} \int_I f(t) \, d\vartheta_z^I(t)$$

provided the Lebesgue-Stieltjes integrals on the right-hand side exist and their sum is meaningful. (Clearly, this definition does not depend upon the particular choice of ϑ_z^I.) If F is a function on C we put $f(t) = F(\psi(t))$, $a \leqq t \leqq b$ and define

$$W^\psi(z; F) = w^\psi(z; f) .$$

2·2. Remark. Let $z \notin C$ and denote by $\vartheta_z(t)$ a continuous single-valued argument of $\psi(t) - z$ on $\langle a, b \rangle$. In order that

$$w^{\psi}(z; f) = \int_a^b f(t) \, \mathrm{d}\vartheta_z(t)$$

exist for any continuous function f on $\langle a, b \rangle$ it is necessary that

$$v^{\psi}(z) = \operatorname{var}\left[\vartheta_z; \langle a, b \rangle\right] < +\infty$$

(cf. remark 1·3). It is known that $\operatorname{var}\left[\psi; \langle a, b \rangle\right] < +\infty$ provided $v^{\psi}(z)$ is finite for at least 3 z's which are not situated on a single straight-line (cf. [3]). We see that the assumption concerning the rectifiability of ψ made at the beginning of § 2 is quite natural in connection with the investigation of $w^{\psi}(z; f)$.

2·3. Proposition. *Put* $s_{\psi}(t) = \operatorname{var}\left[\psi; \langle a, t \rangle\right]$, $a \leqq t \leqq b$, *and suppose that* f *is integrable*(s_{ψ}) *on* $\langle a, b \rangle$. *Then* $w^{\psi}(z; f)$ *exists for every* $z = x + iy \notin C (x, y \in E_1)$,

$$w^{\psi}(z; f) = - \int_a^b f(t) \frac{\psi_2(t) - y}{|\psi(t) - z|^2} \, \mathrm{d}\psi_1(t) + \int_a^b f(t) \frac{\psi_1(t) - x}{|\psi(t) - z|^2} \, \mathrm{d}\psi_2(t) =$$

$$= \operatorname{Im} \int_a^b \frac{f(t)}{\psi(t) - z} \, \mathrm{d}\psi(t)$$

and $w^{\psi}(z; f)$ *is a harmonic function of the variable* z *on* $E_2 - C$.

Proof. Fix a $z = x + iy \notin C$. Let $\vartheta_z(t)$ be a continuous single-valued argument of $\psi(t) - z$ on $\langle a, b \rangle$. Then

$$\vartheta_z(t) = \vartheta_z(a) + \operatorname{Im} \int_a^t \frac{\mathrm{d}\psi(u)}{\psi(u) - z} = \vartheta_z(a) - \int_a^t \frac{\psi_2(u) - y}{|\psi(u) - z|^2} \, \mathrm{d}\psi_1(u) +$$

$$+ \int_a^t \frac{\psi_1(u) - x}{|\psi(u) - z|^2} \, \mathrm{d}\psi_2(u) = \vartheta_z(a) + \int_a^t H_z(u) \, \mathrm{d}s_{\psi}(u),$$

where we put

$$H_z(u) = \frac{-(\psi_2(u) - y) \, \tau_1(u) + (\psi_1(u) - x) \, \tau_2(u)}{|\psi(u) - z|^2}, \quad \tau_k = \frac{\mathrm{d}\psi_k}{\mathrm{d}s_{\psi}}$$

(the derivative $\mathrm{d}\psi_k / \mathrm{d}s_{\psi}$ is taken in the sense of measure theory — cf. [15], chap. VIII, §§ 4–5). Clearly, $|\tau_k| \leqq 1$ almost everywhere (s_{ψ}) on $\langle a, b \rangle$ so that $H_z(u)$ is almost everywhere bounded and $f(u) \, H_z(u)$ is integrable (s_{ψ}) on $\langle a, b \rangle$. Hence (cf. [15], chap. XI, § 3, exercise 8)

$$w^{\psi}(z; f) = \int_a^b f(t) \, \mathrm{d}\vartheta_z(t) = \int_a^b f(t) \, H_z(t) \, \mathrm{d}s_{\psi}(t) =$$

$$= - \int_a^b f(t) \frac{\psi_2(t) - y}{|\psi(t) - z|^2} \, \mathrm{d}\psi_1(t) + \int_a^b f(t) \frac{\psi_1(t) - x}{|\psi(t) - z|^2} \, \mathrm{d}\psi_2(t) = \operatorname{Im} \int_a^b \frac{f(t)}{\psi(t) - z} \, \mathrm{d}\psi(t).$$

Since $\int_a^b \frac{f(t)\,\mathrm{d}\psi(t)}{\psi(t) - z}$ is an analytic function of the variable z on $E_2 - C$, $w^\psi(z;f)$ is harmonic on $E_2 - C$.

2·4. Theorem. *Let $\eta \in C$, $R > 0$, $\beta \in E_1$ and put*

$$S = \{z;\, z = \eta + r \exp i\beta,\ 0 < r < R\}\,.$$

Suppose that there exists a $\delta > 0$ such that

$$\eta \pm r \exp i\gamma \notin C \quad \textit{whenever} \quad |\gamma - \beta| < \delta,\ 0 < r < R\,. \tag{15}$$

If

$$\limsup_{\substack{z \to \eta \\ z \in S}} |w^\psi(z;f)| < +\infty \tag{16}$$

(in particular, if the limit

$$\lim_{\substack{z \to \eta \\ z \in S}} w^\psi(z;f) \neq \pm\infty$$

exists) for every continuous function f on $\langle a, b\rangle$ then

$$v^\psi(\eta) < +\infty\,, \tag{17}$$

$$\sup_{r > 0} r^{-1}\, u_r^\psi(\eta) < +\infty\,. \tag{18}$$

Proof. Denote by $\mathscr{C}(\langle a, b\rangle)$ the Banach space of all continuous functions f on $\langle a, b\rangle$ with the norm $\|f\| = \max_t |f(t)|$, $a \leqq t \leqq b$. For every $z \in S$, $w^\psi(z,f)$ may be considered as a linear functional on $\mathscr{C}(\langle a, b\rangle)$; its norm is equal to $v^\psi(z)$ ($=$ var $[\vartheta_z;\, \langle a, b\rangle]$, where $\vartheta_z(t)$ is a continuous single-valued argument of $\psi(t) - z$ on $\langle a, b\rangle$). Suppose that (16) holds for every $f \in \mathscr{C}(\langle a, b\rangle)$. Let $\{z_n\}$ be an arbitrary sequence of points in S tending to η as $n \to \infty$. Since $\sup_n |w^\psi(z_n;f)| < +\infty$ for every $f \in \mathscr{C}(\langle a, b\rangle)$ we conclude on account of the well-known Banach-Steinhaus theorem in Functional Analysis (cf. [13], n°s 55 and 31) that

$$\sup_n v^\psi(z_n) < +\infty\,. \tag{19}$$

(19) being valid for any sequence of points $z_n \in S$ tending to η we obtain that, for sufficiently small $R_0 > 0$,

$$+\infty > \sup_r v^\psi(\eta + r \exp i\beta)\,, \quad 0 < r \leqq 2R_0\,. \tag{20}$$

Since $v^\psi(z)$ is lower semicontinuous on E_2 (cf. 1.12 in [5]), (20) implies (17). Applying lemma 1·1 we derive from (20)

$$+\infty > \sup_r r^{-1}\, u_r^\psi(\eta)\,, \quad 0 < r < R_0\,. \tag{21}$$

On the other hand, for $r \geqq R_0$ the following inequalities are true (cf. (7)):

$$r^{-1}\, u_r^{\psi}(\eta) \leqq R_0^{-1}\, u_r^{\psi}(\eta) \leqq R_0^{-1} \operatorname{var}\,[\psi;\, \langle a, b\rangle]\,.$$

This together with (21) concludes the proof.

2·5. Remark. In preceding theorem, it is not necessary to require that (16) should be valid for every $f \in \mathscr{C}(\langle a, b\rangle)$. Let Q be a finite subset in $\langle a, b\rangle$ and let $Z(Q)$ stand for the subspace of all $f \in \mathscr{C}(\langle a, b\rangle)$ vanishing on Q. It is easily seen that (17) and (18) still remain in force if (16) is assumed to hold for every $f \in Z(Q)$ only (cf. lemma 2·5 in [5]).

Taking $Q = \{a, b\}$ we obtain easily the following

2·6. Corollary. *Let us keep the notation and the assumption* (15) *introduced in* 2·4. *Further suppose that the path* ψ *is simple on* $\langle a, b\rangle$ (*which means that* $\psi(u) \neq \psi(v)$ *whenever* $a \leqq u < v \leqq b$, $v - u < b - a$). *If*

$$\limsup_{\substack{z \to \eta \\ z \in S}} |W^{\psi}(z; F)| < +\infty$$

(*in particular, if the limit*

$$\lim_{\substack{z \to \eta \\ z \in S}} W^{\psi}(z; F) \neq \pm\infty$$

exists) *for every continuous function* F *on* C *then* (17) *and* (18) *hold.*

Proof follows at once from the preceding remark and the fact that any $f \in Z(Q)$, where $Q = \{a, b\}$, can be expressed in the form

$$f(t) = F(\psi(t))\,, \quad a \leqq t \leqq b\,,$$

where F is a continuous function on C.

The converse of the above corollary (as well as of theorem 2·4 in case that $\psi^{-1}(\eta)$ is finite) is also true (cf. remark 2·13 below). The proof depends on several simple lemmas.

2·7. Lemma. *Suppose that there exists the limit*

$$\tau_{\psi}^{+}(a) = \lim_{t \to a+} \frac{\psi(t) - \psi(a)}{|\psi(t) - \psi(a)|} = \exp i\alpha$$

($\alpha \in E_1$). *Let* $\alpha < \alpha_1 \leqq \alpha_2 < \alpha + 2\pi$, $R > 0$ *and suppose that the set*

$$P = \{\psi(a) + r \exp i\gamma;\ \alpha_1 \leqq \gamma \leqq \alpha_2,\ 0 \leqq r \leqq R\}$$

is disjoint with $\psi((a, b\rangle)$. *Then there exists a continuous single-valued argument* $\vartheta(t, z)$ *of* $\psi(t) - z$ *on* $(a, b\rangle \times P$ ($t \in (a, b\rangle$, $z \in P$) *and, for every* $r \in (0, R\rangle$ *and* $\gamma \in \langle\alpha_1, \alpha_2\rangle$,

$$\lim_{t \to a+} \vartheta(t, \psi(a) + r \exp i\gamma) - \lim_{t \to a+} \vartheta(t, \psi(a)) = \gamma - \alpha - \pi\,. \tag{22}$$

Further we have for any $a_1 \in (a, b\rangle$

$$(23) \qquad 0 = \lim_{z \to \psi(a)} \operatorname{var}_t \left[\vartheta(t, z) - \vartheta(t, \psi(a)); \langle a_1, b\rangle\right], \quad z \in P .$$

Proof. Fix a $\delta > 0$ small enough to secure that

$$H = \{-r \exp i\alpha;\ r \geqq 0\}$$

be disjoint with

$$\{s \exp i(\alpha + \Theta) - r \exp i\gamma;\ s > 0,\ r \geqq 0,\ |\Theta| < \delta,\ \gamma \in \langle \alpha_1, \alpha_2\rangle\} .$$

Let $A(\zeta)$ be a continuous single-valued argument of ζ on $E_2 - H$, $\alpha - \pi < A(\zeta) < \alpha + \pi$ for every $\zeta \in E_2 - H$. We have a $t_0 \in (a, b\rangle$ such that

$$t \in (a, t_0\rangle \Rightarrow \frac{\psi(t) - \psi(a)}{|\psi(t) - \psi(a)|} \in \{\exp i(\alpha + \Theta);\ |\Theta| < \delta\} .$$

Clearly,

$$\vartheta(t, z) = A(\psi(t) - z) \quad (a < t \leqq t_0,\ z \in P)$$

is a continuous single-valued argument of $\psi(t) - z$ on $(a, t_0\rangle \times P$. The reader will easily verify that $\vartheta(t, z)$ can be extended to a continuous single-valued argument of $\psi(t) - z$ on $(a, b\rangle \times P$ (which will be denoted by $\vartheta(t, z)$ again). We have for $r \in (0, R\rangle$, $\gamma \in \langle \alpha_1, \alpha_2\rangle$ and $t \in (a, t_0\rangle$

$$\psi(t) - (\psi(a) + r \exp i\gamma) = -r \exp i\gamma \left(1 + \frac{\psi(t) - \psi(a)}{-r \exp i\gamma}\right) =$$
$$= r \exp i(\gamma - \pi)(1 + f(t)) ,$$

where $f(t) \to 0$ as $t \to a+$. Hence we obtain

$$\vartheta(t, \psi(a) + r \exp i\gamma) = A(r \exp i(\gamma - \pi)(1 + f(t))) \to$$
$$\to A(r \exp i(\gamma - \pi)) = \gamma - \pi , \quad t \to a+$$

(note that $\alpha - \pi < \gamma - \pi < \alpha + \pi$). Finally,

$$\lim_{t \to a+} \vartheta(t, \psi(a)) = \lim_{t \to a+} A(\psi(t) - \psi(a)) = \lim_{t \to a+} A\left(\frac{\psi(t) - \psi(a)}{|\psi(t) - \psi(a)|}\right) = A(\exp i\alpha) = \alpha$$

and (22) is established. (23) is merely a consequence of proposition 1·12 proved in [5]. (Let us notice that only for the proof of (23) the assumption $\operatorname{var}[\psi; \langle a, b\rangle] < +\infty$ is needed.)

2·8. Lemma. *Let Γ be a non-void set, $R > 0$ and suppose that for every $\gamma \in \Gamma$ and $r \in (0, R\rangle$ there is given a function ϑ_r^γ on $(a, b\rangle$. Let ϑ be a function on $(a, b\rangle$ and let*

$$\lim_{r \to 0+} \vartheta_r^\gamma(b) = \vartheta(b)\,,$$

$$\lim_{r \to 0+} \operatorname{var}\left[\vartheta_r^\gamma - \vartheta;\ \langle a_1, b\rangle\right] = 0 \tag{24}$$

uniformly in $\gamma \in \Gamma$ for every $a_1 \in (a, b\rangle$. Further suppose that

$$\limsup_{r \to 0+} \sup_{\gamma \in \Gamma} \operatorname{var}\left[\vartheta_r^\gamma;\ (a, b\rangle\right] < +\infty\,,$$

$$\lim_{t \to a+} \vartheta_r^\gamma(t) = c(\gamma) \quad \textit{for every} \quad r \in (0, R\rangle \quad \textit{and} \quad \gamma \in \Gamma\,.$$

Then $\operatorname{var}[\vartheta;\ (a, b\rangle] < +\infty$ *and, for every continuous function f on $\langle a, b\rangle$,*

$$\lim_{r \to 0+} \int_{(a,b\rangle} f(t)\, \mathrm{d}\vartheta_r^\gamma(t) = \int_{(a,b\rangle} f(t)\, \mathrm{d}\vartheta(t) + f(a)\left(\vartheta(a+) - c(\gamma)\right) \tag{25}$$

uniformly in $\gamma \in \Gamma$.

Proof. (25) is easily checked if f reduces to a constant on $\langle a, b\rangle$. We may therefore assume that $f(a) = 0$. Given $\varepsilon > 0$ we have an $a_1 \in (a, b\rangle$ such that

$$a \leqq t \leqq a_1 \Rightarrow |f(t)| < \varepsilon\,.$$

Put $k = \limsup_{r \to 0+} \sup_{\gamma \in \Gamma} \operatorname{var}\left[\vartheta_r^\gamma;\ (a, b\rangle\right]$. By (24) also

$$\operatorname{var}\left[\vartheta;\ (a, b\rangle\right] = \lim_{d \to a+} \operatorname{var}\left[\vartheta;\ \langle d, b\rangle\right] \leqq k\,.$$

Hence we obtain

$$\limsup_{r \to 0+} \sup_{\gamma \in \Gamma} \left|\int_{(a,a_1)} f\, \mathrm{d}\vartheta_r^\gamma\right| \leqq \varepsilon k\,,\ \left|\int_{(a,a_1)} f\, \mathrm{d}\vartheta\right| \leqq \varepsilon k\,.$$

Further we have

$$\left|\int_{\langle a_1,b\rangle} f\, \mathrm{d}\vartheta_r^\gamma - \int_{\langle a_1,b\rangle} f\, \mathrm{d}\vartheta\right| \leqq \max_t |f(t)| \operatorname{var}\left[\vartheta_r^\gamma - \vartheta;\ \langle a_1, b\rangle\right] \to 0$$

uniformly in $\gamma \in \Gamma$ as $r \to 0+$. We conclude that

$$\limsup_{r \to 0+} \sup_{\gamma \in \Gamma} \left|\int_{(a,b\rangle} f\, \mathrm{d}\vartheta_r^\gamma - \int_{(a,b\rangle} f\, \mathrm{d}\vartheta\right| \leqq 2\varepsilon k$$

which completes the proof, because ε was an arbitrary positive number.

Now we are able to prove the following

2·9. Theorem. *Let $\eta \in C$, $\psi^{-1}(\eta) = \{a\}$, $v^\psi(\eta) < +\infty$. Then there exists the limit*

$$\tau_\psi^+(a) = \lim_{t \to a+} \frac{\psi(t) - \eta}{|\psi(t) - \eta|} = \exp i\alpha\,.$$

If, further,

$$\sup_{r>0} r^{-1}\, u_r^\psi(\eta) < +\infty \tag{26}$$

then, for every continuous function f on $\langle a, b\rangle$,

$$\lim_{r \to 0+} w^\psi(\eta + r \exp i\gamma; f) = w^\psi(\eta; f) + f(a)\,.(\pi + \alpha - \gamma) \tag{27}$$

uniformly in $\gamma \in \langle \alpha_1, \alpha_2\rangle$ whenever $\alpha < \alpha_1 \leqq \alpha_2 < \alpha + 2\pi$.

Proof. Let $\vartheta(t)$ be a continuous single-valued argument of $\psi(t) - \eta$ on $(a, b\rangle$. Since $+\infty > v^\psi(\eta) = \operatorname{var}[\vartheta; (a, b\rangle]$ (cf. remark 1·3), there exists the limit $\lim_{t \to a+} \vartheta(t) = \alpha$. Clearly, $\lim_{t \to a+} (\psi(t) - \eta)/|\psi(t) - \eta| = \exp i\alpha$. Let $\alpha < \alpha_1 < \alpha + \pi < < \alpha_2 < \alpha + 2\pi$. It is easily seen that there is an $R > 0$ such that

$$P = \{\eta + r \exp i\gamma;\ 0 \leqq r \leqq 2R,\ \alpha_1 \leqq \gamma \leqq \alpha_2\}$$

is disjoint with $\psi((a, b\rangle)$. By lemma 2·7, we have a continuous single-valued argument $\vartheta(t, z)$ of $\psi(t) - z$ on $(a, b\rangle \times P$. We may clearly suppose that $\vartheta(t, \eta) = \vartheta(t)$ so that $\lim_{t \to a+} \vartheta(t, \eta) = \alpha$. Writing $\vartheta_r^\gamma(t)$ for $\vartheta(t, \eta + r \exp i\gamma)$ we have by 2·7

$$\lim_{t \to a+} \vartheta_r^\gamma(t) = \gamma - \pi \quad \text{for every} \quad \gamma \in \langle \alpha_1, \alpha_2\rangle = \Gamma \quad \text{and} \quad r \in (0, 2R\rangle\,, \tag{28}$$

$$\lim_{r \to 0+} \sup_{\gamma \in \Gamma} \operatorname{var}[\vartheta_r^\gamma - \vartheta; \langle a_1, b\rangle] = 0 \quad \text{for every } a_1 \in (a, b\rangle\,. \tag{29}$$

We may assume that R is small enough to secure that, for $k = 1, 2$,

$$0 < r \leqq 2R \Rightarrow \eta \pm r \exp i\alpha_k \notin C$$

(observe that $\pm\exp i\alpha_k \neq \tau_\psi^+(a)$); clearly, also

$$(0 < r \leqq 2R,\ |\gamma - \alpha_k| < \delta) \Rightarrow \eta \pm r \exp i\gamma \notin C$$

for sufficiently small $\delta > 0$. Using lemma 1·1 and (26) we obtain that v_R^ψ is bounded on

$$S_k = \{z;\ z = \eta + r \exp i\alpha_k,\ 0 \leqq r < R\} \quad (k = 1, 2)\,.$$

Let us fix a $t_1 \in (a, b)$ such that

$$t \in \langle a, t_1\rangle \Rightarrow |\psi(t) - \eta| < \tfrac{1}{2}R$$

and put $\varphi = \psi|_{\langle a,t_1\rangle}$, $\omega = \psi|_{\langle t_1,b\rangle}$. It is easily seen that, for $k = 1, 2$,

$$0 \leqq r \leqq \tfrac{1}{2}R \Rightarrow v^{\varphi}(\eta + r \exp i\alpha_k) \leqq v_R^{\psi}(\eta + r \exp i\alpha_k)\,,$$

so that v^{φ} must be bounded on

$$\hat{S}_k = \{z;\ z = \eta + r \exp i\alpha_k,\ 0 \leqq r \leqq \tfrac{1}{2}R\}\,.$$

Put

$$M = \{\eta + \tfrac{1}{2}R \exp i\gamma;\ \alpha_1 \leqq \gamma \leqq \alpha_2\}\,.$$

Since $+\infty > \operatorname{var}[\psi; \langle a, b\rangle] \geqq \operatorname{var}[\varphi; \langle a, t_1\rangle]$ and M has a positive distance from $C \supset \varphi(\langle a, t_1\rangle)$, v^{φ} is bounded on M (cf. remark 1·3). Noting that $\hat{S}_1 \cup \hat{S}_2 \cup M$ is the boundary of

$$G = \{\eta + r \exp i\gamma;\ 0 < r < \tfrac{1}{2}R,\ \alpha_1 < \gamma < \alpha_2\}$$

we conclude on account of 1·5 that v^{φ} must be bounded on $\bar{G} = \{z;\ z \in P,\ |z - \eta| \leqq \leqq \tfrac{1}{2}R\}$. Because $\bar{G}$ has a positive distance from $\omega(\langle t_1, b\rangle)$, v^{ω} is also bounded on $\bar{G}$. Consequently, $v^{\psi} \leqq v^{\varphi} + v^{\omega}$ is bounded on $\bar{G}$. We have for $0 < r \leqq \tfrac{1}{2}R$, $\alpha_1 \leqq \gamma \leqq \alpha_2$ the equality $v^{\psi}(\eta + r \exp i\gamma) = \operatorname{var}[\vartheta_r^{\gamma}; (a, b\rangle]$ (cf. remark 1·3) so that

$$\limsup_{r\to 0+}\ \sup_{\gamma\in\Gamma} \operatorname{var}[\vartheta_r^{\gamma}; (a, b\rangle] < +\infty\,. \tag{30}$$

Taking into account that $\vartheta(t, z)$ is continuous on $(a, b\rangle \times P$ we obtain

$$\lim_{r\to 0+}\ \sup_{\gamma\in\Gamma} |\vartheta(b) - \vartheta_r^{\gamma}(b)| = 0\,. \tag{31}$$

By (28)–(31) we are justified to use 2·8 whence, for every continuous f on $\langle a, b\rangle$,

$$w^{\psi}(\eta + r \exp i\gamma; f) = \int_{(a,b\rangle} f(t)\,\mathrm{d}\vartheta_r^{\gamma}(t) \to \int_{(a,b\rangle} f(t)\,\mathrm{d}\vartheta(t) +$$

$$+ f(a)\,.\,(\alpha - \gamma + \pi) = w^{\psi}(\eta; f) + f(a)\,.\,(\alpha - \gamma + \pi) \quad (r \to 0+)$$

unformly in $\gamma \in \Gamma$. Thus (27) is proved.

An analogous theorem concerning $\lim\limits_{z\to\psi(b)} w^{\psi}(z; f)$ is also true.

2.10. Theorem. *Let $\eta \in C$, $\psi^{-1}(\eta) = \{b\}$. If*

$$v^{\psi}(\eta) < +\infty \tag{32}$$

then there exists the limit

$$\tau_{\psi}^{-}(b) = \lim_{t\to b-} \frac{\psi(t) - \eta}{|\eta - \psi(t)|} = \exp i\alpha\,.$$

If, further, (26) *holds then, for every continuous function f on* $\langle a, b\rangle$,

$$w^{\psi}(\eta + r \exp i\gamma; f) \to w^{\psi}(\eta; f) + f(b) \,.\, (\gamma - \alpha - \pi)$$

as $r \to 0+$ *uniformly in* $\gamma \in \langle \alpha_1, \alpha_2 \rangle$ *whenever*

$$\alpha < \alpha_1 \leqq \alpha_2 < \alpha + 2\pi \,.$$

Proof is easily derived from 2·9 where ψ is changed for the path $\tilde{\psi}$ on $\langle -b, -a\rangle$ defined by

$$\tilde{\psi}(t) = \psi(-t)\,, \quad -b \leqq t \leqq -a\,;$$

then $w^{\tilde{\psi}}(z;\tilde{f}) = -w^{\psi}(z;f)$ provided $\tilde{f}(t) = f(-t)$ $(t \in \langle -b, -a\rangle)$ and 2·10 follows at once.

Combining 2·9 and 2·10 one obtains

2·11. Theorem. *Let* $\eta \in C$, $\psi^{-1}(\eta) = \{t_0\}$, $a < t_0 < b$. *If* (32) *takes place then there exist the limits*

$$\tau_{\psi}^{+}(t_0) = \lim_{t \to t_0+} \frac{\psi(t) - \eta}{|\psi(t) - \eta|} = \exp i\alpha_+\,,$$

$$\tau_{\psi}^{-}(t_0) = \lim_{t \to t_0-} \frac{\psi(t) - \eta}{|\eta - \psi(t)|} = \exp i\alpha_-\,.$$

We may clearly suppose that $\alpha_+ \leqq \alpha_- < \alpha_+ + 2\pi$. *Put* $\Delta = \pi - (\alpha_- - \alpha_+)$. *If also* (26) *holds then, for every continuous function f on* $\langle a, b\rangle$,

(33) $$\lim_{r \to 0+} w^{\psi}(\eta + r \exp i\gamma; f) = w^{\psi}(\eta; f) + f(t_0) \,.\, (\pi + \Delta)$$

uniformly in $\gamma \in E$ *for every compact* $E \subset (\alpha_+, \alpha_-)$,

(34) $$\lim_{r \to 0+} w^{\psi}(\eta + r \exp i\gamma; f) = w^{\psi}(\eta; f) - f(t_0) \,.\, (\pi - \Delta)$$

uniformly in $\gamma \in F$ *for every compact* $F \subset (\alpha_-, \alpha_+ + 2\pi)$.

Proof. Put $\varphi = \psi|_{\langle a, t_0\rangle}$, $\omega = \psi|_{\langle t_0, b\rangle}$. Let us first consider the case $\alpha_- < \gamma < \alpha_+ + 2\pi$. By 2·10

(35) $$\lim_{r \to 0+} w^{\varphi}(\eta + r \exp i\gamma; f) = w^{\varphi}(\eta; f) + f(t_0) \,.\, (\gamma - \alpha_- - \pi)$$

uniformly in $\gamma \in F$ for every compact $F \subset (\alpha_-, \alpha_+ + 2\pi) \subset (\alpha_-, \alpha_- + 2\pi)$. In a similar way 2·9 yields

(36) $$\lim_{r \to 0+} w^{\omega}(\eta + r \exp i\gamma; f) = w^{\omega}(\eta; f) + f(t_0) \,.\, (\pi + \alpha_+ - \gamma)$$

uniformly in $\gamma \in F$ for every compact $F \subset (\alpha_-, \alpha_+ + 2\pi)$. On account of (35), (36) we obtain (34) (note that $w^{\varphi}(\ldots; f) + w^{\omega}(\ldots; f) = w^{\psi}(\ldots; f)$).

Let now $\alpha_+ < \gamma < \alpha_-$. Then $\alpha_- < \gamma + 2\pi < \alpha_- + 2\pi$ and we derive from 2·10

$$(37) \qquad \lim_{r \to 0+} w^{\varphi}(\eta + r \exp i\gamma; f) = \lim_{r \to 0+} w^{\varphi}(\eta + r \exp i(\gamma + 2\pi); f) =$$

$$= w^{\varphi}(\eta; f) + f(t_0) \cdot (\gamma + 2\pi - \alpha_- - \pi) = w^{\varphi}(\eta; f) + f(t_0) \cdot (\gamma + \pi - \alpha_-)$$

uniformly in $\gamma \in E$ for every compact $E \subset (\alpha_+, \alpha_-) \subset (\alpha_- - 2\pi, \alpha_-)$. 2·9 implies

$$(38) \qquad \lim_{r \to 0+} w^{\omega}(\eta + r \exp i\gamma; f) = w^{\omega}(\eta; f) + f(t_0) \cdot (\pi + \alpha_+ - \gamma)$$

uniformly in $\gamma \in E$ for every compact $E \subset (\alpha_+, \alpha_-)$. (37) plus (38) gives (33) and the proof is complete.

As a corollary of 2·9–2·11 one obtains

2·12. Theorem. *Let $\eta \in C$, $\psi^{-1}(\eta) = \{t_1 < \ldots < t_n\}$. If (32) takes place then there exist the limits*

$$\tau_{\psi}^{+}(t_k) = \lim_{t \to t_k+} \frac{\psi(t) - \eta}{|\psi(t) - \eta|} \quad \text{for every} \quad t_k < b\,,$$

$$\tau_{\psi}^{-}(t_k) = \lim_{t \to t_k-} \frac{\psi(t) - \eta}{|\eta - \psi(t)|} \quad \text{for every} \quad t_k > a\,.$$

Let us agree to write $\tau_{\psi}^{-}(t_1) = \tau_{\psi}^{+}(a)$ in case $t_1 = a$, $\tau_{\psi}^{+}(t_n) = \tau_{\psi}^{-}(b)$ in case $t_n = b$ and put $T = \bigcup_{k=1}^{n} \{\tau_{\psi}^{+}(t_k), \tau_{\psi}^{-}(t_k)\}$. If, further, (26) holds then, for every continuous function f on $\langle a, b\rangle$, $w^{\psi}(\eta + r\zeta; f)$ tends to a limit as $r \to 0+$ uniformly in $\zeta \in K$ for every compact $K \subset \{\zeta; \zeta \in E_2, |\zeta| = 1\} - T$. If $\psi^{-1}(\eta) \subset (a, b)$ or $\psi(a) = \psi(b)$ then $\lim_{r \to 0+} w^{\psi}(\eta + r\zeta; f)$ is constant on every component of $\{\zeta; |\zeta| = 1\} - T$.

Proof may be left to the reader.

2·13. Remark. The value of $\lim_{r \to 0+} w^{\psi}(\eta + r\zeta; f)$ in preceding theorem can be calculated by means of theorems 2.9–2.11. Since $W^{\psi}(z; F)$ is merely a particular case of $w^{\psi}(z; f)$ (cf. definition 2·1) theorems 2·9–2·12 include analogous results concerning non-tangential limits of the logarithmic potential $W^{\psi}(z; F)$; their formulation may be omitted here. In particular, we see that the converse of 2·6 is true. (By 2·12, also the converse of 2·4 is true provided the set $\psi^{-1}(\eta)$ is finite.)

3

3·1. Remark. In present paragraph we investigate the quantities $v^{\psi}(\eta)$ and $u_r^{\psi}(\eta)$ which proved to be useful in § 2. The reader will easily construct examples showing that for a simple rectifiable path φ on $\langle a, b\rangle$ and an $\eta \in \varphi(\langle a, b\rangle)$ every of the relations

$$v^{\varphi}(\eta) < +\infty\,, \quad v^{\varphi}(\eta) = +\infty\,,$$

may hold simultaneously with any of the relations

$$\sup_{r>0} r^{-1}\, u_r^{\varphi}(\eta) < +\infty\,, \quad \sup_{r>0} r^{-1}\, u_r^{\varphi}(\eta) = +\infty\,.$$

3·2. Notation. If M is a subset in E_2 or in E_1 we denote by λM its length (= Hausdorff linear measure) as defined in [14], chap. II, § 8. If $M \subset E_1$ then λM equals the outer Lebesgue measure of M. If φ is a (real- or complex-valued) function on an interval J, $L \subset J$ and $\zeta \in \varphi(J)$ we denote by $N_\varphi(\zeta; L)$ the number (possibly zero or infinite) of points in $\varphi^{-1}(\zeta) \cap L$. Given $G \subset J$ open in J we put

$$\operatorname{var}[\varphi; G] = \sum_I \operatorname{var}[\varphi; I]\,,$$

I ranging over all components of G. Further we define for any $H \subset J$

$$\operatorname{var}[\varphi; H] = \inf_G \operatorname{var}[\varphi; G]\,, \quad G \text{ open in } J,\ G \supset H\,.$$

The following known theorem will be employed below:

3·3. Lemma. *Let φ be a continous (real- or complex-valued) function on an arbitrary interval J and let $G \subset J$ be a set open in J. Then $N_\varphi(\zeta; G)$, considered as a function of the variable ζ on $\varphi(J)$, is measurable (λ) and*

$$\int_{\varphi(J)} N_\varphi(\zeta; G)\, d\lambda(\zeta) = \operatorname{var}[\varphi; G]\,. \tag{39}$$

If $\operatorname{var}[\varphi; J] < +\infty$ *then*

$$\int_{\varphi(J)} N_\varphi(\zeta; H)\, d\lambda(\zeta) = \operatorname{var}[\varphi; H] \tag{40}$$

for every G_δ-set $H \subset J$.

Proof. For a compact interval $\langle a, b\rangle \subset J$ the formula

$$\int_{\varphi(J)} N_\varphi(\zeta; \langle a, b\rangle)\, d\lambda(\zeta) = \operatorname{var}[\varphi; \langle a, b\rangle]$$

was established by S. Banach provided φ is real-valued (cf. [8], chap. VIII, § 5) and generalized to the case that φ maps J into an Euclidean or a metrical space by several authors (cf. [6] for the bibliography on the subject). Since any interval $I \subset J$ can be expressed as a union of a non-decreasing sequence $\{I_n\}$ of compact intervals and $N_\varphi(\zeta; I_n) \nearrow N_\varphi(\zeta; I)$ as $n \to \infty$ we obtain

$$\int_{\varphi(J)} N_\varphi(\zeta; I)\, d\lambda(\zeta) = \operatorname{var}[\varphi; I]\,.$$

Given a set $G \subset J$ open in J we have

$$\int_{\varphi(J)} N_\varphi(\zeta; G)\, d\lambda(\zeta) = \sum_I \int_{\varphi(J)} N_\varphi(\zeta; I)\, d\lambda(\zeta) = \sum_I \operatorname{var}[\varphi; I] = \operatorname{var}[\varphi; G],$$

I ranging over all components of G. Finally, let

$$+\infty > \operatorname{var}[\varphi; J]\,\Big(= \int_{\varphi(J)} N_\varphi(\zeta; J)\, d\lambda(\zeta),$$

so that $+\infty > N_\varphi(\zeta; J)$ almost everywhere (λ) on $J)$ and let $H \subset J$ be a G_δ. Then $\operatorname{var}[\varphi; H] = \lim_{n\to\infty} \operatorname{var}[\varphi; G_n]$, where G_n is a non-increasing sequence of open subsets in J, $\bigcap_n G_n = H$. It is easily seen that

$$(\zeta \in \varphi(J),\ N_\varphi(\zeta; J) < +\infty) \Rightarrow N_\varphi(\zeta; G_n) \searrow N_\varphi(\zeta; H) \quad \text{as} \quad n \to \infty,$$

whence

$$\lim_{n\to\infty} \int_{\varphi(J)} N_\varphi(\zeta; G_n)\, d\lambda(\zeta) = \int_{\varphi(J)} N_\varphi(\zeta; H)\, d\lambda(\zeta)$$

and (40) follows at once.

As a direct corollary we obtain

3·4. Lemma. *Let φ have the same meaning as in* 3·3 *and suppose that* $\operatorname{var}[\varphi; J] < +\infty$. *Then, for $M \subset J$, the following equivalence is true*:

$$\lambda\, \varphi(M) = 0 \Leftrightarrow \operatorname{var}[\varphi; M] = 0.$$

Proof. If $\operatorname{var}[\varphi; M] = 0$ then there is a G_δ-set $H \supset M$ $(H \subset \langle a, b\rangle)$ such that $\operatorname{var}[\varphi; H] = 0$. By 3·3 (note that $N_\varphi(\zeta; H) \geqq 1$ provided $\zeta \in \varphi(H)$), $\operatorname{var}[\varphi; H] = \int_{\varphi(J)} N_\varphi(\zeta; H)\, d\lambda(\zeta) \geqq \lambda\, \varphi(H) \geqq \lambda\, \varphi(M)$, whence $\lambda\, \varphi(M) = 0$.

Conversely, let $\lambda\, \varphi(M) = 0$. Then there is a G_δ-set $L \supset \varphi(M)$ with $\lambda L = 0$. Put $H = \varphi^{-1}(L)$; clearly, H is a G_δ and $H \supset M$. Since $N_\varphi(\zeta; H) = 0$ for $\zeta \notin L$ we obtain

$$0 = \int_{\varphi(J)} N_\varphi(\zeta; H)\, d\lambda(\zeta) = \operatorname{var}[\varphi; H] \geqq \operatorname{var}[\varphi; M]$$

so that $\operatorname{var}[\varphi; M] = 0$.

3·5. Notation. As in § 2, we shall assume throughout that $\psi = \psi_1 + i\psi_2$ (ψ_1, ψ_2 real-valued) is a path on $\langle a, b\rangle$,

$$\operatorname{var}[\psi; \langle a, b\rangle] < +\infty, \quad \psi(\langle a, b\rangle) = C.$$

On account of 3·4 we obtain the following lemma which could also be derived from a more general theorem due to Г. Е. Перевалов [11].

3·6. Lemma. *For $M \subset \langle a, b \rangle$ the equivalence*

$$\lambda\, \psi_1(M) + \lambda\, \psi_2(M) = 0 \Leftrightarrow \operatorname{var}\,[\psi; M] = 0$$

is true.

Proof.

$\operatorname{var}\,[\psi; M] = 0 \Leftrightarrow \operatorname{var}\,[\psi_1; M] + \operatorname{var}\,[\psi_2; M] = 0 \Leftrightarrow \lambda\, \psi_1(M) + \lambda\, \psi_2(M) = 0$.
We shall also need the following

3·7. Lemma. *Let f be a finite Borel measurable function on $\langle a, b \rangle$ and let σ be a completely finite measure defined on Borel subsets in $\langle a, b \rangle$. Then*

$$\lambda\{y;\, y \in E_1,\, \limsup_{r \to 0+} r^{-1} \sigma f^{-1}(\langle y - r, y + r \rangle) = +\infty\} = 0\,.$$

Proof. Fix an $\varepsilon > 0$. With every $y \in L = \{y;\, y \in E_1,\, \limsup_{r \to 0+} r^{-1} \sigma f^{-1}(\langle y - r, y + r \rangle) = +\infty\}$ we associate the set $\mathscr{R}(y)$ of all $r > 0$ fulfilling

$$r^{-1} \sigma f^{-1}(\langle y - r, y + r \rangle) > \varepsilon^{-1}$$

or, which is the same,

$$r < \varepsilon \sigma f^{-1}(\langle y - r, y + r \rangle)\,.$$

The system $\mathfrak{S}$ of all $\langle y - r, y + r \rangle$ ($y \in L$, $r \in \mathscr{R}(y)$) covers L in the sense of Vitali (cf. [15], chap. X, § 2). Consequently, there exists a sequence $\langle y_1 - r_1, y_1 + r_1 \rangle$, $\langle y_2 - r_2, y_2 + r_2 \rangle, \ldots$ of mutually disjoint intervals belonging to $\mathfrak{S}$ such that

$$\lambda(L - \bigcup_n \langle y_n - r_n, y_n + r_n \rangle) = 0\,.$$

We have thus

$$\lambda L \leqq 2 \sum_n r_n < 2\varepsilon \sum_n \sigma f^{-1}(\langle y_n - r_n, y_n + r_n \rangle) \leqq 2\varepsilon \sigma \langle a, b \rangle\,.$$

Since ε was an arbitrary positive number we conclude that $\lambda L = 0$.

Remark. The above lemma could also be derived from the known fact that every non-decreasing function is almost everywhere differentiable (cf. [8], chap. VIII, § 2; [15], chap. X, § 5).

Now we are able to prove the following

3·8. Proposition. *Given $\zeta \in C$ and $r > 0$ put*

$$G_r^\zeta = \{t;\, t \in \langle a, b \rangle,\, |\psi(t) - \zeta| < r\}\,.$$

Then

$$\lambda\{\zeta;\, \zeta \in C,\, \sup_{r > 0} r^{-1} \operatorname{var}\,[\psi; G_r^\zeta] = +\infty\} = 0\,.$$

Proof. It is well-known that the set-function var $[\psi; M] = \sigma M$, considered on the system of all Borel sets $M \subset \langle a, b \rangle$, is a (completely finite) measure. Given $t \in \langle a; b \rangle$ and $r > 0$ put $U_r^t = \{u; u \in \langle a, b \rangle, |\psi(u) - \psi(t)| < r\}$ $(= G_r^{\psi(t)})$ and denote by Q the set of all $t \in \langle a, b \rangle$ with $+\infty = \limsup_{r \to 0+} r^{-1}$ var $[\psi; U_r^t]$. If $t \in Q$ and $y = \psi_1(t)$ then $U_r^t \subset \psi_1^{-1}(\langle y - r, y + r \rangle)$ so that

$$\sigma \psi_1^{-1}(\langle y - r, y + r \rangle) \geqq \sigma U_r^t = \text{var}\,[\psi; U_r^t]\,,$$

$$\limsup_{r \to 0+} r^{-1} \sigma \psi_1^{-1}(\langle y - r, y + r \rangle) = +\infty\,.$$

We conclude on account of 3·7 that $\lambda\, \psi_1(Q) = 0$. In exactly the same way $\lambda\, \psi_2(Q) = 0$ whence, by 3·4 and 3·6, $\lambda\, \psi(Q) = 0$. We have for every $\zeta \in C - \psi(Q)$

$$\limsup_{r \to 0+} r^{-1}\, \text{var}\,[\psi; G_r^\zeta] < +\infty$$

which means that, for sufficiently small $r_0 > 0$,

$$0 < r < r_0 \Rightarrow r^{-1}\, \text{var}\,[\psi; G_r^\zeta] \leqq \text{const} < +\infty\,;$$

on the other hand,

$$r \geqq r_0 \Rightarrow r^{-1}\, \text{var}\,[\psi; G_r^\zeta] \leqq r_0^{-1}\, \text{var}\,[\psi; \langle a, b \rangle]$$

so that $\sup_{r > 0} r^{-1}$ var $[\psi; G_r^\zeta] < +\infty$. We see that

$$\{\zeta; \zeta \in C, \sup_{r > 0} r^{-1}\, \text{var}\,[\psi; G_r^\zeta] = +\infty\} \subset \psi(Q)$$

and the proof is complete.

3·9. Theorem. *Let G_r^ζ have the same meaning as in* 3·8. *Then, for every $\zeta \in C$ and $r > 0$,*

$$u_r^\psi(\zeta) \leqq \text{var}\,[\psi; G_r^\zeta] \leqq r v_r^\psi(\zeta) + u_r^\psi(\zeta)\,.$$

Proof. Fix a $\zeta \in C$, $r > 0$ and denote by $\mathfrak{S}$ the system of all components of $\{t; t \in G_r^\zeta, \psi(t) \neq \zeta\}$. We have by remark 1·3

$$u_r^\psi(\zeta) = \sum_I \text{var}_t\,[|\psi(t) - \zeta|; I]\,, \quad I \in \mathfrak{S}\,,$$

$$v_r^\psi(\zeta) = \sum_I \text{var}_t\,[\vartheta_\zeta^I(t); I]\,, \quad I \in \mathfrak{S}\,,$$

where $\vartheta_\zeta^I(t)$ stands for a continuous single-valued argument of $\psi(t) - \zeta$ on I. Clearly,

$$\text{var}_t\,[|\psi(t) - \zeta|; I] \leqq \text{var}\,[\psi; I]\,, \quad (I \in \mathfrak{S})$$

whence

$$u_r^\psi(\zeta) \leqq \sum_{I \in \zeta} \mathrm{var}\,[\psi; I] \leqq \mathrm{var}\,[\psi; G_r^\zeta]\,.$$

We have by 3·4

$$\mathrm{var}\,[\psi; \psi^{-1}(\zeta)] = 0\,.$$

Taking into account that, for every $I \in \mathfrak{S}$,

$$\psi(t) = \zeta + |\psi(t) - \zeta| \exp i\,\vartheta_\zeta^I(t)\,, \quad t \in I\,,$$

we conclude that

$$\mathrm{var}\,[\psi; I] \leqq \mathrm{var}_t\,[|\psi(t) - \zeta|; I] + r\,\mathrm{var}_t\,[\exp i\,\vartheta_\zeta^I(t); I] \leqq$$
$$\leqq \mathrm{var}_t\,[|\psi(t) - \zeta|; I] + r\,\mathrm{var}_t\,[\vartheta_\zeta^I(t); I]$$

(cf. also proof of 1·5 in [4]) so that

$$\mathrm{var}\,[\psi; G_r^\zeta] \leqq \mathrm{var}\,[\psi; \psi^{-1}(\zeta)] + \sum \mathrm{var}\,[\psi; I] \leqq u_r^\psi(\zeta) + r\,v_r^\psi(\zeta)$$

and the proof is complete.

As a direct corrolary of 3·8 and 3·9 we obtain

3·10. Theorem.

$$\lambda\{\zeta;\, \zeta \in C,\, \sup_{r>0} r^{-1}\, u_r^\psi(\zeta) = +\infty\} = 0\,.$$

3·11. Remark. We have just proved that the condition (26) is fulfilled almost everywhere (λ) provided ψ is rectifiable. On the other hand, (32) need not be satisfied almost everywhere on C. We shall construct an example of a continuous function f of bounded variation on $\langle 0, 1\rangle$ such that, for the path defined by

$$\psi(t) = t + i f(t)\,, \quad 0 \leqq t \leqq 1\,,$$

the equality $v^\psi(\psi(t)) = +\infty$ holds almost everywhere on $\langle 0, 1\rangle$.

Let us agree to adopt the following

3·12. Notation. If h is a function on $\langle a, b\rangle$, $x, y \in E_1$ and if I is an interval contained in $\langle a, b\rangle$ we put

$$\pi^h[x + iy; I] = \mathrm{var}_t \left[\mathrm{arctg}\, \frac{h(t) - y}{t - x};\quad I \cap (x, b\rangle\right] +$$
$$+ \mathrm{var}_t \left[\mathrm{arctg}\, \frac{h(t) - y}{t - x};\, I \cap \langle a, x)\right].$$

3·13. Remark. If h is continuous on $\langle a, b \rangle$ and

$$\psi(t) = t + i\, h(t)\,, \quad a \leqq t \leqq b\,,$$

then

$$\pi^h[x + i\, h(x); \langle a, b \rangle] = v^{\psi}(\psi(x))\,, \quad a \leqq x \leqq b\,.$$

Indeed, $\vartheta(t) = \operatorname{arctg}\{[h(t) - h(x)]/[t - x]\}$ is a continuous single-valued argument of $\psi(t) - \psi(x)$ on $(x, b\rangle$ and $\pi + \vartheta(t)$ is a continuous single-valued argument of $\psi(t) - \psi(x)$ on $\langle a, x)$ so that, by remark 1·3,

$$v^{\psi}(\psi(x)) = \operatorname{var}[\vartheta; \langle a, x)] + \operatorname{var}[\vartheta; (x, b\rangle] = \pi^h[x + i\, h(x); \langle a, b \rangle]\,.$$

Before going into the construction of the promised example we shall prove the following

3·14. Lemma. *Let h be a continuous function on $\langle a, b \rangle$, $0 < q < 1$ and suppose that*

$$h\left(a + \frac{2j-1}{2n}(b-a)\right) = \frac{2q}{n} \quad (j = 1, \ldots, n)\,, \tag{41}$$

$$h\left(a + \frac{2j}{2n}(b-a)\right) = 0 \quad (j = 0, \ldots, n)\,. \tag{42}$$

Then

$$\pi^h[x + iy; \langle a, b \rangle] \geqq q\, \frac{b-a}{1 + (b-a)^2} \sum_{s=2}^{n} s^{-1}$$

provided $a \leqq x \leqq b$, $0 \leqq y \leqq 2q/n$.

Proof. Put, for the sake of simplicity, $a(s) = a + s(b-a)(2n)^{-1}$ and suppose that $x \in \langle a(2k), a(2k+2)\rangle$. Fix a $j \neq k$, $0 \leqq j < n$. Let us agree to write $\bar{j} = j + 1$ if $a(2j) = x$, $\bar{j} = j$ otherwise. Then $\pi^h[x + iy; \langle a(2j), a(2j+2)\rangle] \geqq$

$$\operatorname{arctg}\left|\frac{h(a(2j+1)) - y}{a(2j+1) - x} - \operatorname{arctg}\frac{h(a(2\bar{j})) - y}{a(2\bar{j}) - x}\right| \geqq$$

$$\geqq \operatorname{arctg}\frac{2qn^{-1} - y}{|a(2j+1) - x|} + \operatorname{arctg}\frac{y}{|a(2\bar{j}) - x|}\,.$$

Noting that

$$\max\left(|a(2j+1) - x|, |a(2\bar{j}) - x|\right) \leqq \left(|j - k| + 1\right)(b-a)\, n^{-1}\,,$$

$$\max\left(y, 2qn^{-1} - y\right) \geqq qn^{-1}\,,$$

we conclude that $\pi^h [x + iy; \langle a(2j), a(2j + 2)\rangle \geqq$

$$\geqq \operatorname{arctg} \frac{q}{(b - a)(|j - k| + 1)} \geqq q \frac{b - a}{1 + (b - a)^2} \frac{1}{|j - k| + 1}.$$

Writing $\sum_j^*$ for the sum extended over $j = 0, \dots, n - 1, j \neq k$, we have

$$\pi^h[x + iy; \langle a, b\rangle] \geqq \sum_j^* \pi^h[x + iy; \langle a(2j), a(2j + 2)\rangle] \geqq$$

$$\geqq q \frac{b - a}{1 + (b - a)^2} \sum_j^* \frac{1}{|j - k| + 1} \geqq q \frac{b - a}{1 + (b - a)^2} \sum_{s=2}^{n} s^{-1}.$$

3·15. Example. We shall denote by φ the well-known Cantor's function on $\langle 0, 1\rangle$ (cf. [15], chap. X, § 5, exerc. 1; [8], chap. VIII, § 2, example on pp. 215–217); φ is continuous and non-decreasing on $\langle 0, 1\rangle$ and remains constant of every component of $\langle 0, 1\rangle - \mathscr{C}$, where $\mathscr{C}$ stands for the Cantor ternary set. Given a positive integer n, $q \in (0, 1)$ and a bounded interval J with end-points $a < b$ we denote by φ_{nq}^J the function defined on $\bar{J} = \langle a, b\rangle$ as follows:

$$\varphi_{nq}^J(a + x) = \frac{2q}{n} \varphi\left(\frac{2n}{b - a}\left(x - 2j \frac{b - a}{2n}\right)\right) \quad \text{for } \frac{2j}{2n}(b - a) \leqq x \leqq \leqq \frac{2j + 1}{2n}(b - a),$$

$$\varphi_{nq}^J(a + x) = \frac{2q}{n} \varphi\left(\frac{2n}{b - a}\left(\frac{2j + 2}{2n}(b - a) - x\right)\right) \quad \text{for } \frac{2j + 1}{2n}(b - a) \leqq \leqq x \leqq \frac{2j + 2}{2n}(b - a),$$

$j = 0, \dots, n - 1$. It is easily seen that φ_{nq}^J is continuous on $\bar{J}$ and has the following properties (A), (B):

(A) $$\max \varphi_{nq}^J = 2qn^{-1},$$

(B) $$\operatorname{var}[\varphi_{nq}^J; J] = 4q.$$

Now we proceed as follows. We put $J = \langle 0, 1\rangle$ and fix a $q > 0$ with

(C_1) $$4q < 2^{-1}.$$

Further we find a positive integer n such that

(D_1) $$q \frac{1}{1 + 1} \sum_{s=2}^{n} s^{-1} > 1.$$

This having been done we put $h_1 = \varphi^J_{nq}$. We denote by $\mathfrak{S}_1$ the system of all components of the interior of $\{x;\ h_1'(x) = 0\}$ and put $\mathfrak{S}_0 = \{(0, 1)\}$. Suppose now that, for a given positive integer k, functions $h_1, \ldots, h_k$ on $\langle 0, 1\rangle$ and systems $\mathfrak{S}_0, \ldots, \mathfrak{S}_k$ consisting each of mutually disjoint open intervals have already been constructed such that

(E) h_k remains constant on every $I \in \mathfrak{S}_k$,

(F) every $I \in \mathfrak{S}_k$ is contained in a $J \in \mathfrak{S}_{k-1}$,

(G) if $I \in \mathfrak{S}_k$ and $I \subset J \in \mathfrak{S}_{k-1}$ then h_k does not assume the value $\sup h_k(J)$ on I.

(This is really the case for $k = 1$.) Next we define h_{k+1} in the following manner. We denote by M_k the union of $\mathfrak{S}_k$ and put $h_{k+1}(x) = 0$ for $x \in \langle 0, 1\rangle - M_k$. Given $I \in \mathfrak{S}_k$, $h_k(I) = \{c\}$, we take $J \in \mathfrak{S}_{k-1}$ with $J \supset I$ and fix a $q > 0$ such that

$$(C_{k+1}) \qquad c + 2q < \sup h_k(J)\,, \quad 4q < 2^{-(k+1)} \,.\, \lambda I\,.$$

Further we fix n large enough to secure

$$(D_{k+1}) \qquad q \frac{\lambda I}{1 + (\lambda I)^2} \sum_{s=2}^{n} s^{-1} > k + 1$$

and define $h_{k+1}(x) = \varphi^I_{nq}(x)$ for $x \in I$. Thus h_{k+1} is defined on $\langle 0, 1\rangle$ and we denote by $\mathfrak{S}_{k+1}$ the system of all components of the interior of $\{x;\ h_{k+1}'(x) = 0\}$. Repeating this procedure infinitely many times we obtain the sequences $\{h_k\}_{k=1}^{\infty}$, $\{\mathfrak{S}_k\}_{k=0}^{\infty}$. Every h_k is continuous and non-negative. We have $\lambda M_k = 1$, $M_{k-1} \supset M_k (k = 1, 2, \ldots)$ so that also $M = \bigcap_k M_k$ has measure 1. We have for every k

$$\sup h_k(\langle 0, 1\rangle) \leqq \operatorname{var}\left[h_k;\ \langle 0, 1\rangle\right] \leqq 2^{-k} \,.\, \sum \lambda I\,, \quad I \in \mathfrak{S}_k$$

(compare (A), (B) and (C_k)). Consequently, the function

$$f = \sum_{k=1}^{\infty} h_k$$

is continuous on $\langle 0, 1\rangle$ and $\operatorname{var}\left[f;\ \langle 0, 1\rangle\right] \leqq \sum_{k=1}^{\infty} 2^{-k} \leqq 1$. Let us fix a point $x \in M$ and denote by $J_l(x)$ the interval of $\mathfrak{S}_l$ containing x. Let k be an arbitrary positive integer. We have for any $s \geqq 0$ (compare (C_{k+s+1}) and (A))

$$h_{k+s}(x) + \sup h_{k+s+1}(J_{k+s}(x)) < \sup h_{k+s}(J_{k+s-1}(x))$$

whence

$$\sum_{s=0}^{p} h_{k+s}(x) + \sup h_{k+p+1}(J_{k+p}(x)) < \sup h_k(J_{k-1}(x))\,.$$

Making $p \to \infty$ we obtain

$$\sum_{s=0}^{\infty} h_{k+s}(x) \leqq \sup h_k(J_{k-1}(x))$$

or, which is the same,

$$f(x) - \sum_{r=1}^{k-1} h_r(x) \leqq \sup h_k(J_{k-1}(x)) . \tag{43}$$

Let $J_{k-1}(x) = (a, b)$. Every h_r with $1 \leqq r < k$ is constant on (a, b), so that $\sum_{r=1}^{k-1} h_r(t) = \alpha$ for every $t \in (a, b)$. Further we have for suitable n, q

$$h_k(t) = \varphi_{nq}^{(a,b)}(t) , \quad t \in (a, b) .$$

Consequently,

$$\sup h_k(J_{k-1}(x)) = 2qn^{-1} , \tag{44}$$

$$h_k\left(a + \frac{2j}{2n}(b - a)\right) = 0 \quad (0 \leqq j \leqq n) ,$$

$$h_k\left(a + \frac{2j + 1}{2n}(b - a)\right) = 2qn^{-1} \quad (0 \leqq j \leqq n - 1) .$$

Since M_k does not contain any of the points

$$a + \frac{2j}{2n}(b - a) \quad (j = 0, \ldots, n) , \quad a + \frac{2j + 1}{2n}(b - a) \quad (j = 0, \ldots, n - 1) ,$$

we have

$$\sum_{p=k+1}^{\infty} h_p\left(a + \frac{2j}{2n}(b - a)\right) = 0 = \sum_{p=k+1}^{\infty} h_p\left(a + \frac{2j + 1}{2n}(b - a)\right) .$$

We see that the function $f - \alpha = \sum_{p=k}^{\infty} h_p = h$ satisfies the assumptions (41), (42) of lemma 3·14. Further, h is continous and $0 \leqq h \leqq 2qn^{-1}$ on $\langle a, b \rangle$ (cf. (43), (44)). By lemma 3·14 we conclude that

$$\pi^h[x + i\,h(x); \langle a, b \rangle] \geqq q \frac{b - a}{1 + (b - a)^2} \sum_{s=2}^{n} s^{-1} .$$

Noting that q, n correspond to h_k we obtain on account of (D_k)

$$\pi^h[x + i\,h(x); \langle a, b \rangle] > k .$$

Since $f = h + \alpha$ on $\langle a, b \rangle$ we have also

$$\pi^f[x + if(x); \langle 0, 1 \rangle] \geqq \pi^f[x + if(x); \langle a, b \rangle] = \pi^h[x + i\,h(x); \langle a, b \rangle] > k .$$

Let us recall that k was an arbitrary positive integer and x was an arbitrary point in M. Consequently,

$$\pi^f[x + i\,f(x); \langle 0, 1\rangle] = +\infty$$

for every $x \in M$.

3·16. Remark. As preceding example shows there are simple rectifiable paths ψ for which such a set $N \subset C(= \psi(\langle a, b\rangle))$ can be assigned that the following conditions (α), (β) take place:

(α) $$\lambda N > 0\,.$$

(β) For every $\eta \in N$ there is a continuous function F on C such that

$$+\infty = \limsup_{z \to \eta} |W^\psi(z, F)|\,, \quad z \in S(\eta)\,,$$

whatever be non-tangential segment $S(\eta) = \{\eta + r \exp i\beta;\, 0 < r < R\}$.

Since

$$W^\psi(z, F) = \operatorname{Im} \int_\psi \frac{F(\zeta)}{\zeta - z}\,d\zeta\,, \quad z \in E_2 - C$$

(F is assumed to be real-valued), a similar remark applies also to the integral

$$\int_\psi \frac{F(\zeta)}{\zeta - z}\,d\zeta$$

itself. For references on Cauchy's type integrals the reader is referred to the work of Н. И. Мусхелишвили [7], И. И. Привалов [12] (red. А. И. Маркушевич), Г. Ц. Тумаркин and С. Я. Хавинсон [16].

Non-tangential limits of the logarithmic potential of the double distribution with a summable density on a curve fulfilling the Ljapunoff condition were studied by W. Nikliborc and W. Stożek in [9], [10].

In connection with investigations concerning potentials of the double distribution in the Euclidean 3-space E_3, a quantity v_P^0 analogous to our $v^\psi(P)$ was introduced by Ю. Д. Бураго, В. Г. Мазья and В. Д. Сапожникова in [1]; in particular, they announced a theorem showing that the boundedness of v_P^0 on the whole of E_3 is sufficient for the existence of limits of the potential with any continuous density (cf. also remarks 2·11–2·13 in [2]).

References

[1] *Ю. Д. Бураго, В. Г. Мазья, В. Д. Сапожникова:* О потенциале двойного слоя для нерегулярных областей. Доклады Ак. наук СССР т. *147* (1962), № 3, 523–524.

[2] *J. Král:* On the logarithmic potential, Comment. Math. Univ. Carolinae *3* (1962), No 1, 3–10.

[3] *J. Král:* On cyclic and radial variations of a plane path. Comment. Math. Univ. Carolinae. *4* (1963), № 1, 3–9.

[4] *J. Král:* Some inequalities concerning the cyclic and radial variations of a plane path-curve. Czech. Math. Journal *14* (89), 1964, 271—280.

[5] *J. Král:* On the logarithmic potential of the double distribution. Czech. Math. Journal *14* (89), 1964, 306–321.

[6] *J. Král:* Poznámka o lineární míře a délce cesty v metrickém prostoru. Acta Univ. Carolinae 1963, № 1, 1–10.

[7] *Н. И. Мусхелишвили:* Синуглярные интегральные уравнения. Москва 1962 (1^{st} ed. appeared also in English: *N. I. Muskhelishvili:* Singular Integral Equations, Groningen 1953).

[8] *I. P. Natanson:* Theorie der Funktionen einer reellen Veränderlichen. Berlin 1954 (И. П. Натансон: Теория функций вещественной переменной).

[9] *W. Nikliborc, W. Stożek:* Sur les potientels logarithmiques des doubles couches. C. R. 1933, t. *197*, p. 808.

[10] *W. Nikliborc, W. Stożek:* Über die Grenzwerte des logarithmischen Potentials der Doppelbelegung. Fundamenta Mathematicae, t. *XXII* (1934), 109–135.

[11] *Г. Е. Перевалов:* О мере множеств, лежащих на плоских континуумах. Сибирский мат. журнал, т. *III* (1962), № 4, 575–581.

[12] *И. И. Привалов:* Граничные свойства аналитических функций (ред. *А. И. Маркушевич*). Москва-Лениград 1950 (*I. I. Priwalow:* Randeigenschaften analytischer Funktionen).

[13] *F. Riesz-B. Sz. Nagy:* Leçons d'Analyse Fonctionelle. Budapest 1952.

[14] *S. Saks:* Theory of the integral. New York 1937.

[15] *R. Sikorski:* Funkcje rzeczywiste I. Warszawa 1958.

[16] *Г. Ц. Тумаркин* и *С. Я. Хавинсон:* Степенные ряды и их обобщения. Проблема моногенности. Граничные свойства. В сборнике: Математика в СССР за сорок лет 1917—1957, Москва 1959.

УГЛОВЫЕ ПРЕДЕЛЬНЫЕ ЗНАЧЕНИЯ ЛОГАРИФМИЧЕСКОГО ПОТЕНЦИАЛА

ЙОСЕФ КРАЛ (Josef Král), Прага

Предположим, что ψ_1, ψ_2 — непрерывные функции с ограниченным изменением на отрезке $\langle a, b\rangle$ и положим $\psi = \psi_1 + i\psi_2$, $C = \psi(\langle a, b\rangle)$. Евклидову плоскость E_2 отождествим с множеством комплексных чисел. Пусть f — непрерывная действительная функция на $\langle a, b\rangle$. Если $z \in E_2$, то обозначим через $\mathfrak{S}$ систему всех компонент множества $\langle a, b\rangle - \psi^{-1}(z)$, для каждого $I \in \mathfrak{S}$ закрепим однозначную непрерывную ветвь $\vartheta_z^I(t)$ аргумента $\psi(t) - z$ на I и положим по определению

$$w^{\psi}(z; f) = \sum_{I \in \mathfrak{S}} \int_I f(t)\, \mathrm{d}\vartheta_z^I(t)$$

в предположении, что имеет смысл сумма интегралов Лебега-Стильтьеса в правой части равенства. Для $z \notin C$, очевидно, $w^{\psi}(z; f) = \operatorname{Im} \int_a^b f(t)/(\psi(t) - z)\,.$ $.\, \mathrm{d}\psi(t)$ всегда существует.

Пусть $\eta \in C$. Для $\alpha \in \langle 0, 2\pi \rangle$ обозначим через $\mu^{\psi}(\alpha; \eta)$ число $(0 \leqq \mu^{\psi}(\alpha; \eta) \leqq +\infty)$ всех $t \in \langle a, b \rangle$, для которых $\psi(t)$ находится на полупрямой $\{\zeta; \zeta = \eta + r \exp i\alpha, r > 0\}$; функция $\mu^{\psi}(\alpha; \eta)$ измерима по Лебегу относительно α, что оправдывает нас полагать

$$v^{\psi}(\eta) = \int_0^{2\pi} \mu^{\psi}(\alpha; \eta) \, d\alpha .$$

Аналогично мы определяем

$$u_r^{\psi}(\eta) = \int_0^r v^{\psi}(\varrho; \eta) \, d\varrho ,$$

где $v^{\psi}(\varrho; \eta)$ равняется числу всех $t \in \langle a, b \rangle$, для которых $\psi(t)$ находится на окружности $\{\zeta; |\zeta - \eta| = \varrho\}$.

Теорема 1. *Пусть* $S = \{z;\ z = \eta + r \exp i\beta,\ 0 < r < R\}$ *и предположим, что имеется* $\delta > 0$ *такое, что*

$$(|\gamma - \beta| < \delta,\ 0 < r < R) \Rightarrow \eta \pm r \exp i\gamma \notin C .$$

Если для каждой непрерывной функции f на $\langle a, b \rangle$ имеет место соотношение

$$+\infty > \limsup_{z \to \eta} |w^{\psi}(z; f)| , \quad z \in S$$

(в частности, если существует конечный предел

$$\lim_{z \to \eta} w^{\psi}(z; f) , \quad z \in S) ,$$

то необходимо

$$v^{\psi}(\eta) < +\infty , \tag{1}$$

$$\sup_{r > 0} r^{-1} u_r^{\psi}(\eta) < +\infty . \tag{2}$$

Теорема 2. *Предположим, что множество* $\psi^{-1}(\eta) = \{t_1 < \ldots < t_n\}$ *конечно. Если имеет место* (1), *то существуют пределы*

$$\tau_{\psi}^{+}(t_k) = \lim_{t \to t_k +} \frac{\psi(t) - \eta}{|\psi(t) - \eta|} \quad \text{для } t_k < b ,$$

$$\tau_{\psi}^{-}(t_k) = \lim_{t \to t_k -} \frac{\psi(t) - \eta}{|\eta - \psi(t)|} \quad \text{для } t_k > a ;$$

положим ещё $\tau_{\psi}^{-}(t_1) = \tau_{\psi}^{+}(a)$ *в случае* $t_1 = a$, $\tau_{\psi}^{+}(t_n) = \tau_{\psi}^{-}(b)$ *в случае* $t_n = b$, $T = \bigcup_{k=1}^{n} \{\tau_{\psi}^{+}(t_k), \tau_{\psi}^{-}(t_k)\}$. *Если, кроме того, имеет место* (2), *то — для каждой* *непрерывной функции f на $\langle a, b\rangle$ — $w^{\psi}(\eta + r\zeta; f)$ стремится при $r \to 0+$ к определенному пределу равномерно относительно $\zeta \in K$ для каждого компактного множества $K \subset \{\zeta; \zeta \in E_2, |\zeta| = 1\} - T$. В случае $n = 1$, $t_1 = a$, $\tau_{\psi}^{+}(a) = \exp i\alpha$ имеет место формула*

(3) $$\lim_{r \to 0+} w^{\psi}(\eta + r \exp i\gamma; f) = w^{\psi}(\eta; f) + f(a) \cdot (\pi + \alpha - \gamma), \quad \alpha < \gamma < \alpha + 2\pi,$$

в случае $n = 1$, $t_1 = b$, $\tau_{\psi}^{-}(b) = \exp i\alpha$ имеет место формула

(4) $$\lim_{r \to 0+} w^{\psi}(\eta + r \exp i\gamma; f) = w^{\psi}(\eta; f) + f(b) \cdot (\gamma - \alpha - \pi), \quad \alpha < \gamma < \alpha + 2\pi,$$

в случае $n = 1$, $a < t_1 < b$, $\tau_{\psi}^{+}(t_1) = \exp i\alpha_{+}$, $\tau_{\psi}^{-}(t_1) = \exp i\alpha_{-}$ $(\alpha_{+} \leqq \alpha_{-} < \alpha_{+} + 2\pi)$ справедливы формулы

(5) $$\lim_{r \to 0+} w^{\psi}(\eta + r \exp i\gamma; f) = w^{\psi}(\eta; f) + f(t_1) \cdot (\pi + \Delta), \quad \alpha_{+} < \gamma < \alpha_{-},$$

(6) $$\lim_{r \to 0+} w^{\psi}(\eta + r \exp i\gamma; f) = w^{\psi}(\eta; f) - f(t_1) \cdot (\pi - \Delta), \quad \alpha_{-} < \gamma < \alpha_{+} + 2\pi,$$

где $\Delta = \pi - (\alpha_{-} - \alpha_{+})$. На основе (3)–(6) *легко вычислить* $\lim\limits_{r \to 0+} w^{\psi}(\eta + r\zeta; f)$ *и в общем случае $n > 1$.*

Обозначая через λ линейную меру Хаусдорфа на C, имеем $\lambda\{\zeta;\ \zeta \in C, \sup\limits_{r>0} r^{-1} u_r^{\psi}(\zeta) = +\infty\} = 0$. Построен пример функции ψ_2 (непрерывной и с ограниченным изменением) на $\langle 0, 1\rangle$ такой, что для $\psi(t) = t + i\,\psi(t)$ справедливо $v^{\psi}(\psi(t)) = +\infty$ для почти всех $t \in \langle 0, 1\rangle$.

Если F — непрерывная функция на C, то соответсвующий логарифмический потенциал двойного слоя

$$W_F^{\psi}(z) = \operatorname{Im} \int_{\psi} \frac{F(\zeta)}{\zeta - z}\, d\zeta$$

сводится к $w^{\psi}(z; \ldots)$ равенством

$$W_F^{\psi}(z) = w^{\psi}(z; f), \quad \text{где} \quad f(t) = F(\psi(t)), \quad a \leqq t \leqq b,$$

и предшествующие теоремы содержат в себе аналогичные утверждения о поведении $W_F^{\psi}(z)$. В частности, если путь ψ оказывается простым, то (1) и (2) являются необходимыми и достаточными условиями для существования угловых предельных значений $W_F^{\psi}(z)$ в точке η для каждой непрерывной функции F на C.