Otto Vejvoda

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PERIODIC SOLUTIONS OF A LINEAR AND WEAKLY NONLINEAR WAVE EQUATION IN ONE DIMENSION, I

OTTO VEJVODA, Praha

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The existence of periodic solutions of a linear wave equation (0.1) or a weakly nonlinear wave equation (0.2) with boundary conditions (0.3) is studied.

Introduction. Many physical phenomena are described by a linear wave equation (in one space dimension)

$$u_{tt} - a^2 u_{xx} = h(t, x) \tag{0.1}$$

(a being a real constant) or by a weakly nonlinear wave equation

$$u_{tt} - a^2 u_{xx} = h(t, x) + \varepsilon f(t, x, u, u_t, u_x, \varepsilon) \tag{0.2}$$

(ε being a small real parameter) with boundary conditions

$$u(t, 0) = u(t, l) = 0 \tag{0.3}$$

and with initial conditions

$$u(0, x) = \varphi(x), \quad u_t(0, x) = \psi(x). \tag{0.4}$$

Now, if $h(t, x)$ and $f(t, x, u, u_t, u_x, \varepsilon)$ are periodic in t with period ω, the question arises if there exists for suitably chosen initial conditions (0.4) an ω-periodic (in t) solution of (0.1), (0.3) or (0.2), (0.3).

By an appropriate choice of units of t and x, $a = 1$ and $l = \pi$ may be attained.

Let $\bar{u}$ be a solution of

$$u_{tt} - u_{xx} = h(t, x), \quad u(t, 0) = u(t, \pi) = 0. \tag{0.5}$$

Then by a substitution $v = u - \bar{u}$, the system of equations

$$\begin{gathered} u_{tt} - u_{xx} = h(t, x) + \varepsilon f(t, x, u, u_t, u_x, \varepsilon), \\ u(t, 0) = u(t, \pi) = 0 \end{gathered} \tag{0.6}$$

is transformed into the system

$$v_{tt} - v_{xx} = \varepsilon f_1(t, x, v, v_t, v_x, \varepsilon), \quad v(t, 0) = v(t, \pi) = 0 . \tag{0.7}$$

In this form, a weakly nonlinear case will be always studied. (If h and f are ω periodic in t and (0.5) has no ω-periodic solution then (0.6) may have evidently no ω-periodic solution depending continuously on ε for small ε.) It will be seen that we must distinguish three different cases: (i) the totally resonance case when $\omega = 2\pi n$, n a natural number (the totally critical case in the terminology of general boundary-value problems in [1]), (ii) the resonance case when $\omega = 2\pi p/q$, p and q natural numbers, $q \neq 1$ (the critical case), (iii) the nonresonance case when $\omega = 2\pi\alpha$, α an irrational number (the noncritical case). In the sequel we shall succeed to treat by the Poincaré method in a satisfactory manner only the cases (i) and (ii). In the case (iii) we are unable to overcome some difficulties arising from the number theory. Hitherto, the case (i) was briefly studied by J. Kurzweil in [2]. (Kurzweil's method besides the existence of a periodic solution reveals its asymptotic stability.) Further, a special problem of the case (ii) was several times investigated by Soviet mathematicians ([3]–[10]).

The bibliography on periodic solutions of some related problems may be found in [1].

In this paper, in paragraph 1 the existence of periodic solutions in the linear case is treated. In paragraph 2 some auxiliary theorems from functional analysis are introduced. In paragraph 3 theorems for a classical and generalized solution of a weakly nonlinear mixed problem given by (0.2), (0.3) and (0.4) are derived. In paragraph 4 the existence of periodic solutions of (0.7) in totally resonance case and resonance case is investigated. In paragraph 5 two particular cases are treated in more detail.

1. PERIODIC SOLUTIONS OF A LINEAR WAVE EQUATION

1.1. Mixed problem. Let the mixed linear problem ($\mathscr{M}$) be given:

$$u_{tt} - u_{xx} = f(t, x), \tag{1.1.1}$$

$$u(t, 0) = u(t, \pi) = 0, \tag{1.1.2}$$

$$u(0, x) = \varphi(x), \quad u_t(0, x) = \psi(x); \tag{1.1.3}$$

let the following conditions be fulfilled:

($\mathscr{A}_1$) the function $f(t, x)$ is of class C^0 in t and of class C^1 in x for $t \in \mathfrak{T} \equiv \langle 0, \infty)$ and $x \in \mathfrak{X} \equiv \langle 0, \pi \rangle$;

($\mathscr{A}_2$) the function $f(t, x)$ satisfies the relations

$$f(t, 0) = f(t, \pi) = 0;$$

$(\mathscr{A}_3)$ the function $\varphi(x)$ is of class C^2 and the function $\psi(x)$ is of class C^1 for $x \in \mathfrak{X}$;

$(\mathscr{A}_4)$ the functions φ and ψ satisfy the relations

$$\varphi(0) = \varphi''(0) = \psi(0) = 0\,, \quad \varphi(\pi) = \varphi''(\pi) = \psi(\pi) = 0\,.$$

(The necessity of $\varphi(0) = \psi(0) = \varphi(\pi) = \psi(\pi) = 0$ for the existence of a classical solution $u^*(t, x)$, i.e. of class C^2 in t and x follows immediately from the equation (1.1.2).)

Let us continue the functions f, φ and ψ in the variable x on the whole x-axis as odd and 2π-periodic functions, i.e.

$$f(t, x) = -f(t, -x) = f(t, x + 2\pi)\,, \tag{1.1.4}$$

$$\varphi(x) = -\varphi(-x) = \varphi(x + 2\pi)\,, \quad \psi(x) = -\psi(-x) = \psi(x + 2\pi)\,. \tag{1.1.5}$$

We shall denote these continued functions by the same symbols f, φ and ψ. Now, we may verify easily that $f(t, x)$ is of class C^0 in t and of class C^1 in x, $\varphi(x) \in C^2$, $\psi(x) \in C^1$ for $t \in \mathfrak{T}$ and $x \in \mathfrak{R} = (-\infty, \infty)$. (Here the necessity of $\varphi''(0) = \varphi''(\pi) = 0$ arises for the continuity of $\varphi''(x)$.) Let us put

$$s(x) = \frac{1}{2}\left[\varphi(x) + \int_0^x \psi(\xi)\,\mathrm{d}\xi + c\right], \tag{1.1.6}$$

where c is a fixed real number. Then

$$s(-x) = \frac{1}{2}\left[-\varphi(x) + \int_0^x \psi(\xi)\,\mathrm{d}\xi + c\right]. \tag{1.1.6'}$$

On the other hand, by (1.1.6) and (1.1.6′) functions $\varphi(x)$ and $\psi(x)$ are uniquely determined as

$$\varphi(x) = s(x) - s(-x)\,, \quad \psi(x) = s'(x) - s'(-x)\,. \tag{1.1.7}$$

Let us note that the set $\mathfrak{S}_2$ of functions s defined by (1.1.6) (where φ and ψ fulfil $(\mathscr{A}_3)$, $(\mathscr{A}_4)$ and (1.1.5)) is the set of all 2π-periodic functions of class C^2. Indeed, the function $s(x)$ defined by (1.1.6) is evidently 2π-periodic and of class C^2. Conversely, $s(x)$ being any 2π-periodic function of class C^2, it may be verified easily that functions φ and ψ defined by (1.1.7) satisfy conditions $(\mathscr{A}_3)$, $(\mathscr{A}_4)$ and (1.1.5).

Theorem 1.1.1. *Let the mixed problem $(\mathscr{M})$ be given. Let the conditions $(\mathscr{A}_1)-(\mathscr{A}_4)$ be fulfilled.*

Then there exists a unique solution $u = u^*(t, x)$ of this problem of class C^2 in x and t for $x \in \mathfrak{X}$ and $t \in \mathfrak{T}$ and this solution is given by the formula

$$u^*(t, x) = s(x + t) - s(-x + t) + \frac{1}{2}\int_0^t \int_{x-t+\vartheta}^{x+t-\vartheta} f(\vartheta, \xi)\,\mathrm{d}\xi\,\mathrm{d}\vartheta\,. \tag{1.1.8}$$

(To emphasize the dependence of the solution (1.1.8) on s we shall often write it in an operator form $U(s)(t, x)$.)

Proof. We find by differentiating (1.1.8)

$$(1.1.9_1)\quad \frac{\partial u^*}{\partial t}(t, x) = s'(x + t) - s'(-x + t) + \frac{1}{2}\int_0^t [f(\vartheta, x + t - \vartheta) + f(\vartheta, x - t + \vartheta)]\, d\vartheta,$$

$$(1.1.9_2)\quad \frac{\partial^2 u^*}{\partial t^2}(t, x) = s''(x + t) - s''(-x + t) + f(t, x) + \frac{1}{2}\int_0^t \left[\frac{\partial f}{\partial x}(\vartheta, x + t - \vartheta) - \frac{\partial f}{\partial x}(\vartheta, x - t + \vartheta)\right] d\vartheta,$$

$$(1.1.9_3)\quad \frac{\partial u^*}{\partial x}(t, x) = s'(x + t) + s'(-x + t) + \frac{1}{2}\int_0^t [f(\vartheta, x + t - \vartheta) - f(\vartheta, x - t + \vartheta)]\, d\vartheta,$$

$$(1.1.9_4)\quad \frac{\partial^2 u^*}{\partial x^2}(t, x) = s''(x + t) - s''(-x + t) + \frac{1}{2}\int_0^t \left[\frac{\partial f}{\partial x}(\vartheta, x + t - \vartheta) - \frac{\partial f}{\partial x}(\vartheta, x - t + \vartheta)\right] d\vartheta,$$

whence (1.1.1, 2, 3) follows immediately. The uniqueness of the solution is a consequence of the energy equality

$$(1.1.10)\quad \int_0^\pi \int_0^t u_t(\vartheta, \xi) f(\vartheta, \xi)\, d\vartheta\, d\xi = \int_0^\pi \int_0^t \{u_t(\vartheta, \xi)\, [u_{tt}(\vartheta, \xi) - u_{xx}(\vartheta, \xi)]\}\, d\vartheta\, d\xi =$$
$$= -\int_0^t [u_t(\vartheta, \xi)\, u_x(\vartheta, \xi)]_0^\pi\, d\vartheta + \frac{1}{2}\int_0^\pi [u_t^2(\vartheta, \xi) + u_x^2(\vartheta, \xi)]_0^t\, d\xi =$$
$$= \frac{1}{2}\int_0^\pi [u_t^2(t, \xi) + u_x^2(t, \xi)]\, d\xi - \frac{1}{2}\int_0^\pi [\psi^2(\xi) + \varphi'^2(\xi)]\, d\xi .$$

Indeed, if there would be two solutions $u_1(t, x)$ and $u_2(t, x)$ of the problem $(\mathscr{M})$ then their difference $v = u_1 - u_2$ would fulfil the equations

$$(1.1.11)\quad v_{tt} - v_{xx} = 0, \quad v(t, 0) = v(t, \pi) = 0, \quad v(0, x) = v_t(0, x) = 0 .$$

Inserting v into (1.1.10) instead of u and the null-function instead of f, φ and ψ, we get

$$0 = \int_0^\pi [v_t^2(t, \xi) + v_x^2(t, \xi)]\, d\xi$$

which yields

$$v_t(t, x) \equiv 0 \equiv v_x(t, x)$$

and in virtue of $v(t, 0) = 0$ also $v(t, x) \equiv 0$. This completes the proof.

Remark 1.1.1. Let us note that $(1.1.9_2)$ shows the necessity of the condition $(\mathscr{A}_2)$. In fact, inserting $x = 0$, $x = \pi$ respectively, into $(1.1.9_2)$ and taking into account that $\partial f/\partial x\,(t, x)$ is even and 2π-periodic in x, we get

$$0 = u_{tt}(t, 0) = f(t, 0)\,, \quad 0 = u_{tt}(t, \pi) = f(t, \pi)\,.$$

(The necessity of $(\mathscr{A}_3)$ if f fulfils $(\mathscr{A}_1)$ is evident from $(1.1.9_2)$ or $(1.1.9_4)$, too.)

1.2. Adjoint boundary-value problem and the Green formula. Let the boundary-value problem $(\mathscr{B})$ with periodic "essential" boundary conditions be given

$$L(u) \equiv u_{tt} - u_{xx} = f(t, x)\,, \tag{1.2.1}$$

$$u(t, 0) = u(t, \pi) = 0\,, \tag{1.2.2}$$

$$u(\omega, x) - u(0, x) = u_t(\omega, x) - u_t(0, x) = 0\,, \tag{1.2.3}$$

where the function $f(t, x)$ fulfils the conditions $(\mathscr{A}_1)$ and $(\mathscr{A}_2)$ and moreover it is ω-periodic in t, i.e.

$$f(t + \omega, x) - f(t, x) = 0\,. \tag{1.2.4}$$

It is clear that under the condition (1.2.4) every solution of the problem $(\mathscr{B})$ is ω-periodic for $t \geqq 0$. The boundary conditions (1.2.2) will be called nonessential while the boundary conditions (1.2.3) will be called essential. (Note that usually only the conditions (1.2.2) are considered and they are simply called boundary conditions.)

As a boundary-value problem $(\mathscr{B}^*)$ adjoint to $(\mathscr{B})$ we define

$$L(v) \equiv v_{tt} - v_{xx} = 0\,, \tag{1.2.5}$$

$$v(t, 0) = v(t, \pi) = 0\,, \tag{1.2.6}$$

$$v(\omega, x) - v(0, x) = v_t(\omega, x) - v_t(0, x) = 0\,. \tag{1.2.7}$$

(The problem $(\mathscr{B})$ is thus self-adjoint.) Evidently,

$$v\,L(u) - u\,L(v) \equiv \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial t}\,, \tag{1.2.8}$$

where

$$P = -\,vu_x + uv_x\,, \quad Q = vu_t - uv_t\,. \tag{1.2.9}$$

Let us now recall the Green formula.

Lemma 1.2.1. *Let $\mathfrak{G}$ be an open domain in the half-plane $-\infty < x < \infty$, $0 \leqq t < \infty$, bounded by a piecewise smooth curve $\mathfrak{C}$. Denote $\overline{\mathfrak{G}} = \mathfrak{G} \cup \mathfrak{C}$. Let $u(t, x)$, $v(t, x)$ be of class C^2 in $\overline{\mathfrak{G}}$. Then it holds*

$$(1.2.10) \quad \iint_{\mathfrak{G}} [v(\vartheta, \xi)\, L(u)\,(\vartheta, \xi) - u(\vartheta, \xi)\, L(v)\,(\vartheta, \xi)]\, \mathrm{d}\vartheta\, \mathrm{d}\xi = \int_{\mathfrak{C}} [P\, \mathrm{d}\vartheta - Q\, \mathrm{d}\xi]\,.$$

Hence, it follows

Lemma 1.2.2. *Let $u(t, x)$, $v(t, x)$ be solutions of $(\mathscr{B})$ and $(\mathscr{B}^*)$, respectively. Then it holds*

$$(1.2.11) \qquad \int_0^\omega \int_0^\pi v(\vartheta, \xi) f(\vartheta, \xi)\, \mathrm{d}\xi\, \mathrm{d}\vartheta = 0\,.$$

Proof. By (1.2.10), (1.2.2), (1.2.3), (1.2.6), (1.2.7)

$$\int_0^\omega \int_0^\pi v(\vartheta, \xi) f(\vartheta, \xi)\, \mathrm{d}\xi\, \mathrm{d}\vartheta =$$

$$= \int_0^\omega [P(\vartheta, \pi) - P(\vartheta, 0)]\, \mathrm{d}\vartheta - \int_0^\pi [-Q(0, \xi) + Q(\omega, \xi)]\, \mathrm{d}\xi = 0\,.$$

Lemma 1.2.2 may be formulated as follows:

Corollary 1.2.1. *The boundary-value problem $(\mathscr{B})$ has a solution only if the function f is orthogonal on the rectangle $0 \leqq x \leqq \pi$, $0 \leqq t \leqq \omega$ to any solution of the adjoint problem $(\mathscr{B}^*)$.*

Remark 1.2.1. The equation (1.2.1) describes some physical phenomena in which other nonessential boundary conditions than those of (1.2.2) may appear more significant, e.g.

$$u(t, 0) = u_x(t, \pi) = 0\,.$$

We find easily that this modified boundary-value problem is again self-adjoint.

1.3. Existence of periodic solutions. Let us now investigate the existence of solutions of the problem $(\mathscr{B})$, i.e. the existence of ω-periodic solutions of (1.2.1), (1.2.2) under the assumption (1.2.4). First, let us somewhat modify the essential boundary conditions (1.2.3). According to (1.2.2) the first condition in (1.2.3) is evidently equivalent to

$$(1.3.1) \qquad u_x(\omega, x) - u_x(0, x) = 0\,.$$

Now substituting the solution (1.1.8) of the mixed problem ($\mathscr{M}$) into ($1.2.3_2$) and (1.3.1) we get

(1.3.2) $$s'(x + \omega) - s'(-x + \omega) + \\ + \frac{1}{2}\int_0^{\omega} [f(\vartheta, x + \omega - \vartheta) + f(\vartheta, x - \omega + \vartheta)]\, d\vartheta - s'(x) + s'(-x) = 0\,,$$

$$s'(x + \omega) + s'(-x + \omega) + \\ + \frac{1}{2}\int_0^{\omega} [f(\vartheta, x + \omega - \vartheta) - f(\vartheta, x - \omega + \vartheta)]\, d\vartheta - s'(x) - s'(-x) = 0\,.$$

Adding and subtracting these two equations we obtain the equivalent system

(1.3.3) $$s'(x + \omega) - s'(x) + \frac{1}{2}\int_0^{\omega} f(\vartheta, x + \omega - \vartheta)\, d\vartheta = 0\,,$$

(1.3.4) $$s'(-x + \omega) - s'(-x) - \frac{1}{2}\int_0^{\omega} f(\vartheta, x - \omega + \vartheta)\, d\vartheta = 0\,.$$

Putting $-x$ instead of x into (1.3.4) and taking into account that $f(\vartheta, -x - \omega + \vartheta) = - f(\vartheta, x + \omega - \vartheta)$ we see that (1.3.4) is a consequence of (1.3.3). Thus, (1.3.2) is equivalent to a single equation (1.3.3). Hence, (1.3.3) *represents a necessary and sufficient condition that* (1.2.1) *with* (1.2.2) *have an ω-periodic solution.* We must now distinguish three cases: (i) $\omega = 2\pi n$, (ii) $\omega = 2\pi p/q$, (iii) $\omega = 2\pi\alpha$, where n, p and $q \neq 1$ are natural numbers and α is an irrational number.

Theorem 1.3.1. *Let the problem ($\mathscr{B}$) be given. Let the following assumptions be fulfilled.*

(i) *The conditions ($\mathscr{A}_1$), ($\mathscr{A}_2$) and (1.2.4) are satisfied.*

(ii) *$\omega = 2\pi n$, n a natural number.*

Then there exist $2\pi n$-periodic solutions of ($\mathscr{B}$) if and only if

(iii) $$\int_0^{2\pi n} f(\vartheta, x - \vartheta)\, d\vartheta = 0\,.$$

These solutions are given by the formula (1.1.8) *where $s(x)$ is any 2π-periodic function of class C^2.*

Proof. According to $s(x + 2\pi n) - s(x) \equiv 0$ the condition (1.3.3) (equivalent to (1.2.3)) is satisfied if and only if (iii) holds. Thus, by Theorem 1.1.1 $u^*(t, x)$ defined by (1.1.8) is a solution of ($\mathscr{B}$) if and only if f satisfies (iii).

Let us show that the assumption (iii) is equivalent to (1.2.11). First, let us evaluate several integrals. Let φ and ψ be functions fulfilling conditions ($\mathscr{A}_3$) and ($\mathscr{A}_4$) and f a function $2\pi n$-periodic in t and fulfilling ($\mathscr{A}_1$), ($\mathscr{A}_2$).

Functions $\varphi(x)$, $\psi(x)$ and $f(t, x)$ be continued onto the whole x-axis as odd and 2π-periodic functions. Denote

$$\Psi(x) = \int_0^x \psi(\xi)\,\mathrm{d}\xi + c \tag{1.3.5}$$

and note that $\Psi(x)$ is even and 2π-periodic. Then

$$\begin{aligned}
&\int_0^{2\pi n}\int_0^{\pi} f(t, x)\,\varphi(x + t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^{2\pi n}\int_t^{t+\pi} f(t, \xi - t)\,\varphi(\xi)\,\mathrm{d}\xi\,\mathrm{d}t = \\
&= \int_0^{\pi}\int_0^{\xi} f(t, \xi - t)\,\varphi(\xi)\,\mathrm{d}t\,\mathrm{d}\xi + \sum_{k=1}^{2n-1}\int_{k\pi}^{(k+1)\pi}\int_{\xi-\pi}^{\xi} f(t, \xi - t)\,\varphi(\xi)\,\mathrm{d}t\,\mathrm{d}\xi + \\
&+ \int_{2\pi n}^{(2n+1)\pi}\int_{\xi-\pi}^{2\pi n} f(t, \xi - t)\,\varphi(\xi)\,\mathrm{d}t\,\mathrm{d}\xi = \\
&= \int_0^{\pi}\Big\{\varphi(\xi)\Big[\int_0^{\xi} f(t, \xi - t)\,\mathrm{d}t + \sum_{k=1}^{n-1}\int_{\xi+(2k-1)\pi}^{\xi+2k\pi} f(t, \xi - t)\,\mathrm{d}t + \\
&+ \int_{\xi+(2n-1)\pi}^{2\pi n} f(t, \xi - t)\,\mathrm{d}t\Big] + \varphi(\xi + \pi)\sum_{k=1}^{n}\int_{\xi+(2k-2)\pi}^{\xi+(2k-1)\pi} f(t, \xi + \pi - t)\,\mathrm{d}t\Big\}\,\mathrm{d}\xi = \\
&= \int_0^{\pi}\varphi(\xi)\Big[\int_0^{\xi} f(t, \xi - t)\,\mathrm{d}t + \sum_{k=1}^{n-1}\int_{\xi+(2k-1)\pi}^{\xi+2k\pi} f(t, \xi - t)\,\mathrm{d}t + \\
&+ \int_{\xi+(2n-1)\pi}^{2\pi n} f(t, \xi - t)\,\mathrm{d}t + \sum_{k=1}^{n}\int_{-\xi+(2k-1)\pi}^{-\xi+2k\pi} f(t, \xi + t)\,\mathrm{d}t\Big]\,\mathrm{d}\xi\,.
\end{aligned} \tag{1.3.6}$$

Similarly

$$\begin{aligned}
&\int_0^{2\pi n}\int_0^{\pi} f(t, x)\,\varphi(x - t)\,\mathrm{d}x\,\mathrm{d}t = \\
&= \int_0^{\pi}\varphi(\xi)\Big[\int_0^{-\xi+\pi} f(t, \xi + t)\,\mathrm{d}t + \sum_{k=1}^{n-1}\int_{-\xi+2k\pi}^{-\xi+(2k+1)\pi} f(t, \xi + t)\,\mathrm{d}t + \\
&+ \int_{-\xi+2\pi n}^{2\pi n} f(t, \xi + t)\,\mathrm{d}t + \sum_{k=1}^{n}\int_{\xi+(2k-2)\pi}^{\xi+(2k-1)\pi} f(t, \xi - t)\,\mathrm{d}t\Big]\,\mathrm{d}\xi\,,
\end{aligned} \tag{1.3.7}$$

$$\begin{aligned}
&\int_0^{2\pi n}\int_0^{\pi} f(t, x)\,\Psi(x + t)\,\mathrm{d}x\,\mathrm{d}t = \\
&= \int_0^{\pi}\Psi(\xi)\Big[\int_0^{\xi} f(t, \xi - t)\,\mathrm{d}t + \sum_{k=1}^{n-1}\int_{\xi+(2k-1)\pi}^{\xi+2k\pi} f(t, \xi - t)\,\mathrm{d}t + \\
&+ \int_{\xi+(2n-1)\pi}^{2\pi n} f(t, \xi - t)\,\mathrm{d}t - \sum_{k=1}^{n}\int_{-\xi+(2k-1)\pi}^{-\xi+2k\pi} f(t, \xi + t)\,\mathrm{d}t\Big]\,\mathrm{d}\xi\,,
\end{aligned} \tag{1.3.8}$$

$$(1.3.9)\quad \int_0^{2\pi n}\int_0^{\pi} f(t,x)\,\Psi(x-t)\,dx\,dt =$$

$$= \int_0^{\pi}\Psi(\xi)\Bigg[\int_0^{-\xi+\pi} f(t,\xi+t)\,dt + \sum_{k=1}^{n-1}\int_{-\xi+2k\pi}^{-\xi+(2k+1)\pi} f(t,\xi+t)\,dt +$$

$$+ \int_{-\xi+2\pi n}^{2\pi n} f(t,\xi+t)\,dt - \sum_{k=1}^{n}\int_{\xi+(2k-2)\pi}^{\xi+(2k-1)\pi} f(t,\xi-t)\,dt\Bigg]\,d\xi\,.$$

Of course any solution of the adjoint boundary-value problem $(\mathscr{B}^*)$ is a linear combination of $\varphi(x+t)+\varphi(x-t)$ and $\Psi(x+t)-\Psi(x-t)$. Hence the condition (1.2.11) by (1.3.6)–(1.3.9) yields

$$(1.3.10_1)\quad \int_0^{2\pi n}\int_0^{\pi} f(t,x)\,[\varphi(x+t)+\varphi(x-t)]\,dx\,dt =$$

$$= \int_0^{\pi}\varphi(x)\Bigg[\int_0^{x} f(t,x-t)\,dt + \sum_{k=1}^{n-1}\int_{x+(2k-1)\pi}^{x+2k\pi} f(t,x-t)\,dt +$$

$$+ \int_{x+(2n-1)\pi}^{2\pi n} f(t,x-t)\,dt + \sum_{k=1}^{n}\int_{-x+(2k-1)\pi}^{-x+2k\pi} f(t,x+t)\,dt + \int_0^{-x+\pi} f(t,x+t)\,dt +$$

$$+ \sum_{k=1}^{n-1}\int_{-x+2k\pi}^{-x+(2k+1)\pi} f(t,x+t)\,dt + \int_{-x+2\pi n}^{2\pi n} f(t,x+t)\,dt +$$

$$+ \sum_{k=1}^{n}\int_{x+(2k-2)\pi}^{x+(2k-1)\pi} f(t,x-t)\,dt\Bigg]\,dx = 0$$

and

$$(1.3.10_2)\quad \int_0^{2\pi n}\int_0^{\pi} f(t,x)\,[\Psi(x+t)+\Psi(x-t)]\,dx\,dt =$$

$$= \int_0^{\pi}\Psi(x)\Bigg[\int_0^{x} f(t,x-t)\,dt + \sum_{k=1}^{n-1}\int_{x+(2k-1)\pi}^{x+2k\pi} f(t,x-t)\,dt +$$

$$+ \int_{x+(2n-1)\pi}^{2\pi n} f(t,x-t)\,dt - \sum_{k=1}^{n}\int_{-x+(2k-1)\pi}^{-x+2k\pi} f(t,x+t)\,dt - \int_0^{-x+\pi} f(t,x+t)\,dt -$$

$$- \sum_{k=1}^{n-1}\int_{-x+2k\pi}^{-x+(2k+1)\pi} f(t,x+t)\,dt - \int_{-x+\pi n2}^{2\pi n} f(t,x+t)\,dt +$$

$$+ \sum_{k=1}^{n}\int_{x+(2k-2)\pi}^{x+(2k-1)\pi} f(t,x-t)\,dt\Bigg]\,dx = 0\,.$$

In virtue of the arbitrariness of φ and Ψ we get readily from (1.3.10) after an elementary procedure

$$\int_0^{x} f(t,x-t)\,dt + \sum_{k=1}^{n-1}\int_{x+(2k-1)\pi}^{x+2k\pi} f(t,x-t)\,dt + \int_{x+(2n-1)\pi}^{2n\pi} f(t,x-t)\,dt +$$

$$+ \sum_{k=1}^{n}\int_{x+(2k-2)\pi}^{x+(2k-1)\pi} f(t,x-t)\,dt = \int_0^{2n\pi} f(t,x-t)\,dt = 0$$

and

$$\int_0^{-x+\pi} f(t, x + t)\,dt + \sum_{k=1}^{n-1} \int_{-x+2k\pi}^{-x+(2k+1)\pi} f(t, x + t)\,dt + \int_{-x+2n\pi}^{2n\pi} f(t, x + t)\,dt +$$

$$+ \sum_{k=1}^{n} \int_{-x+(2k-1)\pi}^{-x+2k\pi} f(t, x + t)\,dt = \int_0^{2n\pi} f(t, x + t)\,dt = 0\,.$$

Since the second of these two equations is equivalent to the first one the assertion is proved.

Thus, Corollary 1.2.1 and Theorem 1.3.1 may be joined to a single

Corollary 1.3.1. *Let the problem* $(\mathscr{B})$ *be given. Let the assumptions* (i) *and* (ii) *of Theorem* 1.3.1 *be fulfilled.*

Then the problem $(\mathscr{B})$ *has a solution if and only if the function* f *is orthogonal to any solution of the adjoint problem* $(\mathscr{B}^*)$ *on the rectangle* $0 \leqq x \leqq \pi$, $0 \leqq t \leqq \leqq 2\pi n$.

Corollary 1.3.2. *Let the problem* $(\mathscr{B})$ *be given. Let the following assumptions be fulfilled: the function* $f(x)$ *is for* $x \in \mathfrak{X}$ *of class* C^1 *and* $f(0) = f(\pi) = 0$.

Then every solution of the problem $(\mathscr{B})$ *with* $f = f(x)$ *is* 2π*-periodic.*

Proof. In fact, we have to prove the assumption (iii) from Theorem 1.3.1. Since $f(x)$ is continued by (1.1.4), we get immediately

$$\int_0^{2\pi} f(x - \vartheta)\,d\vartheta = 0\,.$$

Let us note that some of these solutions may be considered as ω-periodic, where ω is an arbitrary real number – namely those which do not depend on t. We get them by solving the boundary-value problem

$$-\,v''(x) = f(x)\,, \quad v(0) = v(\pi) = 0\,.$$

Theorem 1.3.2. *Let the problem* $(\mathscr{B})$ *be given. Let the following assumptions be fulfilled.*

(i) *The conditions* $(\mathscr{A}_1)$, $(\mathscr{A}_2)$ *and* (1.2.4) *are satisfied.*

(ii) $\omega = 2\pi p/q$, p and $q \neq 1$ *are natural, relatively prime numbers.*

Then $2\pi p/q$*-periodic solutions of the problem* $(\mathscr{B})$ *exist if and only if*

(1.3.11) (iii) $$\int_0^{\omega} \sum_{j=1}^{q} f(\vartheta, x + j\omega - \vartheta)\,d\vartheta \equiv 0\,.$$

These solutions are defined by (1.1.8) *where* $s(x) = s_1(x) + s_2(x)$, $s_1(x)$ *is any* $2\pi/q$*-periodic function of class* C^2 *and*

$$(1.3.12) \qquad s_2(x) = \frac{1}{2q}\int_0^x \int_0^\omega \sum_{j=1}^{q-1} (q - j) f(\vartheta, x + j\omega - \vartheta)\, d\vartheta$$

(*i.e. a particular solution of* (1.3.3)).

Proof. Let us investigate equation (1.3.3) or, writing

$$s'(x) = r(x)\,, \quad -\frac{1}{2}\int_0^\omega f(\vartheta, x + \omega - \vartheta)\, d\vartheta = F(x)\,,$$

equation

$$(1.3.3') \qquad r\left(x + \frac{2\pi p}{q}\right) - r(x) = F(x)\,.$$

Putting successively $x + 2\pi pj/q$, $j = 1, 2, \ldots, q - 1$, instead of x into (1.3.3'), summing all these equations and making use of $s(x + 2\pi p) - s(x) \equiv 0$, we get a neccessary condition for solvability of (1.3.3')

$$(1.3.11') \qquad \sum_{j=0}^{q-1} F\left(x + j\frac{2\pi p}{q}\right) \equiv 0\,.$$

Let us show that this condition is equivalent to

$$(1.3.13) \qquad \int_0^\pi \int_0^\omega f(\vartheta, \xi) \sin lq\xi \cos lq\vartheta\, d\vartheta\, d\xi = 0\,,$$

$$\int_0^\pi \int_0^\omega f(\vartheta, \xi) \sin lq\xi \sin lq\vartheta\, d\vartheta\, d\xi = 0 \qquad (l = 1, 2, \ldots)\,.$$

Indeed, under the stated assumptions the function $f(t, x)$ may be developped into the convergent Fourier series

$$f(t, x) = \sum_{k=1}^{\infty} a_k(t) \sin kx\,, \quad a_k(t) = \frac{2}{\pi}\int_0^\pi f(t, \xi) \sin k\xi\, d\xi\,; \quad k = 1, 2, \ldots\,.$$

Now, by elementary calculations (r being a natural number)

$$(1.3.14) \quad \sum_{j=1}^{r} \sin lq\left(x + j\frac{2\pi p}{q}\right) = r \sin lq\,x\,, \quad \sum_{j=1}^{r} \cos lq\left(x + j\frac{2\pi p}{q}\right) = r \cos lq\,x$$

and for $m \neq lq$,

$$(1.3.15_1) \qquad \sum_{j=1}^{r} \sin m(x + j\omega) = \frac{1}{\sin \frac{m\omega}{2}} \sin m\left(x + \frac{r+1}{2}\omega\right) \sin \frac{mr\omega}{2},$$

$$(1.3.15_2) \qquad \sum_{j=1}^{r} \cos m(x + j\omega) = \frac{1}{\sin \frac{m\omega}{2}} \cos m\left(x + \frac{r+1}{2}\omega\right) \sin \frac{mr\omega}{2}$$

and hence for $m \neq lq$, l a natural number and $\omega = 2\pi p/q$,

$$(1.3.16) \qquad \sum_{j=1}^{q} \sin m(x + j\omega) = 0\,, \quad \sum_{j=1}^{q} \cos m(x + j\omega) = 0\,.$$

This shows by (1.3.11) that the expansion

$$\int_0^{\omega} \sum_{k=1}^{\infty} \sum_{j=1}^{q} a_k(\vartheta) \sin k(x + j\omega - \vartheta)\, d\vartheta$$

must not contain non-zero coefficients with $k = lq$. Since

$$\int_0^{\omega} a_{lq}(\vartheta) \cos lq\,(\omega - \vartheta)\, d\vartheta = \int_0^{\omega} a_{lq}(\vartheta) \cos lq\, \vartheta\, d\vartheta\,,$$

$$\int_0^{\omega} a_{lq}(\vartheta) \sin lq\,(\omega - \vartheta)\, d\vartheta = -\int_0^{\omega} a_{lq}(\vartheta) \sin lq\, \vartheta\, d\vartheta$$

even this is ensured by (1.3.13).

Now, we shall show that the condition (1.3.11′) is sufficient, too.

Denote

$$(1.3.17) \qquad S_k(x) = \sum_{j=0}^{k} F(x + j\omega)\,.$$

By preceding calculations under our assumptions

$$(1.3.18) \qquad S_{q-1}(x) \equiv 0\,.$$

Clearly,

$$(1.3.19) \qquad S_k(x + \omega) = S_{k+1}(x) - S_0(x)\,.$$

Putting

$$(1.3.20) \qquad r(x) = -\frac{1}{q} \sum_{k=0}^{q-2} S_k(x) = -\frac{1}{q} \sum_{k=0}^{q-1} S_k(x)\,,$$

we obtain

$$r(x + \omega) - r(x) = \frac{1}{q} \sum_{0}^{q-2} [-S_k(x + \omega) + S_k(x)] =$$

$$= \frac{1}{q} [-\sum_{k=1}^{q-2} S_k(x) + (q - 1) S_0(x) + \sum_{0}^{q-2} S_k(x)] = F(x)$$

and thus (1.3.20) is a solution of (1.3.3′).

Inserting into (1.3.20) for $S_k(x)$ and $F(x)$ and integrating with respect to x, we get $s_2(x)$. ($f(t, x)$ being odd in x, this integral is again 2π-periodic.) The function $s_1(x) + s_2(x)$, where $s_1(x)$ is the general 2π-periodic solution of the homogeneous equation $s_1'(x + \omega) - s_1'(x) = 0$, is obviously the general solution of (1.3.3) in our case. It is a simple calculation to see that $s_1(x)$ is an arbitrary $2\pi/q$-periodic function of class C^2, which completes the proof.

Corollary 1.3.3. *Let the problem* $(\mathscr{B})$ *be given. Let the assumptions* (i), (ii) *of Theorem* 1.3.2 *be fulfilled.*

Then the problem $(\mathscr{B})$ *has a solution if and only if the function* f *is orthogonal to every solution of the adjoint problem* $(\mathscr{B}^*)$ *on the rectangle* $0 \leqq x \leqq \pi$, $0 \leqq t \leqq \leqq 2\pi p/q$.

Proof. The complete set of orthogonal solutions of $(\mathscr{B}^*)$ is evidently given by

(1.3.21) $$\sin lq\, x \cos lq\, \vartheta\,, \quad \sin lq\, x \sin lq\, \vartheta\,, \quad l = 1, 2, \ldots\,.$$

Thus conditions (1.3.13) express exactly orthogonality of f to (1.3.21) on $\langle 0, \pi \rangle \times \times \langle 0, 2\pi p/q \rangle$.

Finally let us treat briefly the case $\omega = 2\pi\alpha$, α an irrational number. The necessary condition (1.2.11) is satisfied in a trivial way since the unique solution of the adjoint boundary-value problem $(\mathscr{B}^*)$ is evidently $v(t, x) \equiv 0$. In spite of it, the solution of the problem $(\mathscr{B})$ does not always exist. Indeed, let us consider the equation

(1.3.22) $$s'(x + \omega) - s'(x) = -\frac{1}{2} \int_0^{\omega} f(\vartheta, x + \omega - \vartheta)\, d\vartheta = F(x)\,.$$

Since the function $F(x)$ is of class C^1 and 2π-periodic it may be written in the form of a Fourier series

(1.3.23) $$F(x) = \sum_{-\infty}^{\infty} c_k e^{ikx}\,, \quad c_k = \frac{1}{2\pi} \int_0^{2\pi} F(\xi)\, e^{-ik\xi}\, d\xi\,.$$

Since the solution $s(x)$ of (1.3.22) is sought in the class C^2, $s'(x)$ is due to be developable in a Fourier series, too,

(1.3.24) $$s'(x) = \sum_{-\infty}^{\infty} \alpha_k e^{ikx}\,, \quad \alpha_k = \frac{1}{2\pi} \int_0^{2\pi} s'(\xi)\, e^{-ik\xi}\, d\xi\,.$$

Inserting (1.3.23) and (1.3.24) into (1.3.22) we get

$$\alpha_k = \frac{c_k}{e^{ik\omega} - 1}. \tag{1.3.25}$$

By a known theorem from the number theory the expression $e^{ik\omega} - 1$ can be made arbitrarily small for infinitely many k and thus the series (1.3.24), where the α_k are given by (1.3.25), is divergent in general. (The reader may verify easily that the special form of $F(x)$ given by (1.3.22) cannot change anything on this fact. Moreover, the series (1.3.24) with α_k as in (1.3.25) is in general not even $(C, 1)$-summable, see [11].)

Of course, for special classes of irrational numbers α and functions $f(t, x)$ the existence of a $2\pi\alpha$-periodic solution of class C^2 of (1.3.22) may be proved. Thus, e.g. if (a) α is an arbitrary irrational number and $f(t, x)$ is a trigonometrical polynomial in x or (b) α is an algebraic number of degree m and $f(t, x)$ is of class C^{m+3}. In case (a) the assertion is clear (see (1.3.25)). As to (b) by a known theorem to any real irrational algebraic number α of degree m there exists such a constant $c > 0$ that

$$\left|\alpha - \frac{p}{q}\right| \geqq \frac{c}{q^m}, \quad p, q \text{ arbitrary natural numbers}.$$

On the other hand, if $f(t, x)$ is of class C^r in x, then $c_k = O(k^{-r})$ whence the statement follows readily.

2. AUXILIARY THEOREMS OF FUNCTIONAL ANALYSIS

Definition 2.1. A complete linear normed space will be called a Banach space (briefly a B-space).

By a linear operator will be meant an algebraically linear and bounded operator.

The space of all linear operators mapping a B-space $\mathfrak{B}_1$ into a B-space $\mathfrak{B}_2$ will be denoted $[\mathfrak{B}_1 \to \mathfrak{B}_2]$. In the sequel $S(u_0; \varrho, \mathfrak{U})$ (or briefly $S(u_0; \varrho)$ if no confusion may arise) will denote the set of all points u from the B-space $\mathfrak{U}$ such that $\|u - u_0\| < \varrho$. The closure of a set $\mathfrak{M}$ will be denoted $\overline{\mathfrak{M}}$.

Definition 2.2. Let the operator P map an open set $\mathfrak{D}$ of a B-space $\mathfrak{B}_1$ onto a set $\mathfrak{R}$ of a B-space $\mathfrak{B}_2$. Let $u_0 \in \mathfrak{D}$. Let the limit

$$\lim_{t \to 0} \frac{1}{t} [P(u_0 + t\bar{u}) - P(u_0)] = \delta P(u_0; \bar{u}) = P'_u(u_0)(\bar{u}),$$

where $\delta u = \bar{u}$ is any point of $\mathfrak{B}_1$ and $P'_u(u_0) \in [\mathfrak{B}_1 \to \mathfrak{B}_2]$, exist. Then P is said to be $\mathscr{G}$-differentiable at the point u_0, $\delta P(u_0; \bar{u})$ is called a Gâteaux differential of P at u_0 and $P'_u(u_0)$ is called the $\mathscr{G}$-derivative of P at u_0.

Remark 2.1. Partial derivatives $P'_u(u_0, v_0)$ and $P'_v(u_0, v_0)$ at the point (u_0, v_0) of an operator $P(u, v)$ mapping an open set $\mathfrak{D}$ of a direct product $\mathfrak{U} \times \mathfrak{V}$ of two B-spaces $\mathfrak{U}$ and $\mathfrak{V}$ into a B-space $\mathfrak{Z}$ are defined in a quite analogous way.

Lemma 2.1. *Let P_1 map an open set $\mathfrak{D}_1$ of a B-space $\mathfrak{B}_1$ into an open set $\mathfrak{D}_2$ of a B-space $\mathfrak{B}_2$ and let the operator P_2 map $\mathfrak{D}_2$ into a B-space $\mathfrak{B}_3$. Let P_i $(i = 1, 2)$ have a continuous $\mathcal{G}$-derivative P'_i on $\mathfrak{D}_i$. Then the composed operator P_2P_1 has a continuous $\mathcal{G}$-derivative $P'_2P'_1$ on $\mathfrak{D}_1$.*

Theorem 2.1. *Let the equation*

$$P(u, s)(\varepsilon) \equiv -u + L(s) + \varepsilon R(u)(\varepsilon) = 0 \tag{2.1}$$

be given, where $P(u, s)(\varepsilon)$ maps the direct product $\mathfrak{U} \times \mathfrak{S}$ into $\mathfrak{U}$ for every value of the numerical parameter ε from $\mathfrak{E} = \langle 0, \varepsilon_0 \rangle$, $\varepsilon_0 > 0$.

Let $L \in [\mathfrak{S} \to \mathfrak{U}]$. Let $R(u)(\varepsilon)$ be continuous in u and ε and have a $\mathcal{G}$-derivative $R'_u(u)(\varepsilon)$ continuous in u and ε for any $u \in \mathfrak{U}$ and $\varepsilon \in \mathfrak{E}$.

Then to every $\tilde{s} \in \mathfrak{S}$ there exist numbers δ and ε^*, $\delta > 0$, $0 < \varepsilon^* \leqq \varepsilon_0$ such that the equation (2.1) has a unique solution $U(s)(\varepsilon) \in \mathfrak{U}$ for each $s \in S(\tilde{s}; \delta)$ and $\varepsilon \in \langle 0, \varepsilon^* \rangle$. This solution has a $\mathcal{G}$-derivative $U'_s(s)(\varepsilon)$ continuous in s and ε.

Proof. Choose $\tilde{s} \in \mathfrak{S}$. In virtue of the continuity of $R'_u(u)(\varepsilon)$ in u and ε there exist to a prescribed number ϱ numbers η and ε_1, $0 < \varepsilon_1 \leqq \varepsilon_0$ such that

$$\|R'_u(u)(\varepsilon)\| \leqq \varrho \quad \text{for} \quad \|u - L(\tilde{s})\| < \eta \quad \text{and} \quad 0 \leqq \varepsilon \leqq \varepsilon_1 .$$

Let $\varepsilon_2 < 1/\varrho$. According to the continuity of $R(L(s))(\varepsilon)$ in s and ε there exist ε_3, $0 < \varepsilon_3 \leqq \min(\varepsilon_1, \varepsilon_2)$ and $\delta > 0$ such that

$$\left\|\frac{\varepsilon R(L(s))(\varepsilon)}{1 - \varepsilon\varrho}\right\| + \|L(s - \tilde{s})\| \leqq \eta \quad \text{for} \quad s \in S(\tilde{s}; \delta) \quad \text{and} \quad 0 \leqq \varepsilon \leqq \varepsilon_3 .$$

Now, it may be verified that successive approximations defined by

$$U^{(0)}(s) = L(s), \quad U^{(n+1)}(s)(\varepsilon) = L(s) + \varepsilon R(U^{(n)}(s))(\varepsilon)$$

stay in the sphere $S(L(\tilde{s}); \eta)$ and converge to a limit operator $U(s)(\varepsilon) \in \mathfrak{U}$.

Similarly as in [12] it may be shown that the found operator $U(s)(\varepsilon)$ has for $0 \leqq \varepsilon \leqq \varepsilon^*$, $0 < \varepsilon^* \leqq \varepsilon_3$ a continuous $\mathcal{G}$-derivative $(\delta s = \bar{s})$

$$U'_s(s)(\varepsilon)(\bar{s}) = [I - \varepsilon R'_u(U(s)(\varepsilon))(\varepsilon)]^{-1} L(\bar{s}) \tag{2.2}$$

(I being the identical operator).

Since $U^{(n)}(s)(\varepsilon)$ are continuous in ε and converge to $U(s)(\varepsilon)$ uniformly with respect to s and ε, the limit operator $U(s)(\varepsilon)$ is continuous in ε and by (2.2) $U'_s(s)(\varepsilon)(\bar{s})$ is continuous in ε as well.

Theorem 2.2. *Let the equation*

$$G(p)(\varepsilon) = 0 \tag{2.3}$$

be given, where $G(p)(\varepsilon)$ *maps a B-space* $\mathfrak{P}$ *into a B-space* $\mathfrak{Q}$ *for all* $\varepsilon \in \mathfrak{E} = \langle 0, \varepsilon_0 \rangle$, $\varepsilon_0 > 0$. *Let the following assumptions be fulfilled.*

(i) *The equation*

$$G(p_0)(0) = 0 \tag{2.4}$$

has a solution $p_0 = p_0^* \in \mathfrak{P}$.

(ii) *The operator* $G(p)(\varepsilon)$ *is continuous in* p *and* ε *and has a* $\mathscr{G}$*-derivative* $G_p'(p)(\varepsilon)$ *continuous in* p *and* ε *for* $p \in S(p_0^*; \delta)$ $(\delta > 0$ *being a suitably chosen number such that* $S(p_0^*; \delta) \subset \mathfrak{P})$ *and* $\varepsilon \in \mathfrak{E}$.

(iii) *There exists*

$$H = [G_p'(p_0^*)(0)]^{-1} \in [\mathfrak{Q} \to \mathfrak{P}] .$$

Then there exists $\varepsilon^* > 0$ *such that the equation* (2.3) *has for* $0 \leqq \varepsilon \leqq \varepsilon^*$ *a unique solution* $p = p^*(\varepsilon) \in \mathfrak{P}$, *continuous in* ε *such that* $p^*(0) = p_0^*$.

Proof. Define the successive approximations by

$$p^{(0)} = p_0^* ,$$
$$p^{(n+1)}(\varepsilon) = p^{(n)}(\varepsilon) - H[G(p^{(n)}(\varepsilon))(\varepsilon)] .$$

Then by the assumption (ii) and (iii)

$$\|p^{(n)} - p^{(n-1)} - H[G(p^{(n)})(\varepsilon) - G(p^{(n-1)})(\varepsilon)]\| =$$
$$= \left\| p^{(n)} - p^{(n-1)} - H\left[\int_0^1 G_p'(p^{(n-1)} + \alpha(p^{(n)} - p^{(n-1)}))(\varepsilon)\, d\alpha (p^{(n)} - p^{(n-1)})\right] \right\| \leqq$$
$$\leqq \|H\| \cdot \left\| \int_0^1 [G_p'(p_0^*)(0) - G_p'(p^{(n-1)} + \alpha(p^{(n)} - p^{(n-1)}))(\varepsilon)]\, d\alpha \right\| \cdot \|p^{(n)} - p^{(n-1)}\| .$$

Again by (ii) choose numbers δ_1, $0 < \delta_1 < \delta$ and ε_1, $0 < \varepsilon_1 \leqq \varepsilon_0$ so that

$$\|H\| \cdot \|[G_p'(p_0^*)(0) - G_p'(p)(\varepsilon)\| \leqq k < 1$$

for $p \in S(p_0^*; \delta_1)$ and $0 \leqq \varepsilon \leqq \varepsilon_1$.

Now, choose ε^*, $0 < \varepsilon^* \leqq \varepsilon_1$ so that

$$\|H\| \cdot \|G(p_0^*)(\varepsilon)\| \leqq (1 - k)\,\delta_1 .$$

Then, it may be easily verified that all approximations $p^{(n)}(\varepsilon)$ stay for $0 \leqq \varepsilon \leqq \varepsilon^*$ in the sphere $S(p_0^*; \delta_1)$ and converge to an element $p^*(\varepsilon) \in \mathfrak{P}$ uniformly with respect to ε, which completes the proof.

Remark 2.2. If the operators $R(u)(\varepsilon)$, $G(p)(\varepsilon)$ are analytic in both arguments in a suitable region, it may be found (similarly as in [12]) that the solutions whose existence has been proved in Theorems 2.1 and 2.2, are analytic in ε, too.

Lemma 2.2. *Let the linear operator L map a B-space $\mathfrak{P}$ into a B-space $\mathfrak{Q}$. Then a linear inverse operator L^{-1} exists if and only if:*

(i) *the equation*

$$L(p) = q \tag{2.5}$$

has a unique solution $p = P(q) \in \mathfrak{P}$ for arbitrary $q \in \mathfrak{Q}$,

(ii) *there exists a constant $k > 0$ such that $\|P(q)\| \leqq k\|q\|$.* (Cf. [12].)

3. EXISTENCE OF A SOLUTION OF A WEAKLY NONLINEAR MIXED PROBLEM

3.1. Classical solutions. Let the mixed problem $(\mathcal{M})$

$$u_{tt} - u_{xx} = \varepsilon f(t, x, u, u_t, u_x, \varepsilon)\,, \tag{3.1.1}$$

$$u(t, 0) = u(t, \pi) = 0\,, \tag{3.1.2}$$

$$u(0, x) = \varphi(x),\ u_t(0, x) = \psi(x) \tag{3.1.3}$$

be given. Choose a fixed number $T > 0$ and denote

$$\mathfrak{T} = \langle 0, T\rangle,\ \mathfrak{X} = \langle 0, \pi\rangle,\ \mathfrak{R} = (-\infty, \infty) \quad \text{and} \quad \mathfrak{E} = \langle 0, \varepsilon_0\rangle,\ \varepsilon_0 > 0\,.$$

Further denote $u = u_0$, $u_t = u_1$, $u_x = u_2$ when it will be useful. Suppose the following conditions be fulfilled.

$(\mathscr{C}_1)$ The function $f(t, x, u_0, u_1, u_2, \varepsilon)$ is on the interval $\mathfrak{M} = \mathfrak{T} \times \mathfrak{X} \times \mathfrak{R}^3 \times \mathfrak{E}$ together with its partial derivatives

$$\frac{\partial f}{\partial x},\ \frac{\partial f}{\partial u_i},\ \frac{\partial^2 f}{\partial x\, \partial u_i},\ \frac{\partial^2 f}{\partial u_i\, \partial u_j},\quad i, j = 0, 1, 2\,, \tag{3.1.4}$$

continuous in all its variables.

$(\mathscr{C}_2)$ The function $f(t, x, u_0, u_1, u_2, \varepsilon)$ satisfies the equalities $f(t, 0, 0, 0, u_2, \varepsilon) = f(t, \pi, 0, 0, u_2, \varepsilon) = 0$.

$(\mathscr{C}_3)$ The function $\varphi(x)$ is of class C^2 and the function $\psi(x)$ is of class C^1 for $x \in \mathfrak{X}$.

$(\mathscr{C}_4)$ The functions φ and ψ satisfy the equalities

$$\varphi(0) = \varphi''(0) = \psi(0) = 0\,,\quad \varphi(\pi) = \varphi''(\pi) = \psi(\pi) = 0\,.$$

Now, continue the functions f, φ and ψ in the variable x from the interval $\mathfrak{X}$ onto the interval $\mathfrak{R}$ by the relations

$$\begin{aligned} f(t, x, u_0, u_1, u_2, \varepsilon) &= -f(t, -x, -u_0, -u_1, u_2, \varepsilon) = \\ &= f(t, x + 2\pi, u_0, u_1, u_2, \varepsilon)\,, \end{aligned} \tag{3.1.5}$$

$$\varphi(x) = -\varphi(-x) = \varphi(x + 2\pi)\,,\quad \psi(x) = -\psi(-x) = \psi(x + 2\pi)\,. \tag{3.1.6}$$

For functions continued in this way keep the same notation, i.e. f, φ and ψ. Let us denote $\mathfrak{M}^*$ the set of points $(t, x, u_0, u_1, u_2, \varepsilon)$ such that for $x \neq n\pi$ (n being an integer) there is $t \in \mathfrak{T}$, $u_0, u_1, u_2 \in \mathfrak{R}$, $\varepsilon \in \mathfrak{E}$ and for $x = n\pi$ there is $t \in \mathfrak{T}$, $u_0 = u_1 = 0$, $u_2 \in \mathfrak{R}$, $\varepsilon \in \mathfrak{E}$. Evidently, by $(\mathscr{C}_2)$ the continued function f is on $\mathfrak{M}^*$ up to points $(t, n\pi, 0,0\, u_2, \varepsilon)$ of the same class in all variables as the original function f on $\mathfrak{M}$ and on the whole set $\mathfrak{M}^*$ it is of class C^0 in t and ε, of class C^1 in x, u_0, u_1 and u_2. Further, by $(\mathscr{C}_4)$ the continued functions φ or ψ, respectively, are on $\mathfrak{R}$ of class C^2 or C^1, respectively.

Denoting again

$$s(x) = \frac{1}{2}\left[\varphi(x) + \int_0^x \psi(\xi)\,\mathrm{d}\xi + c\right], \tag{3.1.7}$$

where c is an arbitrary number, we have

$$s(-x) = \frac{1}{2}\left[-\varphi(x) + \int_0^x \psi(\xi)\,\mathrm{d}\xi + c\right], \quad s(x + 2\pi) = s(x) \tag{3.1.8}$$

and

$$\varphi(x) = s(x) - s(-x)\,, \tag{3.1.9$_1$}$$

$$\psi(x) = s'(x) - s'(-x)\,. \tag{3.1.9$_2$}$$

Using the result of sec. 1.1 we find easily that every solution of the problem $(\mathscr{M})$ satisfies the integro-differential equation

$$\begin{aligned} u(t, x) = {} & s(x + t) - s(-x + t) + \\ & + \frac{1}{2}\,\varepsilon \int_0^t \int_{x-t+\vartheta}^{x+t-\vartheta} f(\vartheta, \xi, u(\vartheta, \xi), u_t(\vartheta, \xi), u_x(\vartheta, \xi), \varepsilon)\,\mathrm{d}\xi\,\mathrm{d}\vartheta \end{aligned} \tag{3.1.10}$$

and on the other hand it may be easily verified that every solution of (3.1.10) which is of class C^2 in t and x is a solution of $(\mathscr{M})$, too. Indeed, the fact that every such solution of (3.1.10) fulfils (3.1.1) and 3.1.3) is found immediately. With help of Remark 3.1.1 it may be proved easily that it satisfies (3.1.2), too.

We shall seek the solution of (3.1.10) in the B-space $\mathfrak{U}_2$ of functions $u(t, x)$ which are of class C^2 on $\mathfrak{T} \times \mathfrak{R}$. The norm in $\mathfrak{U}_2$ is given by

$$\|u\| = \sup_{t\in\mathfrak{T},\ x\in\mathfrak{R}} \left(|u|, |u_t|, |u_x|, \quad |u_{tt}|, |u_{tx}|, |u_{xx}|\right). \tag{3.1.11}$$

Further, let us denote $\mathfrak{S}_2$ the B-space of functions $s(x)$ which are 2π-periodic and of class C^2. The norm in $\mathfrak{S}_2$ is given by

$$\|s\| = \max_{0\leqq x\leqq 2\pi} \left(|s(x)|, |s'(x)|, |s''(x)|\right). \tag{3.1.12}$$

Let us prove the following

Theorem 3.1.1. *Let the problem* $(\mathcal{M})$ *be given. Let the conditions* $(\mathcal{C}_1)-(\mathcal{C}_4)$ *be fulfilled.*

Then being given a function $\tilde{s} \in \mathfrak{S}_2$ *and a number* $T > 0$, *there exist numbers* $\delta > 0$ *and* ε^*, $0 < \varepsilon^* \leqq \varepsilon_0$ *such that the problem* $(\mathcal{M})$ *for* $0 \leqq \varepsilon \leqq \varepsilon^*$ *and for all* $s \in S(\tilde{s}; \delta; \mathfrak{S}_2)$ *has a unique solution* $u^*(\varepsilon)(t, x) = U(s)(\varepsilon)(t, x) \in \mathfrak{U}_2$. *The operator* U *is continuously* $\mathcal{G}$-*differentiable in* s *and continuous in* ε *while*

$$u^*(0)(t, x) = U(s)(0)(t, x) = s(x + t) - s(-x + t). \tag{3.1.13}$$

Proof. Write the equation (3.1.10) in the form

$$P(u, s)(\varepsilon) = 0 \tag{3.1.10'}$$

where the operator P is defined by

$$P(u, s)(\varepsilon)(t, x) = -u(t, x) + s(x + t) - s(-x + t) + \\ + \frac{1}{2}\varepsilon \int_0^t \int_{x-t+\vartheta}^{x+t-\vartheta} f(\vartheta, \xi, u(\vartheta, \xi), u_t(\vartheta, \xi), u_x(\vartheta, \xi), \varepsilon)\, d\xi\, d\vartheta. \tag{3.1.10''}$$

Let us show that the equation (3.1.10′) fulfils the assumptions of Theorem 2.1. First, let us verify in detail that P maps $\mathfrak{U}_2 \times \mathfrak{S}_2$ into $\mathfrak{U}_2$. Indeed, denoting

$$v(t, x) = P(u, s)(\varepsilon)(t, x) \tag{3.1.14}$$

and

$$F(u)(\varepsilon)(t, x) = f(t, x, u, u_t, u_x, \varepsilon) \tag{3.1.15}$$

(if no confusion can arise we shall write briefly $F(u)(t, x)$ instead of $F(u)(\varepsilon)(t, x)$), we have

$$v_t(t, x) = -u_t(t, x) + s'(x + t) - s'(-x + t) + \\ + \frac{1}{2}\varepsilon \int_0^t [F(u)(\vartheta, x + t - \vartheta) + F(u)(\vartheta, x - t + \vartheta)]\, d\vartheta, \\ v_x(t, x) = -u_x(t, x) + s'(x + t) + s'(-x + t) + \\ + \frac{1}{2}\varepsilon \int_0^t [F(u)(\vartheta, x + t - \vartheta) - F(u)(\vartheta, x - t + \vartheta)]\, d\vartheta, \tag{3.1.16}$$

$$v_{tt}(t, x) = -u_{tt}(t, x) + s''(x + t) - s''(-x + t) + \varepsilon F(u)(t, x) + \\ + \frac{1}{2}\varepsilon \int_0^t [F_x(u)(\vartheta, x + t - \vartheta) - F_x(u)(\vartheta, x - t + \vartheta)]\, d\vartheta, \tag{3.1.17}$$

$$v_{tx}(t, x) = -u_{tx}(t, x) + s''(x + t) + s''(-x + t) +$$
$$+ \frac{1}{2} \varepsilon \int_0^t [F_x(u)(\vartheta, x + t - \vartheta) + F_x(u)(\vartheta, x - t + \vartheta)] \, d\vartheta ,$$
$$v_{xx}(t, x) = -u_{xx}(t, x) + s''(x + t) - s''(-x + t) +$$
$$+ \frac{1}{2} \varepsilon \int_0^t [F_x(u)(\vartheta, x + t - \vartheta) - F_x(u)(\vartheta, x - t + \vartheta)] \, d\vartheta .$$

where of course

$$F_x(u)(t, x) = \frac{\partial f}{\partial x}(t, x, u_0, u_1, u_2 \, \varepsilon) + \sum_{i=0}^{2} \frac{\partial f}{\partial u_i}(t, x, u_0, u_1, u_2, \varepsilon) \frac{\partial u_i}{\partial x}(t, x) .$$

(Observe that all integrals in (3.1.10, 16, 17) are actually continuous in t and x on $\mathfrak{T} \times \mathfrak{R}$).

Putting

(3.1.18) $$L(s)(t, x) = s(x + t) - s(-x + t) ,$$

(3.1.19) $$R(u)(\varepsilon)(t, x) = \frac{1}{2} \int_0^t \int_{x-t+\vartheta}^{x+t-\vartheta} F(u)(\varepsilon)(\vartheta, \xi) \, d\xi \, d\vartheta$$

there is $L(s) \in [\mathfrak{S}_2 \to \mathfrak{U}_2]$ and $R(u)(\varepsilon)$ is continuous in u and ε for all $u \in \mathfrak{U}_2$ and $\varepsilon \in \mathfrak{E}$. Further, the $\mathscr{G}$-derivative $R'_u(u)(\varepsilon)$ given by ($\delta u = \bar{u}$)

(3.1.20) $$R'_u(u)(\varepsilon)(\bar{u}) = \frac{1}{2} \int_0^t \int_{x-t+\vartheta}^{x+t-\vartheta} F'_u(u)(\varepsilon)(\bar{u})(\vartheta, \xi) \, d\xi \, d\vartheta =$$
$$= \int_0^t \int_{x-t+\vartheta}^{x+t-\vartheta} \sum_{i=0}^{2} \frac{\partial f}{\partial u_i}(\vartheta, \xi, u_0(\vartheta, \xi), u_1(\vartheta, \xi), u_2(\vartheta, \xi), \varepsilon) \, \bar{u}_i(\vartheta, \xi) \, d\xi \, d\vartheta$$

is continuous in u and ε in virtue of ($\mathscr{C}_1$). This completes the proof of our theorem.

Remark 3.1.1. Note that the solution of (3.1.10) consequently to the way of continuation of functions f, φ and ψ, is in the variable x odd and 2π-periodic, i.e.

(3.1.21) $$u^*(t, x) = - u^*(t, -x) = u^*(t, x + 2\pi)$$

and of course

(3.1.22) $$u_t^*(t, x) = - u_t^*(t, -x) = u_t^*(t, x + 2\pi) ,$$
$$u_x^*(t, x) = u_x^*(t, -x) = u_x^*(t, x + 2\pi) .$$

In fact, inserting $- u^*(t, -x)$, $u^*(t, x + 2\pi)$ into (3.1.10) we find easily that these functions are also solutions of this equation and in virtue of the uniqueness of the solution of (3.1.10) the assertion follows readily.

Denote $\tilde{\mathfrak{U}}_2$ and $\tilde{\mathfrak{S}}_2$ subspaces of $\mathfrak{U}_2$ and $\mathfrak{S}_2$, respectively. (In general, $\tilde{\mathfrak{C}}$ will always denote a subspace of a space $\mathfrak{C}$.)

Remark 3.1.2. In paragraph 4 it will turn out that it is often useful to consider only some subspaces $\tilde{\mathfrak{U}}_2$ and $\tilde{\mathfrak{S}}_2$ of $\mathfrak{U}_2$ and $\mathfrak{S}_2$, respectively. Let us note that if $\tilde{s} \in \tilde{\mathfrak{S}}_2$ and the operator $P(u, s)(\varepsilon)$ defined in (3.1.10″) maps $\tilde{\mathfrak{U}}_2 \times \tilde{\mathfrak{S}}_2$ into $\tilde{\mathfrak{U}}_2$ for every $\varepsilon \in \mathfrak{E}$, then again by Theorem 2.1

$$U(s)(\varepsilon)(t, x) \in \tilde{\mathfrak{U}}_2$$

for every $s \in S(\tilde{s}; \delta; \tilde{\mathfrak{S}}_2)$ and for every $\varepsilon \in \mathfrak{E}$.

Remark 3.1.3. The mixed problem $(\mathcal{M}')$ which differs from $(\mathcal{M})$ in equation (3.1.1) being replaced by

$$u_{tt} - u_{xx} = g(t, x) + \varepsilon f(t, x, u_t, u_x, \varepsilon), \tag{3.1.1'}$$

where the function $g(t, x)$ is of class C^0 in $t \in \mathfrak{T}$ and of class C^1 in $x \in \mathfrak{X}$ and $g(t, 0) = g(t, \pi) = 0$, may be carried over to the problem $(\mathcal{M})$ by introducing a new unknown w by putting

$$w = u - \frac{1}{2}\int_0^t \int_{x-t+\vartheta}^{x+t-\vartheta} g(\vartheta, \xi)\, d\xi\, d\vartheta . \tag{3.1.23}$$

Theorem 3.1.2. *Let the problem $(\mathcal{M})$ be given. Let besides the conditions $(\mathcal{C}_1)-(\mathcal{C}_4)$ the assumption*

$(\mathcal{A})$ The function $f(t, x, u_0, u_1, u_2, \varepsilon)$ is analytic in u_i $(i = 0, 1, 2)$ and ε, be fulfilled.

Then the solution $u^*(\varepsilon)(t, x) = U(s)(\varepsilon)(t, x)$ of $(\mathcal{M})$ whose existence is ensured by Theorem 3.1.1 is analytic in ε. Writing $u = \sum_0^\infty \varepsilon \overset{(n)}{u}$, it may be found by solving a recursive system of equations

$$\overset{(0)}{u}_{tt} - \overset{(0)}{u}_{xx} = 0,\ \overset{(0)}{u}(t, 0) = \overset{(0)}{u}(t, \pi) = 0,\ \overset{(0)}{u}(0, x) = \varphi(x),\ \overset{(0)}{u}_t(0, x) = \psi(x), \tag{3.1.24$_0$}$$

$$\overset{(n)}{u}_{tt} - \overset{(n)}{u}_{xx} = F_n(t, x, \overset{(0)}{u}, \ldots, \overset{(n-1)}{u}, \overset{(0)}{u}_t, \ldots, \overset{(n-1)}{u}_t, \overset{(0)}{u}_x, \ldots, \overset{(n-1)}{u}_x, \varepsilon), \tag{3.1.24$_n$}$$

$$\overset{(n)}{u}(t, 0) = \overset{(n)}{u}(t, \pi) = 0,\ \overset{(n)}{u}(0, x) = \overset{(n)}{u}_t(0, x) = 0,\ n = 1, 2, \ldots .$$

The proof may be performed in accordance with Remark 2.2.

Remark 3.1.4. In $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$ the requirements on the function f may be restricted only on the set $t \in \mathfrak{T}$, $x \in \mathfrak{X}$, $|u_0 - \tilde{s}(x + t) + \tilde{s}(-x + t)| < \varrho$, $|u_1 - \tilde{s}'(x + t) + \tilde{s}'(-x + t)| < \varrho$, $|u_2 - \tilde{s}'(x + t) - \tilde{s}'(-x + t)| < \varrho$, $\varepsilon \in \mathfrak{E}$, since obviously by choosing δ and ε sufficiently small we may reach that $u^*(t, x)$, $u_t^*(t, x)$ and $u_x^*(t, x)$ stay for $t \in \mathfrak{T}$ in the prescribed sets.

Remark 3.1.5. If we are interested only in proving the existence and uniqueness of a solution $u^*(t, x) \in \mathfrak{U}_2$ of the problem $(\mathcal{M})$, it is sufficient to suppose that $\partial f/\partial x$, $\partial f/\partial u_i$ $(i = 0, 1, 2)$ are Lipschitzian in u_i on $\mathfrak{M}$ instead of supposing the existence of continuous derivatives

$$\frac{\partial^2 f}{\partial x\, \partial u_i}, \quad \frac{\partial^2 f}{\partial u_i\, \partial u_j}.$$

3.2. Generalized solutions. The equation (3.1.10) which is under assumptions stated above equivalent to the problem $(\mathcal{M})$, enables us to define a generalized solution of this problem.

Definition 3.2.1. A function $u(t, x)$ of class C^1 in t and x on $\mathfrak{T} \times \mathfrak{R}$ which satisfies euqation (3.1.10) is called a (1)-generalized solution of $(\mathcal{M})$.

Let us formulate the following set of conditions:

$(\mathscr{C}'_1)$ Function $f(t, x, u_0, u_1, u_2, \varepsilon)$ is together with its partial derivatives $\partial f/\partial u_i$ $(i = 0, 1, 2)$ continuous in all variables on $\mathfrak{T} \times \mathfrak{X} \times \mathfrak{R}^3 \times \mathfrak{E}$.

$(\mathscr{C}'_3)$ $\varphi(x) \in C^1$, $\psi(x) \in C^0$ for $x \in \mathfrak{X}$.

$(\mathscr{C}'_4)$ $\varphi(0) = \psi(0) = 0$, $\varphi(\pi) = \psi(\pi) = 0$.

Denote $\mathfrak{U}_1$ the B-space of functions $u(t, x)$ which are of class C^1 in t and x on $\mathfrak{T} \times \mathfrak{R}$ while the norm is defined by

$$\|u\|_{\mathfrak{A}_1} = \sup_{t\in\mathfrak{T},\ x\in\mathfrak{R}} \left(|u|, \left|\frac{\partial u}{\partial t}\right|, \left|\frac{\partial u}{\partial x}\right|\right). \tag{3.2.1}$$

Denote $\mathfrak{S}_1(\mathfrak{S}_0)$ a B-space of functions $s(x)$ which are of class $C^1(C^0)$ and 2π-periodic in x on $\mathfrak{R}$ with the norm:

$$\|s\|_{\mathfrak{S}_1} = \max_{0\leq x\leq 2\pi} (|s(x)|, |s'(x)|) \tag{3.2.2}$$

$$\left(\|s\|_{\mathfrak{S}_0} = \max_{0\leq x\leq 2\pi} |s(x)|\right).$$

(Clearly, if φ and ψ fulfil $(\mathscr{C}'_3)$ and $(\mathscr{C}'_4)$ then $s(x)$ defined by (3.1.7) belongs to $\mathfrak{S}_1$. On the other hand if $s \in \mathfrak{S}_1$, then φ and ψ given by (3.1.9) fulfil $(\mathscr{C}'_3)$ and $(\mathscr{C}'_4)$.)

Theorem 3.2.1. *Let the problem $(\mathcal{M})$ be given. Let the conditions $(\mathscr{C}'_1)$, $(\mathscr{C}'_3)$, $(\mathscr{C}'_4)$ be fulfilled.*

Then to a given function $\tilde{s} \in \mathfrak{S}_1$ and to a given number $T > 0$ there exist numbers $\delta > 0$ and ε^*, $0 < \varepsilon^* \leq \varepsilon_0$ such that the problem $(\mathcal{M})$ for any $\varepsilon \in \langle 0, \varepsilon^* \rangle$ and for any $s \in S(\tilde{s}; \delta; \mathfrak{S}_1)$ has a unique (1)-generalized solution

$$u^*(\varepsilon)(t, x) = U(s)(\varepsilon)(t, x) \in \mathfrak{U}_1 \quad \textit{on} \quad \mathfrak{T} \times \mathfrak{R}$$

where $U(s)(\varepsilon)$ is continuously $\mathcal{G}$-differentiable in s and continuous in ε while

$$u^*(0)(t, x) = U(s)(0)(t, x) = s(x + t) - s(-x + t).$$

The proof is quite analogous to the proof of Theorem 3.1.1.

Remark 3.2.1. A theorem similar to Theorem 3.1.2 and remarks similar to all those of section 3.1 may be stated for the (1)-generalized solution, too.

4. PERIODIC SOLUTIONS OF A WEAKLY NONLINEAR WAVE EQUATION

4.1. Total resonance. Let the boundary-value problem $(\mathcal{P})$ with periodic boundary conditions be given by equations

$$u_{tt} - u_{xx} = \varepsilon f(t, x, u, u_t, u_x, \varepsilon), \tag{4.1.1}$$

$$u(t, 0) = u(t, \pi) = 0, \tag{4.1.2}$$

$$u(2\pi n, x) - u(0, x) = u_t(2\pi n, x) - u_t(0, x) = 0, \tag{4.1.3}$$

where n is a natural number. Suppose the function f fulfils besides the conditions $(\mathcal{C}_1)$, $(\mathcal{C}_2)$ the condition of $2\pi n$-periodicity in t

$$f(t + 2\pi n, x, u_0, u_1, u_2, \varepsilon) = f(t, x, u_0, u_1, u_2, \varepsilon). \tag{4.1.4}$$

Let us use the same notations as in the foregoing paragraph.

Let φ and ψ be arbitrary fixed functions fulfilling conditions $(\mathcal{C}_3)$ and $(\mathcal{C}_4)$. Take the solution $U(s)(\varepsilon)(t, x)$ of the problem $(\mathcal{M})$ corresponding to these functions φ and ψ (whose existence for $\varepsilon \in \mathfrak{E}_1 = \langle 0, \varepsilon_1 \rangle$ is ensured by Theorem (3.1.1)) and write that this solution satisfies conditions (4.1.3):

$$\begin{aligned} U(s)(\varepsilon)(2\pi n, x) - U(s)(\varepsilon)(0, x) &= 0, \\ U_t(s)(\varepsilon)(2\pi n, x) - U_t(s)(\varepsilon)(0, x) &= 0. \end{aligned} \tag{4.1.5}$$

Making use of the identity

$$\begin{aligned} U(s)(\varepsilon)(t, x) &= s(x + t) - s(-x + t) + \\ &+ \frac{1}{2}\varepsilon \int_0^t \int_{x-t+\vartheta}^{x+t-\vartheta} F(U(s)(\varepsilon)(\vartheta, \xi))(\varepsilon)(\vartheta, \xi)\, d\xi\, d\vartheta \end{aligned} \tag{4.1.6}$$

and repeating the considerations of sec. 1.3 we find easily that equations (4.1.5) are equivalent to a single equation (after dividing the resulting equation by ε and making use of 2π-periodicity of f and $U(s)(\varepsilon)(t, x)$ in x):

$$G(s)(\varepsilon)(x) \equiv \int_0^{2\pi n} F(U(s)(\varepsilon)(\vartheta, x - \vartheta))(\varepsilon)(\vartheta, x - \vartheta)\, d\vartheta = 0. \tag{4.1.7}$$

This is a necessary and sufficient condition for $s(x)$ (and hence, in virtue of (3.1.7) for $\varphi(x)$ and $\psi(x)$) that the solution $U(s)(\varepsilon)(t, x)$ of $(\mathscr{M})$ be a solution of $(\mathscr{P})$. Of course, this condition cannot be verified in practice and thus, we have to replace it by a necessary or by a sufficient condition which are more apt to a practical treatment, as it is usually done in the Poincaré method of a small parameter. Letting $\varepsilon \to 0$ in (4.1.7) we get instantaneously the limit equation

$$(4.1.8) \qquad G(s_0)(0)(x) \equiv \int_0^{2\pi n} F(U(s_0)(0)(\vartheta, x - \vartheta))(0)(\vartheta, x - \vartheta)\, d\vartheta = 0$$

which must have a solution $s_0^*(x) \in \mathfrak{S}_2$ that a solution of $(\mathscr{P})$ exist. Let us prove

Theorem 4.1.1. *Let the problem $(\mathscr{P})$ be given. Let besides the conditions $(\mathscr{C}_1)$, $(\mathscr{C}_2)$ and (4.1.4) the following assumptions be fulfilled:*

(i) *The equation (4.1.8) has a solution $s_0 = s_0^*(x) \in \tilde{\mathfrak{S}}_2$.*

(ii) *There exists the operator*

$$H = [G_s'(s_0^*)(0)]^{-1} \in [\tilde{\mathfrak{S}}_1 \to \tilde{\mathfrak{S}}_2],$$

where $\tilde{\mathfrak{S}}_1 \supset G(\tilde{\mathfrak{S}}_2)(\varepsilon)$.

Then there exists a number $\varepsilon^* > 0$ such that the problem $(\mathscr{P})$ has for any $\varepsilon \in \langle 0, \varepsilon^* \rangle$ a unique solution $U(s^*(\varepsilon))(\varepsilon)(t, x) \in \mathfrak{U}_2$, such that $s^*(0)(x) = s_0^*(x)$, while the function $s^*(\varepsilon)(x) \in \tilde{\mathfrak{S}}_2$ is continuous in ε.

Proof. Let us apply Theorem 2.2 with $p = s$, $\mathfrak{P} = \tilde{\mathfrak{S}}_2$, $\mathfrak{Q} = \tilde{\mathfrak{S}}_1$. Evidently, we have to verify only the assumption (ii) of that theorem. We shall show slightly-more, namely that the assumption (ii) is fulfilled even for $\mathfrak{P} = \mathfrak{S}_2$ and $\mathfrak{Q} = \mathfrak{S}_1$.

In fact, by Theorem 3.1.1 there exist numbers ε_1, $0 < \varepsilon_1 \leqq \varepsilon_0$, and $\delta > 0$ such that the solution $U(s)(\varepsilon)$ of $(\mathscr{M})$ for $0 \leqq \varepsilon \leqq \varepsilon_1$ and $s \in S(s_0^*; \delta)$ is continuous in s and ε and has a $\mathscr{G}$-derivative $U_s'(s)(\varepsilon) \in [\mathfrak{S}_2 \to \mathfrak{U}_2]$ which is also continuous in s and ε. Further, it follows readily that the operator

$$\int_0^{2\pi n} F(u)(\varepsilon)(\vartheta, x - \vartheta)\, d\vartheta$$

has at each point $u \in \mathfrak{U}_2$ a $\mathscr{G}$-differential $(\delta u_i = \bar{u}_i)$

$$\int_0^{2\pi n} \sum_{i=0}^{2} \frac{\partial f}{\partial u_i}(\vartheta, x - \vartheta, u_0(\vartheta, x - \vartheta), u_1(\vartheta, x - \vartheta),$$
$$u_2(\vartheta, x - \vartheta), \varepsilon)\, \bar{u}_i(\vartheta, x - \vartheta)\, d\vartheta \in [\mathfrak{U}_2 \to \mathfrak{S}_1]$$

so that the $\mathscr{G}$-derivative is continuous in u_i and ε. Hence by Lemma 2.1 the operator $G(s)(\varepsilon)$ has a continuous (in s and ε) $\mathscr{G}$-derivative $G_s'(s)(\varepsilon) \in [\mathfrak{S}_2 \to \mathfrak{S}_1]$ for $0 \leqq \varepsilon \leqq \varepsilon_1$ and $s \in S(s_0^*; \delta)$, which completes the proof. (It is easily seen that ε^* may be chosen so that $\|s^*(\varepsilon) - s_0^*\| < \delta$ for $0 \leqq \varepsilon \leqq \varepsilon^*$, δ having the same meaning as in Theorem 3.1.1.)

Theorem 4.1.2. *Let the problem* $(\mathscr{P})$ *be given. Let besides the conditions* $(\mathscr{C}_1')$ *and* (4.1.4) *the following assumptions be fulfilled:*

(i) *The equation* (4.1.8) *has a solution* $s_0 = s_0^*(x) \in \tilde{\tilde{\mathfrak{S}}}_1$.

(ii) *There exists a linear operator*

$$H = [G_s'(s_0^*)(0)]^{-1}$$

which maps $\tilde{\tilde{\mathfrak{S}}}_0$ *onto* $\tilde{\tilde{\mathfrak{S}}}_1$, *while* $G(\tilde{\tilde{\mathfrak{S}}}_1)(\varepsilon) \subset \tilde{\tilde{\mathfrak{S}}}_0$.

Then there exists a number $\varepsilon^* > 0$ *such that* $(\mathscr{P})$ *has a unique* (1)*-generalized solution* $u^*(\varepsilon)(t, x) = U(s^*(\varepsilon))(\varepsilon)(t, x) \in \mathfrak{U}_1$ *such that* $s^*(0)(x) = s_0^*(x)$, *while the function* $s^*(\varepsilon)(x) \in \tilde{\tilde{\mathfrak{S}}}_1$ *is continuous in* ε.

Proof is very similar to the preceding one and may be therefore omitted.

In the two foregoing theorems we supposed that $G(s)(\varepsilon)$ maps $\tilde{\tilde{\mathfrak{S}}}_i$ into $\tilde{\tilde{\mathfrak{S}}}_{i-1} \supset G(\tilde{\tilde{\mathfrak{S}}}_i)(\varepsilon)$, $i = 1, 2$, respectively. This is reasonable only in the cases when $|\partial f/\partial u_1| + |\partial f/\partial u_2| \not\equiv 0$. In the opposite case we must apply one of the following theorems.

First let us formulate the condition:

$(\mathscr{C}_1'')$ The function $f(t, x, u, \varepsilon)$ is together with its partial derivatives

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial u}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x\, \partial u}, \frac{\partial^2 f}{\partial u^2}, \frac{\partial^3 f}{\partial x^2\, \partial u}, \frac{\partial^3 f}{\partial x\, \partial u^2}, \frac{\partial^3 f}{\partial u^3}$$

continuous in all its variables on $\mathfrak{T} \times \mathfrak{X} \times \mathfrak{R} \times \mathfrak{E}$.

Theorem 4.1.3. *Let the problem* $(\mathscr{P})$ *be given with* $f = f(t, x, u, \varepsilon)$. *Let besides the conditions* $(\mathscr{C}_1'')$, $(\mathscr{C}_2)$ *and* (4.1.4) *the following assumptions be fulfilled:*

(i) *The equation* (4.1.8) *has a solution* $s_0 = s_0^*(x) \in \tilde{\tilde{\mathfrak{S}}}_2$.

(ii) *There exists the operator* $H = [G_s'(s_0^*)(0)]^{-1}$ *which maps* $\tilde{\tilde{\mathfrak{S}}}_2$ *onto* $\tilde{\tilde{\mathfrak{S}}}_2$, *where* $\tilde{\tilde{\mathfrak{S}}}_2 \supset G(\tilde{\tilde{\mathfrak{S}}}_2)(\varepsilon)$.

Then there exists $\varepsilon^* > 0$, *such that the problem* $(\mathscr{P})$ *has for any* $\varepsilon \in \langle 0, \varepsilon^* \rangle$ *a unique solution* $u^*(\varepsilon)(t, x) = U(s^*(\varepsilon))(\varepsilon)(t, x) \in \mathfrak{U}_2$, *such that* $s^*(0)(x) = s_0^*(x)$, *while the function* $s^*(\varepsilon) \in \tilde{\tilde{\mathfrak{S}}}_2$ *is continuous in* ε.

Theorem 4.1.4. *Let the problem* $(\mathscr{P})$ *be given with* $f = f(t, x, u, \varepsilon)$. *Let besides the conditions* $(\mathscr{C}_1)$ *and* (4.1.4) *the following assumptions be fulfilled.*

(i) *The equation* (4.1.8) *has a solution* $s_0 = s_0^*(x) \in \tilde{\tilde{\mathfrak{S}}}_1$.

(ii) *There exists the operator* $H = [G'_s(s_0^*)(0)]^{-1}$ *which maps* $\tilde{\tilde{\mathfrak{S}}}_1$ *onto* $\mathfrak{S}_1$, *where* $\tilde{\tilde{\mathfrak{S}}}_1 \supset G(\mathfrak{S}_1)(\varepsilon)$.

Then there exists a number $\varepsilon^* > 0$ *such that the problem* $(\mathscr{P})$ *has a unique* (1)-*generalized solution*

$$u^*(\varepsilon)(t, x) = U(s^*(\varepsilon))(\varepsilon)(t, x) \in \mathfrak{U}_1 \quad \textit{for any} \quad \varepsilon \in \langle 0, \varepsilon^* \rangle\,,$$

such that $s^*(0)(x) = s_0^*(x)$, *while the function* $s^*(\varepsilon)(x) \in \tilde{\mathfrak{S}}_1$ *is continuous in* ε.

Proof. Assumptions in both last theorems are chosen in such a way that the continuous $\mathscr{G}$-differentiability of $G(s)(\varepsilon)$ in corresponding B-spaces be ensured in a neighborhood of $s = s_0^*$, $\varepsilon = 0$. Details may be omitted.

Theorem 4.1.5. *Let the problem* $(\mathscr{P})$ *be given. Let the assumptions of one of Theorems* (4.1.1, 2, 3, 4) *be fulfilled. Further, let the assumption* $(\mathscr{A})$ *of Theorem* 3.1.2 *be fulfilled.*

Then the solution $u^*(\varepsilon)(t, x)$ *of the problem* $(\mathscr{P})$ *as well as the associated function* $s^*(\varepsilon)(x)$ *whose existence and uniqueness for* $0 \leqq \varepsilon \leqq \varepsilon^*$ *is ensured by corresponding theorem are analytic in* ε. *This solution may be determined by finding succesively* $2\pi n$-*periodic solutions* $\overset{(n)}{u}{}^*(t, x)$ (*and thereby the* n-th *factor* $\overset{(n)}{s}(x)$ *in* $s(\varepsilon)(x) = \sum \varepsilon^n \overset{(n)}{s}(x)$) *of the system*

$$(4.1.9) \qquad \overset{(0)}{u}_{tt} - \overset{(0)}{u}_{xx} = 0\,, \quad \overset{(0)}{u}(t, 0) = \overset{(0)}{u}(t, \pi) = 0\,,$$

$$\overset{(1)}{u}_{tt} - \overset{(1)}{u}_{xx} = f(t, x, \overset{(0)}{u}, \overset{(0)}{u}_t, \overset{(0)}{u}_x, 0)\,, \quad \overset{(1)}{u}(t, 0) = \overset{(1)}{u}(t, \pi) = 0\,,$$

$$\overset{(2)}{u}_{tt} - \overset{(2)}{u}_{xx} = \left(\frac{\partial f}{\partial u_0}\right)^{(0)} \overset{(1)}{u} + \left(\frac{\partial f}{\partial u_1}\right)^{(0)} \overset{(1)}{u}_t + \left(\frac{\partial f}{\partial u_2}\right)^{(0)} \overset{(1)}{u}_x + \left(\frac{\partial f}{\partial \varepsilon}\right)^{(0)},$$

$$\overset{(2)}{u}(t, 0) = \overset{(2)}{u}(t, \pi) = 0\,,$$

. . . .

$$\overset{(n)}{u}_{tt} - \overset{(n)}{u}_{xx} = \left(\frac{\partial f}{\partial u_0}\right)^{(0)} \overset{(n-1)}{u} + \left(\frac{\partial f}{\partial u_1}\right)^{(0)} \overset{(n-1)}{u}_t + \left(\frac{\partial f}{\partial u_2}\right)^{(0)} \overset{(n-1)}{u}_x +$$

$$+ \Phi_n(t, x, \overset{(n-2)}{u}, \ldots, \overset{(0)}{u}, \overset{(n-2)}{u}_t, \ldots, \overset{(0)}{u}_t, \overset{(n-2)}{u}_x, \ldots, \overset{(0)}{u}_x) \quad (n = 3, 4, \ldots)$$

(*where the index* 0 *at the derivatives of* f *denotes that we have to take them at the point* $(t, x, \overset{(0)}{u}, \overset{(0)}{u}_t, \overset{(0)}{u}_x, 0)$) *or, in the case of generalized solutions, of the system*

$$(4.1.10)\quad \overset{(0)}{u}(t, x) = \overset{(0)}{s}(x + t) - \overset{(0)}{s}(-x + t)\,,$$

$$\overset{(1)}{u}(t, x) = \overset{(1)}{s}(x + t) - \overset{(1)}{s}(-x + t) + $$
$$+ \frac{1}{2}\int_0^t \int_{x-t+\vartheta}^{x+t-\vartheta} f(\vartheta, \xi, \overset{(0)}{u}(\vartheta, \xi), \overset{(0)}{u}_t(\vartheta, \xi), \overset{(0)}{u}_x(\vartheta, \xi), 0)\, d\xi\, d\vartheta\,,$$

. . . .

$$\overset{(n)}{u}(t, x) = \overset{(n)}{s}(x + t) - \overset{(n)}{s}(-x + t) + $$
$$+ \frac{1}{2}\int_0^t \int_{x-t+\vartheta}^{x+t-\vartheta} \Psi_n(\vartheta, \xi, \overset{(n-1)}{u}(\vartheta, \xi), \ldots, \overset{(0)}{u}(\vartheta, \xi), \ldots)\, d\xi\, d\vartheta\,.$$

Proof. The first part of the assertion is again an immediate consequence of Remark 2.2 applied to equation 4.1.8, since the operator $U(s)(\varepsilon)$ is already according to Theorem 3.1.2 or Theorem 3.2.3 analytic in ε. Hence, $s^*(\varepsilon)(x)$ being analytic in ε, $U(s^*(\varepsilon))(\varepsilon)(t, x)$ is analytic in ε as well.

The second part is also obvious enough. Note that the finding of $2\pi n$-periodic solutions of the system (4.1.9) alucidates the role of the assumptions (i) and (ii) of Theorem 4.1.1. By Theorem 1.3.1 the assumption (i) of Th. 4.1.1 is a necessary and sufficient condition that the equation ($4.1.9_2$) have a $2\pi n$-periodic solution. On the other hand the assumption (ii) is the simplest sufficient condition ensuring the solvability (in a unique way) with respect to $\overset{(n)}{s}$ of the equation

$$(4.1.11)\quad \int_0^{2\pi n} \Big\{\frac{\partial F}{\partial u_0}(\overset{(0)}{u})(0)(\vartheta, x - \vartheta)\,[\overset{(n)}{s'}(x) - \overset{(n)}{s'}(-x + 2\vartheta) + \overset{(n)}{v}(\vartheta, x - \vartheta)] + $$
$$+ \frac{\partial F}{\partial u_1}(\overset{(0)}{u})(0)(\vartheta, x - \vartheta)\,[\overset{(n)}{s'}(x) - \overset{(n)}{s'}(-x + 2\vartheta) + \overset{(n)}{v}_t(\vartheta, x - \vartheta)] + $$
$$+ \frac{\partial F}{\partial u_2}(\overset{(0)}{u})(0)(\vartheta, x - \vartheta)\,[\overset{(n)}{s}(x) + \overset{(n)}{s}(-x + 2\vartheta) + \overset{(n)}{v}_x(\vartheta, x - \vartheta)] + $$
$$+ \Phi_n(\vartheta, x - \vartheta)\Big\}\, d\vartheta = 0$$

(where $\overset{(n)}{v}(t, x)$ is the particular $2\pi n$-periodic solution of the equation for $\overset{(n)}{u}$ such that $\overset{(n)}{v}(0, x) = \overset{(n)}{v}_t(0, x) = 0$, and the meaning of Φ_n is clear) which represents the necessary and sufficient condition that the equation for $\overset{(n+1)}{u}$ have a $2\pi n$-periodic solution.

Remark 4.1.1. The problem ($\mathscr{P}$) may be solved by another method which enables us to determine the sought solution to an arbitrary degree of accuracy without knowing the explicit solution $U(s)(\varepsilon)(t, x)$ of the associated problem $(\mathscr{M})$. Indeed, let us consider simultaneously the equations

$$P(u, s)(\varepsilon)(t, x) \equiv -u(t, x) + s(x + t) - s(-x + t) +$$

$$+ \frac{1}{2}\varepsilon \int_0^t \int_{x-t+\vartheta}^{x+t-\vartheta} F(u(\vartheta, \xi))(\varepsilon)(\vartheta, \xi)\, d\xi\, d\vartheta = 0\,,$$

$$\tilde{G}(u)(\varepsilon)(x) \equiv \int_0^{2\pi n} F(u(\vartheta, x - \vartheta))(\varepsilon)(\vartheta, x - \vartheta)\, d\vartheta = 0\,.$$

Put

$$\tilde{G} = (P, G),\ p = (u, s),\ q = G(p) = (P(u, s), \tilde{G}(u))\,,$$

$$\mathfrak{P} = \tilde{\mathfrak{U}}_2 \times \tilde{\mathfrak{S}}_2,\ \mathfrak{Q} = \tilde{\mathfrak{U}}_2 \times \tilde{\mathfrak{S}}_1 \quad \text{or} \quad \tilde{\mathfrak{U}}_2 \times \tilde{\mathfrak{S}}_2\,.$$

Then applying again Theorem 2.2 to the equation $G(p)(\varepsilon) = 0$ we may prove all theorems stated above.

4.2. Resonance; general case. Let the problem $(\mathscr{R})$ be given by

(4.2.1) $$u_{tt} - u_{xx} = \varepsilon f(t, x, u, u_t, u_x, \varepsilon)\,,$$

(4.2.2) $$u(t, 0) = u(t, \pi) = 0\,,$$

(4.2.3) $$u(\omega, x) - u(0, x) = u_t(\omega, x) - u_t(0, x) = 0\,,$$

where $\omega = 2\pi p/q$, p and $q \neq 1$ being relatively prime natural numbers and the function f besides $(\mathscr{C}_1)$, $(\mathscr{C}_2)$ fulfils

(4.2.4) $$f(t + \omega, x, u_0, u_1, u_2, \varepsilon) - f(t, x, u_0, u_1, u_2, \varepsilon) = 0\,.$$

Let $U(s)(\varepsilon)(t, x)$ be the solution of the mixed problem $(\mathscr{M})$ associated to $(\mathscr{R})$. By the same reasoning as in sec. 1.3 we get the necessary and sufficient condition for the existence of a solution of $(\mathscr{R})$ in the form

(4.2.6)

$$s'(x + \omega) - s'(x) = -\frac{1}{2}\varepsilon \int_0^{\omega} F(U(s))(\varepsilon)(\vartheta, x + \omega - \vartheta))(\varepsilon)(\vartheta, x + \omega - \vartheta)\, d\vartheta\,.$$

By Theorem 1.3.2 the necessary and sufficient condition for the solvability of (4.2.6) reads

(4.2.7) $$\int_0^{\omega} \sum_{j=1}^{q} F(U(s)(\varepsilon)(\vartheta, x + j\omega - \vartheta))(\varepsilon)(\vartheta, x + j\omega - \vartheta)\, d\vartheta = 0\,.$$

Denoting

$$\Phi(s)(\varepsilon)(x) = \frac{1}{2q}\sum_{j=1}^{q}(q - j)\int_0^x \int_0^{\omega} F(U(s)(\varepsilon)(\vartheta, \xi + j\omega - \vartheta))(\varepsilon)(\vartheta, \xi + j\omega - \vartheta)\, d\vartheta\, d\xi$$

by Theorem 1.3.2 the equation (4.2.6) is equivalent to

$$G_1(s, \sigma)(\varepsilon)(x) \equiv -s(x) + \sigma(x) + \varepsilon\Phi(s)(\varepsilon)(x) = 0\,, \tag{4.2.8}$$

where $\sigma(x)$ is some $2\pi/q$-periodic function of class C^2. Making use of (4.2.8) the equation (4.2.7) may be written as

$$G_2(s, \sigma)(\varepsilon)(x) \equiv \tag{4.2.9}$$

$$\equiv \sum_{j=1}^{q} \int_0^{\omega} F(U(\sigma + \varepsilon\,\Phi(s)(\varepsilon))(\varepsilon)(\vartheta, x + j\omega - \vartheta))(\varepsilon)(\vartheta, x + j\omega - \vartheta)\,\mathrm{d}\vartheta = 0\,.$$

(Clearly in the last three equations we may write $\sum_0^{q-1}$ instead of $\sum_1^q$).

Let us denote $\mathfrak{S}_i(p/q)$ $(i = 0, 1, 2)$ the space of $2\pi p/q$-periodic functions of class C^2 with the norm of $\mathfrak{S}_i$. (We shall write $\mathfrak{S}_i$ instead of $\mathfrak{S}_i(1)$.) It may be easily verified that the operator $G_1(s, \sigma)(\varepsilon)$ maps $\mathfrak{S}_2 \times \mathfrak{S}_2(1/q)$ into $\mathfrak{S}_2$ for every $\varepsilon \in \mathfrak{E}$ and there exist continuous $\mathscr{G}$-derivatives $G'_{1s}(s, \sigma)(\varepsilon) \in [\mathfrak{S}_2 \to \mathfrak{S}_2]$, $G'_{1\sigma}(s, \sigma)(\varepsilon) = I \in [\mathfrak{S}_2(1/q) \to \mathfrak{S}_2(1/q)]$ for every $\varepsilon \in \mathfrak{E}$. First, let us suppose, that $|\partial f/\partial u_1| + |\partial f/\partial u_2| \not\equiv 0$. Then the operator $G_2(s, \sigma)(\varepsilon)$ maps $\mathfrak{S}_2 \times \mathfrak{S}_2(1/q)$ as in $\mathfrak{S}_1$ as in $\mathfrak{S}_1(p/q)$ whence it follows that it maps $\mathfrak{S}_2 \times \mathfrak{S}_2(1/q)$ into $\mathfrak{S}_1(1/q)$ for every $\varepsilon \in \mathfrak{E}$. (In fact, that $G_2(s, \sigma)(\varepsilon)(x)$ belongs to $\mathfrak{S}_1$ is clear. On the other hand, since

$$U(s, \sigma)(\varepsilon)(t, x + (q + 1)\,\omega) = U(s, \sigma)(\varepsilon)(t, x + 2\pi p + \omega) =$$
$$= U(s, \sigma)(\varepsilon)(t, x + \omega)$$

and similarly

$$F(u)(\varepsilon)(t, x + (q + 1)\,\omega) = F(u)(\varepsilon)(t, x + \omega)$$

we have

$$G_2(s, \sigma)(\varepsilon)(x + \omega) =$$

$$= \sum_{j=2}^{q+1} \int_0^{\omega} F(U(\sigma + \varepsilon\Phi)(\varepsilon)(\vartheta, x + j\omega - \vartheta))(\varepsilon)(\vartheta, x + j\omega - \vartheta)\,\mathrm{d}\vartheta =$$

$$= \sum_{j=1}^{q} \int_0^{\omega} F(U(\sigma + \varepsilon\Phi)(\varepsilon)(\vartheta, x + j\omega - \vartheta))(\varepsilon)(\vartheta, x + j\omega - \vartheta)\,\mathrm{d}\vartheta =$$

$$= G_2(s, \sigma)(\varepsilon)(x).)$$

We find readily that the operator $G_2(s, \sigma)(\varepsilon)$ has continuous $\mathscr{G}$-derivatives $G'_{2s}(s, \sigma)(\varepsilon) \in [\mathfrak{S}_2 \to \mathfrak{S}_1(1/q)]$, $G'_{2\sigma}(s, \sigma)(\varepsilon) \in [\mathfrak{S}_2(1/q) \to \mathfrak{S}_1(1/q)]$ at any point $s \in \mathfrak{S}_2$, $\sigma \in \mathfrak{S}_2(1/q)$ for every $\varepsilon \in \mathfrak{E}$.

Further, let $|\partial f/\partial u_1| + |\partial f/\partial u_2| \equiv 0$ and let f fulfil $(\mathscr{C}''_1)$ and $(\mathscr{C}_2)$. Then $G_2(s, \sigma)(\varepsilon)$ maps $\mathfrak{S}_2 \times \mathfrak{S}_2(1/q)$ into $\mathfrak{S}_2(1/q)$ for every $\varepsilon \in \mathfrak{E}$ and there exist continuous $\mathscr{G}$-derivatives $G'_{2s}(s, \sigma)(\varepsilon) \in [\mathfrak{S}_2 \to \mathfrak{S}_2(1/q)]$ and $G'_{2\sigma}(s, \sigma)(\varepsilon) \in [\mathfrak{S}_2(1/q) \to \mathfrak{S}_2(1/q)]$ for every $\varepsilon \in \mathfrak{E}$.

Letting $\varepsilon \to 0$ in (4.2.8) and (4.2.9) we get

$$G_1(s_0, \sigma_0)(0)(x) \equiv -s_0(x) + \sigma_0(x) = 0 \tag{4.2.10}$$

(4.2.11)

$$G_2(s_0, \sigma_0)(0)(x) \equiv \sum_{j=1}^{q} \int_0^{\omega} F(U(\sigma_0)(0)(\vartheta, x + j\omega - \vartheta))(0)(\vartheta, x + j\omega - \vartheta)\,d\vartheta =$$
$$= \sum_{j=1}^{q} \int_0^{\omega} F(\sigma_0(x + j\omega) - \sigma_0(-x - j\omega + 2\vartheta))(0)(\vartheta, x + j\omega - \vartheta)\,d\vartheta = 0\,.$$

Now we can state following two theorems.

Theorem 4.2.1. *Let the problem* $(\mathscr{R})$ *be given with* $|\partial f/\partial u_1| + |\partial f/\partial u_2| \not\equiv 0$. *Let besides the conditions* $(\mathscr{C}_1)$, $(\mathscr{C}_2)$ *and* (4.2.4) *the following assumptions be fulfilled.*

(i) *The operator* $G_1(s, \sigma)(\varepsilon)$ *maps* $\tilde{\tilde{\mathfrak{S}}}_2 \times \tilde{\mathfrak{S}}_2(1/q)$ *into* $\tilde{\tilde{\mathfrak{S}}}_2$ *and* $G_2(s, \sigma)(\varepsilon)$ *maps* $\tilde{\tilde{\mathfrak{S}}}_2 \times \tilde{\mathfrak{S}}_2(1/q)$ *into* $\tilde{\tilde{\mathfrak{S}}}_1(1/q)$ *for every* $\varepsilon \in \mathfrak{E}$.

(ii) *The equation* (4.2.11) *has a solution*

$$\sigma_0 = \sigma_0^*(x) \in \tilde{\tilde{\mathfrak{S}}}_2\left(\frac{1}{q}\right).$$

(iii) *There exists an operator*

$$H_2 = [G'_{2\sigma}(\sigma_0^*, \sigma_0^*)(0)]^{-1} \in \left[\tilde{\mathfrak{S}}_1\left(\frac{1}{q}\right) \to \tilde{\tilde{\mathfrak{S}}}_2\left(\frac{1}{q}\right)\right].$$

Then there exists $\varepsilon^* > 0$ *such that the problem* $(\mathscr{R})$ *has for* $\varepsilon \in \langle 0, \varepsilon^* \rangle$ *a unique solution* $u^*(\varepsilon)(t, x) = U(s^*(\varepsilon))(\varepsilon)(t, x) \in \mathfrak{U}_2$ *such that* $s^*(0)(x) = s_0^*(x) = \sigma_0^*(x)$, *while* $s^*(\varepsilon)$ *is continuous in* ε.

Theorem 4.2.2. *Let the problem* $(\mathscr{R})$ *be given with* $f = f(t, x, u, \varepsilon)$. *Let besides the conditions* $(\mathscr{C}_1'')$, $(\mathscr{C}_2)$ *and* (4.2.4) *the following assumptions be fulfilled.*

(i) *The operator* $G_1(s, \sigma)(\varepsilon)$ *maps* $\tilde{\tilde{\mathfrak{S}}}_2 \times \tilde{\mathfrak{S}}_2(1/q)$ *into* $\tilde{\tilde{\mathfrak{S}}}_2$ *and* $G_2(s, \sigma)(\varepsilon)$ *maps* $\tilde{\tilde{\mathfrak{S}}}_2 \times \tilde{\mathfrak{S}}_2(1/q)$ *into* $\tilde{\tilde{\mathfrak{S}}}_2(1/q)$ *for every* $\varepsilon \in \mathfrak{E}$.

(ii) *The equation* (4.2.11) *has a solution* $\sigma_0 = \sigma_0^*(x) \in \tilde{\tilde{\mathfrak{S}}}_2(1/q)$.

(iii) *There exists an operator*

$$H_2 = [G'_{2\sigma}(\sigma_0^*, \sigma_0^*)(0)]^{-1} \in \left[\tilde{\tilde{\mathfrak{S}}}_2\left(\frac{1}{q}\right) \to \tilde{\tilde{\mathfrak{S}}}_2\left(\frac{1}{q}\right)\right].$$

Then there exists $\varepsilon^* > 0$ *such that the problem* $(\mathscr{R})$ *has for* $\varepsilon \in \langle 0, \varepsilon^* \rangle$ *a unique solution* $u^*(\varepsilon)(t, x) = U(s^*(\varepsilon))(\varepsilon)(t, x) \in \mathfrak{U}_2$ *such that* $s^*(0)(x) = \sigma_0^*(x)$, *while* $s^*(\varepsilon)$ *is continuous in* ε.

Proof of Theorem 4.2.1. Let us make use of Theorem 2.2 where we put $p = (s, \sigma)$, $q = (r, \varrho)$, $G = (G_1, G_2)$, $q = G(p)$, $r = G_1(s, \sigma)$, $\varrho = G_2(s, \sigma)$, $\mathfrak{P} = \tilde{\mathfrak{S}}_2 \times \tilde{\mathfrak{S}}_2(1/q)$, $\mathfrak{Q} = \tilde{\mathfrak{S}}_2 \times \mathfrak{S}_1(1/q)$.

Then according to the assumptions of our theorem and to the considerations above there are fulfilled all assumptions of Theorem 2.2 in the neighborhood of the point $p_0^* = (\sigma_0^*, \sigma_0^*)$ with $H(\tilde{r}, \tilde{\varrho}) = (-\tilde{r} + H_2\tilde{\varrho}, H_2\tilde{\varrho})$, $(\tilde{r}, \tilde{\varrho})$ being any element from $\mathfrak{Q}$.

Proof of Theorem 4.2.2 is quite analogous.

Remark 4.2.1. In this section we forgo the formulation of corresponding theorems for generalized solutions and solutions analytic in ε.

4.3. Resonance; a special case. A special case of the preceding problem has been thoroughly studied by several Soviet mathematicians using other methods (see papers [3]–[10]). Let

$$\omega = 2\pi \frac{p}{q} \quad \text{where} \quad p = 2k - 1,\ q = 2l,\quad k, l = 1, 2, \ldots . \tag{4.3.1}$$

Let $f = f(t, x, u, u_t, \varepsilon)$ satisfy $(\mathscr{C}_1)$, $(\mathscr{C}_2)$ and (4.2.4) with ω from (4.3.1) and let f be symmetric in x with respect to the point $x = \pi/2$, i.e.

$$f(t, x, u_0, u_1, \varepsilon) = f(t, \pi - x, u_0, u_1, \varepsilon) . \tag{4.3.2}$$

We shall show that in this case there always exists a unique solution of the problem $(\mathscr{R})$ in the subspace $\tilde{\mathfrak{U}}_2$ of 2π-periodic and odd in x functions $u(t, x)$ from $\mathfrak{U}_2$, such that

$$u(t, x) = u(t, \pi - x) . \tag{4.3.3}$$

First let us show that if $U(s)(\varepsilon)(t, x)$ has the property (4.3.3) then the equation (4.2.9) is identically satisfied. Indeed, by (3.1.5) and (4.3.2)

$$\begin{aligned} F(U(s)(\varepsilon)(t, x + \pi))(\varepsilon)(t, x + \pi) &= F(U(s)(\varepsilon)(t, -x))(\varepsilon)(t, -x) = \\ &= -F(U(s)(\varepsilon)(t, x))(\varepsilon)(t, x) . \end{aligned} \tag{4.3.4}$$

Then

$$\begin{aligned} &\sum_{j=1}^{2l} F(U(s)(\varepsilon)(\vartheta, x + j\omega - \vartheta))(\varepsilon)(\vartheta, x + j\omega - \vartheta) = \\ &= \sum_{j=1}^{l} F(U(s)(\varepsilon)(\vartheta, x + j\omega - \vartheta))(\varepsilon)(\vartheta, x + j\omega - \vartheta) + \\ &+ \sum_{j=1}^{l} F(U(s)(\varepsilon)(\vartheta, x + (2k-1)\pi + j\omega - \vartheta))(\varepsilon)(\vartheta, x + (2k-1)\pi + j\omega - \vartheta) = \\ &= \sum_{j=1}^{l} F(U(s)(\varepsilon)(\vartheta, x + j\omega - \vartheta))(\varepsilon)(\vartheta, x + j\omega - \vartheta) + \\ &+ \sum_{j=1}^{l} F(U(s)(\varepsilon)(\vartheta, x + \pi + j\omega - \vartheta))(\varepsilon)(\vartheta, x + \pi + j\omega - \vartheta) = 0 . \end{aligned}$$

Now let us show that $U(s)(\varepsilon)(t, x)$ has the property (4.3.3) if and only if

$$(4.3.5) \qquad s(x + \pi) + s(x) = s(\pi) + s(0) .$$

In fact let

$$(4.3.6) \qquad U(s)(t, x) = U(s)(t, \pi - x) .$$

Then making use of the identity (4.1.6) and taking into account

$$\int_0^t \int_{\pi-x-t+\vartheta}^{\pi-x+t-\vartheta} F(U(s)(\vartheta, \xi))(\vartheta, \xi)\, d\xi\, d\vartheta =$$

$$= - \int_0^t \int_{x+t-\vartheta}^{x-t+\vartheta} F(U(s)(\vartheta, \pi - \eta))(\vartheta, \pi - \eta)\, d\eta\, d\vartheta =$$

$$= \int_0^t \int_{x-t+\vartheta}^{x+t-\vartheta} F(U(s)(\vartheta, \eta))(\vartheta, \eta)\, d\eta\, d\vartheta$$

the equality (4.3.6) reduces to

$$(4.3.6') \qquad s(x + t) - s(- x + t) = s(\pi - x + t) - s(-\pi + x + t) .$$

Conversely, if (4.3.6′) holds, we find easily by the successive approximation method that $U(s)(t, x)$ has the property (4.3.6). Putting now in (4.3.6′) $x - t = 0$, $x + t = \xi$, we get

$$s(\xi) - s(0) = s(\pi) - s(-\pi + \xi) ,$$

whence the necessity of (4.3.5) follows. On the other hand from (4.3.5) the equality (4.3.6′) follows readily.

The equality (4.3.5) is identically satisfied for $x = 0$ and hence it is equivalent to

$$(4.3.5') \qquad s'(x + \pi) + s'(x) = 0 .$$

Denote $\tilde{\mathfrak{S}}_2(p/q)$ the subspace of functions $s(x)$ from $\mathfrak{S}_2(p/q)$ which fulfil (4.3.5′).

Now we have to show that functions $s(x) \in \tilde{\mathfrak{S}}_2$ and $\sigma(x) \in \mathfrak{S}_2(1/q)$ may be chosen so that they satisfy the equation (4.2.8). Since any solution $s^*(\varepsilon)(x)$ of (4.2.8) must satisfy (4.3.5′) for $\varepsilon = 0$ it follows that $s^*(0)(x) = \sigma(x)$ must belong to $\tilde{\mathfrak{S}}_2(1/q)$. But by the $2\pi/q$-periodicity of $\sigma(x)$

$$\sigma'(x) = \sigma'(x + l\omega) = \sigma'(x + (2k - 1)\pi) = \sigma'(x + \pi) .$$

On the other hand, by (4.3.5′)

$$\sigma'(x) = -\sigma'(x + \pi)$$

which implies

$$(4.3.7) \qquad \sigma'(x) \equiv 0 \ (\text{or } \sigma(x) \equiv d,\ d \text{ being any real constant}) .$$

Finally let us show that the operator G_1 defined by

$$G_1(s)(\varepsilon)(x) \equiv -s(x) + d + \varepsilon\Phi(s)(\varepsilon)(x)\,, \tag{4.3.8}$$

maps $\tilde{\mathfrak{S}}_2$ into $\tilde{\mathfrak{S}}_2$. We know already that $G_1(s)(\varepsilon)(x) \in \mathfrak{S}_2$ for $s(x) \in \mathfrak{S}_2$ and $\varepsilon \in \mathfrak{E}$. Further,

$$\Phi'(s)(\varepsilon)(x + \pi) + \Phi'(s)(\varepsilon)(x) =$$
$$= \frac{1}{2q}\sum_{j=1}^{q}(q - j)\int_0^{\omega}[F(U(s)(\varepsilon)(\vartheta, x + \pi + j\omega - \vartheta))(\varepsilon)(\vartheta, x + \pi + j\omega - \vartheta) +$$
$$+ F(U(s)(\varepsilon)(\vartheta, x + j\omega - \vartheta))(\varepsilon)(\vartheta, x + j\omega - \vartheta)]\,\mathrm{d}\vartheta = 0$$

whence the assertion results.

Hence, the equation

$$G_1(s)(\varepsilon)(x) = -\,s(x) + d + \varepsilon\,\Phi(s)(\varepsilon)(x) = 0 \tag{4.3.9}$$

satisfies all assumptions of Theorem 2.2 with $p = s$, $\mathfrak{P} = \mathfrak{Q} = \tilde{\mathfrak{S}}_2$.

The solution of our problem $(\mathscr{R})$ is independent on the constant d. Indeed, by (3.1.10) $U(s)(\varepsilon)(t, x) \equiv U(s + s_0)(\varepsilon)(t, x)$ for any constant s_0 and accordingly $\Phi(s)(\varepsilon)(x) = \Phi(s + s_0)(\varepsilon)(x)$. Hence without loss of generality we may take instead of s a new function $s_1 = s - d$ for which we get the equation

$$-s_1(x) + \varepsilon\,\Phi(s_1 + d)(\varepsilon)(x) = -s_1(x) + \varepsilon\,\Phi(s_1)(\varepsilon)(x) = 0\,. \tag{4.3.10}$$

Thus, we may state the following

Theorem 4.3.1. *Let the problem $(\mathscr{R})$ be given with $f = f(t, x, u, u_t, \varepsilon)$ and $\omega = 2\pi(2k - 1)/2l$, $k, l = 1, 2, \ldots$. Let the conditions $(\mathscr{C}_1), (\mathscr{C}_2)$, (4.2.4) and (4.3.2) be fulfilled.*

Then there exists ε^* such that the problem $(\mathscr{R})$ has for $\varepsilon \in \langle 0, \varepsilon^*\rangle$ in $\tilde{\mathfrak{U}}_2$ a unique solution $u^*(\varepsilon)(t, x) = U(s^*(\varepsilon))(\varepsilon)(t, x)$ such that $s^*(\varepsilon)$ is continuous in ε and $s^*(0) \equiv 0$.

Remark 4.3.1. The assumption $(\mathscr{C}_1)$ may be weakened, viz. it is sufficient to suppose the function $f(t, x, u, u_t, \varepsilon)$ is together with its derivatives $\partial f/\partial u$, $\partial f/\partial u_t$ continuous in all variables and Lipschitzian in u and u_t. This turns out, immediately if we solve (4.3.9) by successive approximations.

5. EXAMPLES

5.1. $f = \alpha u + \beta u^3 + h(t, x)$, $\omega = 2\pi$. Let us consider the problem $(\mathscr{P})$ with $n = 1$ and with

$$f = \alpha u + \beta u^3 + h(t, x) \tag{5.1.1}$$

where α, β are constants, $\alpha\beta > 0$ and $h(t, x)$ is of class C^0 in t, of class C^2 in x, 2π-periodic in t and x, odd in x and

$$h(t, x + \pi) = -h(t, x), \tag{5.1.2}$$

$$2\pi\beta\chi(x) = \int_0^{2\pi} h(\vartheta, x - \vartheta)\, d\vartheta \not\equiv 0. \tag{5.1.3}$$

Clearly, the function f satisfies the conditions $(\mathscr{C}_1'')$ and $(\mathscr{C}_2)$ for $(t, x, u, \varepsilon) \in \mathfrak{R} \times \times \mathfrak{R} \times \mathfrak{R} \times \mathfrak{R}$.

Now, the necessary condition (4.1.8) reads

$$\int_0^{2\pi} \{\alpha[s_0(x) - s_0(-x + 2\vartheta)] + \beta[s_0(x) - s_0(-x + 2\vartheta)]^3 + h(\vartheta, x - \vartheta)\}\, d\vartheta = 0. \tag{5.1.4}$$

Dividing by $2\pi\beta$ and putting

$$\int_0^{2\pi} s_0^k(-x + 2\vartheta)\, d\vartheta = \int_0^{2\pi} s_0^k(\xi)\, d\xi = 2\pi I_k \quad (k = 1, 2, 3), \tag{5.1.5}$$

$$\gamma = \frac{\alpha}{\beta} > 0, \tag{5.1.6}$$

(5.1.4) yields

$$G(s_0)(0)(x) \equiv s_0^3(x) - 3I_1 s_0^2(x) + (\gamma + 3I_2)\, s_0(x) - \gamma I_1 - I_3 + \chi(x) = 0, \tag{5.1.7}$$

where by (5.1.2) and (5.1.3)

$$\chi(x + \pi) = -\chi(x). \tag{5.1.8}$$

We shall seek the solution of (5.1.7) in the subspace $\tilde{\mathfrak{S}}_2$ of functions $s(x) \in \mathfrak{S}_2$ such that

$$s(x + \pi) = -s(x). \tag{5.1.9}$$

Then, of course,

$$I_1 = I_3 = 0. \tag{5.1.10}$$

Let us consider for a while the functional I_2 as an absolute positive constant. Denoting

$$3p = \gamma + 3I_2, \quad 2q = \chi(x), \tag{5.1.11}$$

(so that $p^3 + q^2 > 0$)

the unique real solution of (5.1.7) is given by

$$\tilde{s}_0(x; I_2) = [-q + (q^2 + p^3)^{\frac{1}{2}}]^{\frac{1}{3}} + [-q - (q^2 + p^3)^{\frac{1}{2}}]^{\frac{1}{3}}. \tag{5.1.12}$$

In virtue of (5.1.8) $\tilde{s}_0(x; I_2) \in \tilde{\mathfrak{S}}_2$ for each I_2. It remains to show that $I_2 > 0$ may be chosen in such a way that (5.1.5) be satisfied for $k = 2$.

Clearly, for $\gamma + 3I_2 = 0$

$$\int_0^{2\pi} \tilde{s}_0^2(\xi; -\tfrac{1}{3}\gamma)\, \mathrm{d}\xi = \int_0^{2\pi} \chi^{\frac{2}{3}}(\xi)\, \mathrm{d}\xi > 0$$

and for $I_2 \to \infty$ the solution $\tilde{s}_0(x; I_2) \to 0$ uniformly with respect to x so that

$$\int_0^{2\pi} \tilde{s}_0^2(\xi; I_2)\, \mathrm{d}\xi \to 0$$

monotonically with respect to I_2. (Indeed, it is easily found by (5.1.12) that $\partial/\partial I_2 \tilde{s}_0^2(x; I_2) \leqq 0$ for all x and $\partial/\partial I_2\, \tilde{s}_0^2(x; I_2) < 0$ on a set of a positive measure.) Hence, because of

$$-\tfrac{1}{3}\gamma < \int_0^{2\pi} \chi^{\frac{2}{3}}(\xi)\, \mathrm{d}\xi \tag{5.1.13}$$

there exists a unique positive value $I_2 = I_2^*$ such that $s_0^*(x) = \tilde{s}_0(x; I_2^*)$ fulfils

$$\int_0^{2\pi} s_0^{*\,2}(\xi)\, \mathrm{d}\xi = 2\pi I_2^* .$$

Now, having in view the assumption (ii) of Theorem 4.1.3 we have to prove the existence of the inverse operator. It may be easily verified that if $s \in \tilde{\mathfrak{S}}_2$ and

$$f(t, x, u, \varepsilon) = -f(t, x + \pi, -u, \varepsilon)$$

then the solution $u(t, x)$ of (3.1.10) lies in the subspace $\tilde{\mathfrak{U}}_2$ of 2π-periodic and odd in x functions $u(t, x)$ from $\mathfrak{U}_2$, which satisfy the relation

$$u(t, x) = -u(t, x + \pi) .$$

(In fact, observe that if $u(t, x) \in \tilde{\mathfrak{U}}_2$,

$$\int_0^t \int_{\pi+x-t+\vartheta}^{\pi+x+t-\vartheta} f(\vartheta, \xi, u(\vartheta, \xi), \varepsilon)\, \mathrm{d}\xi\, \mathrm{d}\vartheta =$$

$$= \int_0^t \int_{x-t+\vartheta}^{x+t-\vartheta} f(\vartheta, \eta + \pi, u(\vartheta, \eta + \pi), \varepsilon)\, \mathrm{d}\eta\, \mathrm{d}\vartheta =$$

$$= - \int_0^t \int_{x-t+\vartheta}^{x+t-\vartheta} f(\vartheta, \eta, u(\vartheta, \eta), \varepsilon)\, \mathrm{d}\eta\, \mathrm{d}\vartheta .)$$

Then we find easily that our operator $G(s)(\varepsilon)$ maps $\tilde{\mathfrak{S}}_2$ into $\tilde{\mathfrak{S}}_2$. Let us show that at the point $s = s_0^*$, $\varepsilon = 0$, the operator $G_s'(s)(\varepsilon)$ maps $\tilde{\mathfrak{S}}_2$ onto $\tilde{\mathfrak{S}}_2$. Then we shall be able to prove by Lemma 2.2 that there exists

$$H = [G'_s(s_0^*)(0)]^{-1} \in [\tilde{\mathfrak{S}}_2 \to \tilde{\mathfrak{S}}_2].$$

Clearly, ($\delta s = \bar{s}$)

$$G'_s(s_0^*)(0)(\bar{s}) = (3s_0^{*2} + \gamma + 3I_2^*)\,\bar{s} + 3J_2 s_0^* - \\ - (3s_0^{*2} + \gamma)\,J_1 + 3J_2 s_0^* - J_3$$

where

$$J_k = \frac{k}{2\pi}\int_0^{2\pi} s_0^{*k-1}(\xi)\,\bar{s}(\xi)\,d\xi.$$

By (5.1.9)

$$J_1 = J_3 = 0.$$

Thus, by Lemma 2.2 we have to verify that the equation

$$(5.1.14) \qquad (3s_0^{*2}(x) + \gamma + 3I_2^*)\,\bar{s}(x) + 3J_2 s_0^*(x) = r(x)$$

where $r(x)$ is an arbitrary function from $\tilde{\mathfrak{S}}_2$, has a unique solution $\bar{s}^*(x)$ such that

$$(5.1.15) \qquad \|\bar{s}^*(x)\| \leqq k\|r(x)\|, \quad k > 0.$$

Evidently,

$$(5.1.16) \qquad \bar{s}^*(x) = a(x)\,[r(x) - 3J_2^*\,s_0^*(x)],$$

where

$$a(x) = [3s_0^{*2}(x) + \gamma + 3I_2^*]^{-1} > 0,$$

$$J_2^* = \int_0^{2\pi} a(\xi)\,s_0^*(\xi)\,r(\xi)\,d\xi[\pi + 3\int_0^{2\pi} a(\xi)\,s_0^{*2}(\xi)\,d\xi]^{-1}$$

is a unique solution of (5.1.14). The function $\bar{s}^*(x)$ is from $\tilde{\mathfrak{S}}_2$ and fulfils the inequality (5.1.15) with

$$k = \max_{x\in\langle 0,\pi\rangle} (a(x) + 2|a'(x)| + |a''(x)|)\,.$$

$$.\left\{1 + \|s_0^*\|\left[\pi + 3\int_0^{2\pi} a(\xi)\,s_0^{*2}(\xi)\,d\xi\right]^{-1}\int_0^{2\pi}|a(\xi)|\,|s_0^*(\xi)|\,d\xi\right\}.$$

Hence, we may state the following

Theorem 5.1.1. *Given the problem ($\mathscr{P}$) with $n = 1$ and with $f = \alpha u + \beta u^3 + h(t, x)$, where $\alpha\beta > 0$ and $h(t, x)$ is of class C^0 in t, of class C^2 in x, 2π-periodic in t and x, odd in x and fulfils* (5.1.2), (5.1.3).

Then there exists $\varepsilon^* > 0$ such that our problem ($\mathscr{P}$) has for $0 < \varepsilon \leqq \varepsilon^*$ in $\tilde{\mathfrak{U}}_2$ a unique 2π-periodic solution $U(s^*(\varepsilon))(\varepsilon)(t, x)$ such that $s^*(0) = s_0^*$.

Remark 5.1.1. By more detailed calculations it may be shown that there exists a solution of our problem $(\mathscr{P})$ also for sufficiently small nonpositive values of γ.

Remark 5.1.2. Applying Theorem 4.1.1 it may be shown by similar considerations as above that the problem $(\mathscr{P})$ for

$$f = (\alpha + \beta u^2)\, u_t + h(t, x) \tag{5.1.17}$$

has in $\tilde{\mathfrak{U}}_2$ a unique 2π-periodic solution if the constants α, β and the function $h(t, x)$ fulfil following assumptions:

(i) $\alpha\beta > 0$,

(ii) $h(t, x)$ is of class C^0 in t, of class C^1 in x, 2π-periodic in t and x, odd in x and

$$h(t, x + \pi) = -h(t, x)\,, \quad 2\pi\beta\chi(x) = \int_0^{2\pi} h(\vartheta, x - \vartheta)\, \mathrm{d}\vartheta \not\equiv 0\,.$$

The proof will be published elsewhere.

5.2. $f = \alpha u + \beta u^3 + h(t, x)$, $\omega = \pi$. Let us consider the problem $(\mathscr{R})$ with $\omega = \pi$ (i.e. $p = 1$, $q = 2$) and with

$$f = \alpha u + \beta u^3 + h(t, x)\,, \tag{5.2.1}$$

where α, β are constants, $\alpha\beta > 0$ and $h(t, x)$ is of class C^0 in t, of class C^2 in x, π-periodic in t, 2π-periodic and odd in x and

$$h(t, x + \tfrac{1}{2}\pi) = -h(t, x)\,. \tag{5.2.2}$$

$$\pi\beta\chi(x) = \int_0^{\pi} [h(\vartheta, x - \vartheta) + h(\vartheta, x + \pi - \vartheta)]\, \mathrm{d}\vartheta \not\equiv 0\,. \tag{5.2.3}$$

The function f again satisfies conditions $(\mathscr{C}_1'')$ and $(\mathscr{C}_2)$ for $(t, x, u, \varepsilon) \in \mathfrak{R} \times \mathfrak{R} \times \mathfrak{R} \times \mathfrak{R}$. Now, let us write the operators $G_i(s, \sigma)(\varepsilon)$ $(i = 1, 2)$ from (4.2.8) and (4.2.9) in the form $G_i(s, \sigma + \varepsilon c)(\varepsilon)$ $(i = 1, 2)$ where σ is from $\mathfrak{S}_2(\frac{1}{2})$ and c is some constant. (This is possible since c is also from $\mathfrak{S}_2(\frac{1}{2})$.) The necessary condition (4.2.11) in our present case reads

$$\begin{aligned}\int_0^{\pi} \{&\alpha[\sigma_0(x) - \sigma_0(-x + 2\vartheta)] + \beta[\sigma_0(x) - \sigma_0(-x + 2\vartheta)]^3 + \\ &+ h(\vartheta, x - \vartheta) + \alpha[\sigma_0(x + \pi) - \sigma_0(-x - \pi + 2\vartheta)] + \\ &+ \beta[\sigma_0(x + \pi) - \sigma_0(-x - \pi + 2\vartheta)]^3 + h(\vartheta, x + \pi - \vartheta)\}\, \mathrm{d}\vartheta = 0\,.\end{aligned} \tag{5.2.4}$$

Taking into account that σ is a π-periodic function and denoting

$$\int_0^{\pi} \sigma_0^k(-x + 2\vartheta)\, \mathrm{d}\vartheta = \pi I_k\,, \tag{5.2.5}$$

(5.2.6) $$\gamma = \frac{\alpha}{\beta} > 0\,,$$

(5.2.4) yields

(5.2.7) $$G_2(s_0, \sigma_0)(0)(x) \equiv \sigma_0^3(x) - 3I_1\sigma_0^2(x) + (\gamma + 3I_2)\,\sigma_0(x) - \gamma I_1 - I_3 + \chi(x) = 0,$$

where by (5.2.2) and (5.2.3)

(5.2.8) $$\chi(x + \tfrac{1}{2}\pi) = -\chi(x)\,.$$

We shall seek the solution of (5.2.7) in the subspace $\tilde{\mathfrak{S}}_2(\frac{1}{2})$ of functions $\sigma(x)$ from $\mathfrak{S}_2(\frac{1}{2})$ such that

(5.2.9) $$\sigma(x + \tfrac{1}{2}\pi) = -\sigma(x)\,,$$

so that

(5.2.10) $$I_1 = I_3 = 0\,.$$

In a quite analogous way as in the foregoing section we can show that the equation (5.2.7) has a unique solution $\sigma^*(x) \in \tilde{\mathfrak{S}}_2(\frac{1}{2})$.

Let us now examine the existence of the inverse operator H_2. Let $\tilde{\mathfrak{S}}_2$ be the subspace of functions $s(x)$ from $\mathfrak{S}_2$ such that

(5.2.11) $$s(x + \tfrac{1}{2}\pi) = -s(x)$$

and let $\tilde{\mathfrak{U}}_2$ be the subspace of 2π-periodic and odd in x functions $u(t, x)$ from $\mathfrak{U}_2$ such that

(5.2.12) $$u(t, x + \tfrac{1}{2}\pi) = -u(t, x)\,.$$

Then by the inspection of the equation (3.1.10′) it may be verified that the operator $P(u, s)(\varepsilon)$ maps $\tilde{\mathfrak{U}}_2 \times \tilde{\mathfrak{S}}_2$ into $\tilde{\mathfrak{U}}_2$, if

$$f(t, x + \tfrac{1}{2}\pi, -u, \varepsilon) = -f(t, x, u, \varepsilon)\,.$$

Further choosing the constant c in $G_i(s, \sigma + \varepsilon c)(\varepsilon)$

$$c = -\frac{1}{4q}\sum_{j=1}^{q}(q - j)\int_0^{\pi/2}\int_0^{\omega} F(U(s)(\varepsilon)(\vartheta, \xi + j\omega - \vartheta))(\varepsilon)(\vartheta, \xi + j\omega - \vartheta)\,d\vartheta\,d\xi,$$

the operator $G_1(s, \sigma)(\varepsilon)$ maps $\tilde{\mathfrak{S}}_2 \times \tilde{\mathfrak{S}}_2(\frac{1}{2})$ into $\tilde{\mathfrak{S}}_2$.

Finally, the operator $G_2(s, \sigma)(\varepsilon)$ from (4.2.9) maps $\tilde{\mathfrak{S}}_2 \times \tilde{\mathfrak{S}}_2(\frac{1}{2})$ into $\tilde{\mathfrak{S}}_2(\frac{1}{2})$. Similarly as in 5.1, it turns out that there exists the inverse operator

$$H_2 = [G'_{2\sigma}(\sigma_0^*, \sigma_0^*)(0)]^{-1} \in [\tilde{\mathfrak{S}}_2(\tfrac{1}{2}) \to \tilde{\mathfrak{S}}_2(\tfrac{1}{2})]\,.$$

Hence by Theorem 4.2.2 the following theorem is valid:

Theorem 5.2.1. *Given the problem* $(\mathscr{R})$ *with* $f = \alpha u + \beta u^3 + h(t, x)$, *where* $\alpha\beta > 0$ *and* $h(t, x)$ *is of class* C^0 *and* π*-periodic in* t, *of class* C^2, *odd and* 2π*-periodic in* x *and fulfils* (5.2.2), (5.2.3).

Then there exists $\varepsilon^* > 0$ *such that our problem* $(\mathscr{R})$ *has for* $0 < \varepsilon \leqq \varepsilon^*$ *in* $\tilde{\mathfrak{U}}_2$ *a unique* π*-periodic solution* $U(s^*(\varepsilon))(\varepsilon)(t, x)$ *such that* $s^*(0) = s_0^*$.

Bibliography

[1] *O. Vejvoda:* Nonlinear boundary-value problems for differential equations. Differential Equations and Their Applications, Proceedings of the Conference held in Prague in September 1962, 199 – 215.

[2] *J. Kurzweil:* Problems which lead to a generalization of the concept of an ordinary nonlinear differential equation. Differential Equations and Their Applications, Proceedings of the Conference held in Prague in September 1962, 65 – 76.

[3] *Н. А. Артемьев:* Периодические решения одного класса уравнений в частных производных. Изв. АН СССР, сер. матем., *1*, 1937.

[4] *П. В. Соловьев:* Некоторые замечания о периодических решениях нелинейных уравнений гиперболического типа. Изв. АН СССР, сер. матем., *2*, 1939, 149 – 164.

[5] *А. П. Митряков:* О решениях бесконечных систем нелинейных интегральных и интегро-дифференциальных уравнений. Тр. Уз. Гос. Унив., *37*, 1948.

[6] *А. П. Митряков:* О периодических решениях нелинейного уравнения гиперболического типа. Тр. ИММ АН УзССР, *7*, 1949, 137 – 149.

[7] *Г. Т. Соколов:* О периодических решениях одного класса уравнений в частных производных. ДАН УзССР, *12*, 1953, 3 – 7.

[8] *В. Н. Карп:* О периодических решениях одного нелинейного уравнения гиперболического типа. ДАН УзССР, *5*, 1953.

[9] *В. Н. Карп:* Применение метода волновых областей к решению задачи о вынужденных нелинейных периодических колебаниях струны. Изв. ВУЗ, матем., *6* (*25*), 1961, 51 – 59.

[10] *В. Н. Карп:* О существовании и единственности периодического решения одного нелинейного уравнения гиперболического типа. Изв. ВУЗ, мат., *5*, 1963, 43 – 50.

[11] *L. J. Mordell:* The series $\sum a_n/(1 - xe^{2n\pi i\alpha})$. Journ. of the London Math. Soc., *38* (*1*), 1963, 111 – 117.

[12] *Л. В. Канторович, Г. П. Акилов:* Функциональный анализ в нормированных пространствах. Москва, Гос. издат. физико-матем. лит., 1959.

Резюме

ПЕРИОДИЧЕСКИЕ РЕШЕНИЯ ЛИНЕЙНОГО И СЛАБО НЕЛИНЕЙНОГО ВОЛНОВОГО УРАВНЕНИЯ В ОДНОЙ РАЗМЕРНОСТИ, I

ОТТО ВЕЙВОДА (Otto Vejvoda), Прага

В первом параграфе исследуется существование классического ω-периодического (в t) решения для волнового уравнения

$$u_{tt} - u_{xx} = f(t, x) \tag{1.1.1}$$

с краевыми условиями

$$(1.1.2) \qquad u(t, 0) = u(t, \pi) = 0\,,$$

и доказываются прежде всего две следующие теоремы.

Теорема 1.3.1. *Пусть*

(1) *функция* $f(t, x)$ *класса* C^0 *в* t *и класса* C^1 *в* x *для* $t \geqq 0, 0 \leqq x \leqq \pi$ *и* $f(t, 0) = = f(t, \pi) = 0$;

(2) *функция* $f(t, x)$ ω*-периодична в* t, *где* $\omega = 2\pi n$, n — *натуральное число.*

Тогда существуют $2\pi n$*-периодические решения уравнений* (1.1.1) *и* (1.1.2) *тогда и только тогда, если*

$$(3) \qquad \int_0^{2\pi n} f(\vartheta, x - \vartheta)\,\mathrm{d}\vartheta = 0\,.$$

Эти решения даются формулой

$$(1.1.8) \qquad u(t, x) = s(x + t) - s(-x + t) + \frac{1}{2}\int_0^t \int_{x-t+\vartheta}^{x+t-\vartheta} f(\vartheta, \xi)\,\mathrm{d}\xi\,\mathrm{d}\vartheta\,,$$

где $s(x)$ — *произвольная* 2π*-периодическая функция класса* C^2 *и* f *продолжена в* x *на* $(-\infty, \infty)$ *как* 2π*-периодическая и нечетная функция.*

Теорема 1.3.2. *Пусть*

(1) *то же самое, как в Теореме* 1.3.1;

(2) *функция* $f(t, x)$ ω*-периодична в* t, *где* $\omega = 2\pi p/q$, p *и* $q \neq 1$ — *натуральные несократимые числа.*

Тогда существуют $2\pi p/q$*-периодические решения уравнений* (1.1.1) *и* (1.1.2) *тогда и только тогда, если*

$$(3) \qquad \int_0^{\omega} \sum_{j=1}^{q} f(\vartheta, x + j\omega - \vartheta)\,\mathrm{d}\vartheta = 0\,.$$

Эти решения даются формулой (1.1.8), *в которой* $s(x) = s_1(x) + s_2(x)$, *где функция* f *продолжена в* x *как в Теореме* (1.3.1), $s_1(x)$ — *произвольная* $2\pi/q$*-периодическая функция класса* C^2 *и*

$$(1.3.12) \qquad s_2(x) = \frac{1}{2q}\int_0^x \int_0^{\omega} \sum_{j=1}^{q-1} (q - j) f(\vartheta, x + j\omega - \vartheta)\,\mathrm{d}\vartheta\,.$$

Далее показывается, что если $\omega = 2\pi\alpha$, α-иррациональное число, то классическое $2\pi\alpha$-периодическое решение вообще не существует.

Во втором параграфе приводятся некоторые вспомогательные теоремы из функционального анализа.

В третьем параграфе доказывается

Теорема 3.1.1. *Пусть дана смешанная задача* ($\mathscr{M}$) *уравнением*

$$u_{tt} - u_{xx} = \varepsilon f(t, x, u, u_t, u_x, \varepsilon)\,, \tag{3.1.1}$$

краевыми условиями

$$u(t, 0) = u(t, \pi) = 0 \tag{3.1.2}$$

и начальными условиями

$$u(0, x) = \varphi(x)\,, \quad u_t(0, x) = \psi(x)\,. \tag{3.1.3}$$

Пусть выполнены следующие условия:

(а) *Функция* $f(t, x, u_0, u_1, u_2, \varepsilon)$ *вместе со своими производными* (3.1.4) *непрерывна по всем переменным для* $t \geqq 0$, $0 \leqq x \leqq \pi$, $-\infty < u_i < +\infty$ $(i = 0, 1, 2)$, $0 \leqq \varepsilon \leqq \varepsilon_0$ *и*

$$f(t, 0, 0, 0, u_2, \varepsilon) = f(t, \pi, 0, 0, u_2, \varepsilon) = 0\,;$$

(б) $\varphi(x)$ — *класса* C^2 *и* $\psi(x)$ — *класса* C^1 *для* $0 \leqq x \leqq \pi$ *и*

$$\varphi(0) = \varphi''(0) = \psi(0) = 0\,, \quad \varphi(\pi) = \varphi''(\pi) = \psi(\pi) = 0\,.$$

Тогда для данной 2π-*периодической функции* $\tilde{s}(x)$ *класса* C^2 *и данного числа* $T > 0$ *существуют числа* $\delta > 0$ *и* ε^*, $0 < \varepsilon^* \leqq \varepsilon_0$ *такие, что задача* ($\mathscr{M}$) *имеет для* $0 \leqq \varepsilon \leqq \varepsilon^*$ *и для всех* s *из сферы* $S(\tilde{s}; \delta)$ *единственное классическое решение* $u^*(\varepsilon)(t, x) = U(s)(\varepsilon)(t, x)$. *Оператор* U *непрерывен в* ε *и обладает непрерывной производной Гато относительно* s, *причем*

$$u^*(0)(t, x) = U(s)(0)(t, x) = s(x + t) - s(-x + t)\,. \tag{3.1.13}$$

(Кроме того, доказывается при более слабых условиях существование обобщенного в некотором смысле решения той же задачи.)

Две основные теоремы параграфа четвертого гласят:

Теорема 4.1.1. *Пусть задача* ($\mathscr{P}$) *дана уравнениями* (3.1.1), (3.1.2) *и*

$$u(2\pi n, x) - u(0, x) = 0\,, \quad u_t(2\pi n, x) - u_t(0, x) = 0\,, \tag{4.1.3}$$

где n — *натуральное число.*

Пусть, кроме условия (а) *из теоремы* (3.1.1) *и условия*

$$f(t + 2\pi n, x, u_0, u_1, u_2, \varepsilon) = f(t, x, u_0, u_1, u_2, \varepsilon)\,, \tag{4.1.4}$$

выполнены следующие условия:

(1) *уравнение*

(4.1.8)
$$G(s_0)(0)(x) \equiv$$
$$\equiv \int_0^{2\pi n} f(\vartheta, x - \vartheta, s_0(x) - s_0(-x + 2\vartheta), s_0'(x) - s_0'(-x + 2\vartheta),$$
$$s_0'(x) + s_0'(-x + 2\vartheta), 0)\, d\vartheta = 0$$

имеет решение $s_0 = s_0^*(x)$ *из некоторого подпространства* $\tilde{\mathfrak{S}}_2$ *пространства* $\mathfrak{S}_2$ 2π*-периодических функций класса* C^2.

(2) *Существует линейный оператор* $H = [G_s'(s_0^*)(0)]^{-1} \in [\tilde{\mathfrak{S}}_1 \to \tilde{\mathfrak{S}}_2]$.

($\mathfrak{S}_1$ *— пространство* 2π*-периодических функций класса* C^1 *и его подпространство* $\tilde{\mathfrak{S}}_1 \supset G(\tilde{\mathfrak{S}}_2)(\varepsilon)$.

Тогда существует число $\varepsilon^* > 0$ *такое, что задача* ($\mathscr{P}$) *имеет для всех* $\varepsilon \in \langle 0, \varepsilon^* \rangle$ *единственное классическое решение* $U(s^*(\varepsilon))(\varepsilon)(t, x)$ *такое, что* $s^*(0)(x) = s_0^*(x)$, *причем функция* $s^*(\varepsilon)(x) \in \tilde{\mathfrak{S}}_2$ *непрерывна в* ε.

Теорема 4.1.3. *Пусть дана задача* ($\mathscr{P}$) *с* $f = f(t, x, u, \varepsilon)$, *где функция* f *со своими производными*

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial u}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x\, \partial u}, \frac{\partial^2 f}{\partial u^2}, \frac{\partial^3 f}{\partial x^2\, \partial u}, \frac{\partial^3 f}{\partial x\, \partial u^2}, \frac{\partial^3 f}{\partial u^3}$$

непрерывна по всем переменным, $2\pi n$*-периодична в* t *и*

$$f(t, 0, 0, \varepsilon) = f(t, \pi, 0, \varepsilon) = 0 .$$

Пусть выполнены следующие условия:

(1) *уравнение* (4.1.8) *имеет решение* $s_0 = s_0^*(x) \in \tilde{\mathfrak{S}}_2$;

(2) *существует оператор*

$$H = [G_s'(s_0^*)(0)]^{-1} \in [\tilde{\tilde{\mathfrak{S}}}_2 \to \tilde{\mathfrak{S}}_2],$$

где $\tilde{\tilde{\mathfrak{S}}}_2 \supset G(\tilde{\mathfrak{S}}_2)(\varepsilon)$.

Тогда существует $\varepsilon^* > 0$ *такое, что задача* ($\mathscr{P}$) *имеет для всех* $\varepsilon \in \langle 0, \varepsilon^* \rangle$ *единственное классическое решение* $U(s^*(\varepsilon))(\varepsilon)(t, x)$ *такое, что* $s^*(0)(x) = s_0^*(x)$, *причем функция* $s^*(\varepsilon) \in \tilde{\mathfrak{S}}_2$ *непрерывна в* ε.

Далее доказываются аналогичные теоремы для обобщенных решений и для случая $\omega = 2\pi p/q$. В особенности, в последнем случае исследуется одна отдельная задача, которая была несколько раз рассмотрена в советской литературе ([3] – [10]).

В пятом параграфе применением теорем предыдущего параграфа доказывается существование периодического решения в случае

$$f = \alpha u + \beta u^3 + h(t, x), \quad \omega = 2\pi \quad \text{и} \quad \omega = \pi$$

при некоторых ограничениях на α, β и $h(t, x)$.