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ON THE LOGARITHMIC POTENTIAL OF THE DOUBLE DISTRIBUTION

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Necessary and sufficient conditions are established securing the existence of limits of the logarithmic potential of the double distribution at points of the curve on which the distribution is spread.

INTRODUCTION

Investigation of the logarithmic potential of the double distribution is a classical topic occuring in a number of applications. The Fredholm method for solution of the Dirichlet problem included in most courses of Integral Equations and Partial Differential Equations may serve as an example. It is well known that this method is based on the behaviour of the potential

$$W_F(z) = \operatorname{Im} \int_K \frac{F(\zeta)}{\zeta - z} \, d\zeta$$

as z approaches the curve K on which the double distribution is spread. To secure the existence of limits of $W_F(z)$ as z converges to an arbitrary point in K along any path not meeting K the curve is usually submitted to diverse restrictions. (It has been recognized that, for a simple closed curve K and a continuous function F on it, $W_F(z)$ need not admit a continuous extension from the bounded complementary domain D of K to $\bar{D} = D \cup K$ even if K is smooth, i.e. K possesses a tangent $\tau(\zeta)$ at any $\zeta \in K$ varying continuously as ζ describes K.) Most frequently it is assumed that K fulfils so-called Ljapunoff condition (cf. [8], remark 2, § 7, chap. I; in [3] analogous surfaces in 3-space are considered) or that K is a curve of bounded rotation („Kurve beschränkter Drehung") as introduced by J. RADON in his classical memoir [10] (cf. also [11]. According to the bibliography given in [7], a translation of Radon's memoir appeared in Uspekhi mat. nauk in 1946.) At this junction the work of T. CARLEMAN [2], which is not available to the author, is also usually mentioned.

The present paper aims at showing that simple necessary and sufficient geometric conditions can be established securing the existence of above mentioned limits for any continuous F on a simple oriented curve K of finite length. Let K be such a curve in the Euclidean plane E_2 (which is identified with the set of finite complex numbers). Given $z \notin K$ write $a(z) = \Delta_K \arg(\zeta - z)$ for the increment of the argument of $\zeta - z$ as ζ describes K. Let $C(M)$ be the Banach space of all (real-valued) bounded continuous functions F on $M \subset E_2$ with the norm $\|F\|_M = \sup_z |F(z)|$, $z \in M$. With every $F \in C(E_2)$ we associate a (continuous) function UF on $G = E_2 - K$ defined by

$$U\,F(z) = \operatorname{Im} \int_K \frac{F(\zeta)}{\zeta - z}\, d\zeta - F(z)\, a(z)\,, \quad z \in G\,.$$

If $M \subset G$ is any set containing K in its closure then the following conditions (I), (II) are equivalent to each other:

(I) For every $F \in C(E_2)$, UF is uniformly continuous on M.

(II) $$+\infty > \sup_\zeta v^K(\zeta)\,, \quad \zeta \in K\,,$$

where $v^K(\zeta) = \int_0^{2\pi} \mu^K(\zeta, \alpha)\, d\alpha$ and $\mu^K(\zeta, \alpha)$ stands for the number of points at which K meets the half-line

$$\{z;\ z = \zeta + r \exp i\alpha,\ r > 0\}\,.$$

In particular, if D is a Jordan domain with oriented rectifiable boundary K, then (II) is a necessary and sufficient condition that, for every continuous function F on K, the corresponding potential $W_F(z)$ be extendable from D to a continuous function on $\bar{D} = D \cup K$.

Similar results for non-simple curves are also established and some further results concerning the operator

$$U : F \to UF$$

are obtained.

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In this paragraph some auxiliary results are collected. In particular, theorem 1.11 will be our main tool in § 2.

The term path (on $\langle a, b\rangle$) is taken to mean a continuous complex-valued function on a compact interval $\langle a, b\rangle = \{t;\ a \leqq t \leqq b\}$. Given such a path ψ, $z \in E_2$ and $\alpha \in E_1$ (= the set of finite real numbers) we denote by $\mu^\psi(z, \alpha)$ the number $(0 \leqq \leqq \mu^\psi(z, \alpha) \leqq +\infty)$ of points in

$$\{t;\ t \in \langle a, b\rangle,\ \psi(t) \neq z,\ \psi(t) - z = |\psi(t) - z| \exp i\alpha\}\,.$$

If f is a finite (real- or complex-valued) function on an arbitrary interval J we write var $[f; J]$ for the variation of f on J which is defined as the least upper bound of all the sums

$$\sum_{j=1}^{n} |f(b_j) - f(a_j)|$$

$\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle$ ranging over all finite systems of non-overlapping compact intervals contained in J. If necessary, a more explicit notation of the kind $\text{var}_t\,[f(t); J]$ will be used.

Proof of the following lemma may be left to the reader.

1.1. Lemma. *Let φ be a continuous complex-valued function on the interval J (which need not be closed) and let V be a segment disjoint with $\varphi(J)$. Then there exists a single-valued continuous argument of $\varphi(t) - z$ on $J \times V$, i.e. a continuous real-valued function $\vartheta(t, z)$ on $J \times V$ with*

$$\varphi(t) - z = |\varphi(t) - z| \exp i\,\vartheta(t, z)\,, \quad t \in J\,, \quad z \in V\,.$$

1.2. Lemma. *Let ψ be a path on $\langle a, b\rangle$, $z \in E_2$ and let $\mathfrak{G}$ be the system of all components of $\langle a, b\rangle - \psi^{-1}(z) = \{t; t \in \langle a, b\rangle, \psi(t) \neq z\}$. With every $J \in \mathfrak{G}$ we associate a single-valued continuous argument $\vartheta_J(t)$ of $\psi(t) - z$ on J. Then $\mu^\psi(z, \alpha)$ is Lebesgue measurable with respect to the variable α on $\langle 0, 2\pi\rangle$ and*

$$\int_0^{2\pi} \mu^\psi(z, \alpha)\,\mathrm{d}\alpha = \sum_J \text{var}\,[\vartheta_J; J]\,, \quad J \in \mathfrak{G}\,.$$

Proof of this lemma is easily obtained on account of the well-known Banach theorem on variation of a continuous function (cf. [9], chap. VIII, § 5) and may be found in [5] (cf. lemma 2.2).

By preceding lemma the following definition is justified:

1.3. Definition. *Given a path ψ and $z \in E_2$ we define*

$$v^\psi(z) = \int_0^{2\pi} \mu^\psi(z, \alpha)\,\mathrm{d}\alpha\,.$$

1.4. Notation. Let φ be a continuous complex-valued function on an interval J (which need not be closed) and suppose that $\varphi(t) \neq 0$ for every $t \in J$. Let ϑ be a single-valued continuous argument of φ on J. If $a \leqq b$ are the end-points of J and if there exist finite limits $\vartheta(a+) = \lim_{t \to a+} \vartheta(t)$, $\vartheta(b-) = \lim_{t \to b-} \vartheta(t)$ we put

$$\Delta \arg\,[\varphi; J] = \vartheta(b-) - \vartheta(a+)\,. \tag{1}$$

Clearly, neither the existence nor the value of the difference $\vartheta(b-) - \vartheta(a+)$ depend on the choice of ϑ so that the definition (1) is justified. If necessary, we shall write $\Delta_x \arg\,[\varphi(x); J]$ instead of $\Delta \arg\,[\varphi; J]$.

The following lemma is obvious:

1.5. Lemma. *Let $z^1, z^2, z \in E_2$ and suppose that z does not belong to the segment with end-points z^1, z^2. Then*

$$|\Delta_x \arg\,[z^1 + x(z^2 - z^1); \langle 0, 1\rangle]| < \pi\,.$$

Using 1.5 we obtain

1.6. Lemma. *Let φ be a continuous complex-valued function on the interval J and suppose that the segment V with end-points z^1, z^2 does not meet $\varphi(J)$. Then*

$$|\Delta \arg [\varphi - z^1; J] - \Delta \arg [\varphi - z^2; J]| \leqq 2\pi \tag{2}$$

provided the symbols $\Delta \arg [\varphi - z^k; J]$ $(k = 1, 2)$ are meaningful.

Proof. By lemma 1.1 there is a continuous single-valued argument $\vartheta(t, z)$ of $\varphi(t) - z$ on $J \times V (t \in J, z \in V)$. Fix a $t_0 \in J$. It is easily seen that $\pi + \vartheta(t_0, z^1 + x(z^2 - z^1))$ is a continuous argument of $z^1 + x(z^2 - z^1) - \varphi(t_0)$ on $\{x; 0 \leqq x \leqq 1\}$ so that, by lemma 1.5,

$$|\vartheta(t_0, z^2) - \vartheta(t_0, z^1)| < \pi . \tag{3}$$

Let $u \leqq v$ be an arbitrary pair of points in J. Using (3) with $t_0 = u$ and $t_0 = v$ we derive

$$\begin{aligned} &|\Delta \arg [\psi - z^1; \langle u, v\rangle] - \Delta \arg [\psi - z^2; \langle u, v\rangle]| = \\ &= |\vartheta(v, z^1) - \vartheta(u, z^1) - \vartheta(v, z^2) + \vartheta(u, z^2)| \leqq \\ &\leqq |\vartheta(v, z^2) - \vartheta(v, z^1)| + |\vartheta(u, z^2) - \vartheta(u, z^1)| < 2\pi . \end{aligned}$$

Making $u \to a +$ and $v \to b -$ (where a, b are the end-points of J) we obtain (2).

1.7. Notation. If ψ is a path on $\langle a, b\rangle$ and $z \in E_2$ we write $N_\psi(z)$ for the number (possibly zero or infinite) of points in $\psi^{-1}(z) = \{t; t \in \langle a, b\rangle, \psi(t) = z\}$.

1.8. Lemma. *Let ψ be a path on $\langle a, b\rangle$, $\zeta \in \psi(\langle a, b\rangle)$, $z \in E_2 - \psi(\langle a, b\rangle)$ and suppose that the segment with end-points z, ζ meets $\psi(\langle a, b\rangle)$ at ζ only. Then*

$$|\Delta \arg [\psi - z; \langle a, b\rangle]| \leqq v^\psi(\zeta) + 2\pi(N_\psi(\zeta) + 1) . \tag{4}$$

Proof. To prove this lemma we may clearly suppose that the right-hand side in (4) is finite. We shall first assume that

$$\psi^{-1}(\zeta) \cap (a, b) = \emptyset \tag{5}$$

(so that $\zeta \in \{\psi(a), \psi(b)\}$). Then there is a continuous single-valued argument $\vartheta(t)$ of $\psi(t) - \zeta$ on (a, b). Since $\operatorname{var} [\vartheta; (a, b)] = v^\psi(\zeta) < + \infty$ (cf. 1.2 and 1.3) the limits $\vartheta(a+)$, $\vartheta(b-)$ exist and

$$|\Delta \arg [\psi - \zeta; (a, b)]| = |\vartheta(b-) - \vartheta(a+)| \leqq \operatorname{var} [\vartheta; (a, b)] = v^\psi(\zeta) . \tag{6}$$

By lemma 1.6 we conclude that

$$|\Delta \arg [\psi - z; \langle a, b\rangle]| = |\Delta \arg [\psi - z; (a, b)]| \leqq 2\pi + |\Delta \arg [\psi - \zeta; (a, b)]|$$

which together with (6) implies

$$|\Delta \arg [\psi - z; \langle a, b\rangle]| \leqq 2\pi + v^\psi(\zeta) . \tag{7}$$

Now we shall drop the assumption (5). Since $N_\psi(\zeta) < +\infty$ we may divide $\langle a, b\rangle$ into $p \leqq N_\psi(\zeta) + 1$ non-overlapping intervals $\langle a_j, b_j\rangle$ such that

$$\psi^{-1}(\zeta) \cap (a_j, b_j) = \emptyset\,, \quad 1 \leqq j \leqq p\,.$$

Put $\psi^j = \psi|_{\langle a_j, b_j\rangle}$. If the half-line $\{\zeta + r \exp i\alpha;\ r > 0\}$ does not meet $\bigcup_{j=1}^{p} \{\psi(a_j), \psi(b_j)\}$ (which is true for almost every $\alpha \in \langle 0, 2\pi\rangle$) then $\mu^\psi(\zeta; \alpha) = \sum_{j=1}^{p} \mu^{\psi^j}(\zeta; \alpha)$ whence

$$v^\psi(\zeta) = \sum_{j=1}^{p} v^{\psi^j}(\zeta)\,. \tag{8}$$

Using (7) with ψ^j and $\langle a_j, b_j\rangle$ instead of ψ and $\langle a, b\rangle$ we obtain

$$|\Delta \arg [\psi - z; \langle a_j, b_j\rangle]| \leqq 2\pi + v^{\psi^j}(\zeta)\,, \quad 1 \leqq j \leqq p\,. \tag{9}$$

On account of (8), (9) we have

$$|\Delta \arg [\psi - z; \langle a, b\rangle]| \leqq \sum_{j=1}^{p} |\Delta \arg [\psi - z; \langle a_j, b_j\rangle]| \leqq$$
$$\leqq 2\pi p + v^\psi(\zeta) \leqq 2\pi(N_\psi(\zeta) + 1) + v^\psi(\zeta)$$

and the proof is complete.

The following well-known property of harmonic functions will be used below:

1.9. *Let $G \subset E_2$ be an open set with boundary $B \neq \emptyset$ and let h be a harmonic function on G such that $\lim_{|z|\to+\infty,\, z\in G} h(z) = 0$. Suppose that $\limsup_{z\to\zeta,\, z\in G} h(z) \leqq c \ (\in E_1)$ for every $\zeta \in B$. Then $h(z) \leqq c$ for every $z \in G$.*

1.10. Lemma. *Let ψ be a path on $\langle a, b\rangle$. Then $h(z) = \Delta \arg [\psi - z; \langle a, b\rangle]$, considered as a function of the variable z, is harmonic on $E_2 - \psi(\langle a, b\rangle)$ and $\lim_{|z|\to+\infty} h(z) = 0$.*

Proof. $h(z) = \Delta \arg [\psi - z; \langle a, b\rangle]$ is the imaginary part of the increment (to be denoted by $I(z)$) of $\log(\zeta - z)$ as ζ describes ψ. $I(z)$ being an analytic function of the variable z on $E_2 - \psi(\langle a, b\rangle)$, h is harmonic on $E_2 - \psi(\langle a, b\rangle)$. It is easily seen that, for sufficiently large $|z|$, $|h(z)| = |\Delta \arg |\psi - z; \langle a, b\rangle]|$ is merely the radian measure of the (acute) angle enclosed by the vectors $\psi(a) - z$ and $\psi(b) - z$ whence $h(z) \to 0$ as $|z| \to +\infty$.

Now we are able to prove the main result of this paragraph.

1.11. Theorem. *Let ψ be a path on $\langle a, b\rangle$, $K = \psi(\langle a, b\rangle)$ and suppose that*

$$+\infty > \sup_\zeta (N_\psi(\zeta) + v^\psi(\zeta))\,, \quad \zeta \in K\,.$$

Then $+\infty > \sup_z v^\psi(z)$, $z \in E_2$.

Proof. Put

$$c_1 = \sup_{\zeta \in K} N_\psi(\zeta)\,, \quad c_2 = \sup_{\zeta \in K} v^\psi(\zeta)\,, \quad G = E_2 - K\,.$$

We shall prove that, for any $z_0 \in G$,

$$v^\psi(z_0) \leqq c_2 + 2c_1(c_2 + 2\pi(c_1 + 1)) = c\,. \tag{10}$$

Fix a $z_0 \in G$, a $d \in E_1$ with $d < v^\psi(z_0)$ and denote by $\vartheta(t)$ a continuous single-valued argument of $\psi(t) - z_0$ on $\langle a, b\rangle$. Since

$$\operatorname{var}[\vartheta; \langle a, b\rangle] = v^\psi(z_0) > d$$

we can find a subdivision $\{a = t_0 < \ldots < t_n = b\}$ of $\langle a, b\rangle$ such that $d < \sum_{j=1}^{n} |\vartheta(t_j) - \vartheta(t_{j-1})|$. Put $s_j = \operatorname{sign}(\vartheta(t_j) - \vartheta(t_{j-1}))$ and define

$$h(z) = \sum_{j=1}^{n} s_j \Delta_t \arg[\psi(t) - z; \langle t_{j-1}, t_j\rangle],\ z \in G.$$

Clearly,

$$d < h(z_0) \quad \left(= \sum_{j=1}^{n} |\vartheta(t_j) - \vartheta(t_{j-1})|\right) \tag{11}$$

and h is a harmonic function on G with $\lim_{|z| \to +\infty} h(z) = 0$ (cf. 1.10). We shall prove that, for every $\zeta \in K$,

$$\limsup_{\substack{z \to \zeta \\ z \in G}} h(z) \leqq c\,. \tag{12}$$

Hence it follows by 1.9 that $h \leqq c$ on G; in particular, $d < h(z_0) \leqq c$. Since d was an arbitrary number with $d < v^\psi(z_0)$, (10) will be established.

Fix a $\zeta \in K$, denote by $\mathfrak{G}_1$ the set of all $j \in \{1, \ldots, n\}$ with $\langle t_{j-1}, t_j\rangle \cap \psi^{-1}(\zeta) = \emptyset$ and put $\mathfrak{G}_2 = \{1, \ldots, n\} - \mathfrak{G}_1$. Let $K_k = \bigcup_j \psi(\langle t_{j-1}, t_j\rangle)$, $j \in \mathfrak{G}_k$ and define the function h_k of the variable z on $E_2 - K_k = G_k$ by

$$h_k(z) = \sum_j s_j \Delta \arg[\psi - z; \langle t_{j-1}, t_j\rangle]\,, \quad j \in \mathfrak{G}_k \quad (k = 1, 2)\,.$$

Since $\zeta \in G_1$ and h_1 is continuous on G_1 we have

$$\lim_{\substack{z \to \zeta \\ z \in G}} h_1(z) = h_1(\zeta) \leqq \sum_{j \in \mathfrak{G}_1} |\Delta \arg[\psi - \zeta; \langle t_{j-1}, t_j\rangle]| \leqq \tag{13}$$

$$\leqq (\text{cf. } 1.2 \text{ and } 1.3) \leqq v^\psi(\zeta) \leqq c_2\,.$$

On the other hand, with every $z \in G_2$ we can associate a $\zeta_z \in K_2$ such that the segment with end-points z and ζ_z meets K_2 at ζ_z only. By lemma 1.8 we obtain for every $j \in \mathfrak{G}_2$

$$|\Delta \arg [\psi - z; \langle t_{j-1}, t_j \rangle]| \leqq v^{\psi}(\zeta_z) + 2\pi(N_{\psi}(\zeta_z) + 1) \leqq c_2 + 2\pi(c_1 + 1).$$

The number of points in $\psi^{-1}(\zeta)$ does not exceed c_1 and, consequently, the number of elements in $\mathfrak{G}_2$ does not exceed $2c_1$. We conclude that, for every $z \in G_2$,

$$h_2(z) \leqq \sum_{j \in \mathfrak{G}_2} |\Delta \arg [\psi - z; \langle t_{j-1}, t_j \rangle]| \leqq 2c_1(c_2 + 2\pi(c_1 + 1)).$$

We have thus by (13)

$$\limsup_{\substack{z \to \zeta \\ z \in G}} h(z) = \limsup (h_1(z) + h_2(z)) \leqq \lim_{\substack{z \to \zeta \\ z \in G}} h_1(z) + \\ + \sup_{z \in G} h_2(z) \leqq c_2 + 2c_1(c_2 + 2\pi(c_1 + 1))$$

and (12) is established. Thus the proof is complete.

1.12. Proposition. *Let ψ be a path on $\langle a, b \rangle$, $K = \psi(\langle a, b \rangle)$. Then v^{ψ} is lower semicontinuous on E_2. Further suppose that* var $[\psi; \langle a, b \rangle] < +\infty$. *Given $z \notin K$ put $\varrho(z) = \inf \{|\zeta - z|; \zeta \in K\}$ and denote by $\vartheta_z(t)$ a continuous single-valued argument of $\psi(t) - z$ on $\langle a, b \rangle$. Then*

$$v^{\psi}(z) = \operatorname{var} [\vartheta_z; \langle a, b \rangle] \leqq \varrho^{-1}(z) \operatorname{var} [\psi; \langle a, b \rangle], \quad z \notin K,$$

$$\operatorname{var} [\vartheta_u - \vartheta_v; \langle a, b \rangle] \leqq |u - v| \operatorname{var} [\psi; \langle a, b \rangle] \varrho^{-1}(u) \varrho^{-1}(v), \\ u \notin K, \quad v \notin K.$$

Proof. Fix a $z_0 \in E_2$ and let d be an arbitrary real number with $d < v^{\psi}(z_0)$. It follows easily from lemma 1.2 (cf. also 1.3) that there is a finite system of non-overlapping intervals $\langle a_j, b_j \rangle$ $(j = 1, \ldots, n)$ contained in $\langle a, b \rangle - \psi^{-1}(z_0)$ such that

$$\sum_{j=1}^{n} |\Delta \arg [\psi - z_0; \langle a_j, b_j \rangle]| > d.$$

Clearly,

$$\lim_{z \to z_0} \Delta \arg [\psi - z; \langle a_j, b_j \rangle] = \Delta \arg [\psi - z_0; \langle a_j, b_j \rangle], \quad 1 \leqq j \leqq n,$$

because $z_0 \notin \bigcup_{j=1}^{n} \psi(\langle a_j, b_j \rangle)$. Noting that

$$v^{\psi}(z) \geqq \sum_{j=1}^{n} |\Delta \arg [\psi - z; \langle a_j, b_j \rangle]|$$

whenever $z \notin \bigcup_{j=1}^{n} \psi(\langle a_j, b_j \rangle)$, we conclude that

$$\liminf_{z \to z_0} v^{\psi}(z) \geqq \lim_{z \to z_0} \sum_{j=1}^{n} |\Delta \arg [\psi - z; \langle a_j, b_j \rangle]| = \sum_{j=1}^{n} |\Delta \arg [\psi - z_0; \langle a_j, b_j \rangle]| > d$$

and the lower-semicontinuity of v^{ψ} is established.

Further suppose that $\operatorname{var}[\psi; \langle a, b\rangle] < +\infty$. We have for every $z \notin K$ and $\tau \in \langle a, b\rangle$

$$\Delta_t \arg[\psi(t) - z; \langle a, \tau\rangle] = \operatorname{Im} \int_a^\tau \frac{1}{\psi(t) - z} \, d\psi(t)$$

so that

$$\vartheta_z(\tau) = \vartheta_z(a) + \operatorname{Im} \int_a^\tau \frac{1}{\psi(t) - z} \, d\psi(t), \quad a \leqq \tau \leqq b.$$

Hence we derive for any subdivision $\{a = \tau_0 < \ldots < \tau_n = b\}$ of $\langle a, b\rangle$

$$\sum_{j=1}^{n} |\vartheta_z(\tau_j) - \vartheta_z(\tau_{j-1})| \leqq \sum_{j=1}^{n} \left| \int_{\tau_{j-1}}^{\tau_j} \frac{1}{\psi(t) - z} \, d\psi(t) \right| \leqq \varrho^{-1}(z) \operatorname{var}[\psi; \langle a, b\rangle].$$

Consequently,

$$\operatorname{var}[\vartheta_z; \langle a, b\rangle] = v^\psi(z) \leqq \varrho^{-1}(z) . \operatorname{var}[\psi; \langle a, b\rangle].$$

Using similar arguments we obtain for every pair of points u, v in $E_2 - K$

$$\vartheta_u(\tau) - \vartheta_v(\tau) = \vartheta_u(a) - \vartheta_v(a) + \operatorname{Im} \int_a^\tau \left(\frac{1}{\psi(t) - u} - \frac{1}{\psi(t) - v} \right) d\psi(t), \quad \tau \in \langle a, b\rangle,$$

$$\operatorname{var}[\vartheta_u - \vartheta_v; \langle a, b\rangle] \leqq \sup_{a \leqq t \leqq b} \left| \frac{1}{\psi(t) - u} - \frac{1}{\psi(t) - v} \right| . \operatorname{var}[\psi; \langle a, b\rangle] \leqq$$
$$\leqq |u - v| \varrho^{-1}(u) . \varrho^{-1}(v) . \operatorname{var}[\psi; \langle a, b\rangle].$$

In particular,

$$|v^\psi(u) - v^\psi(v)| = |\operatorname{var}[\vartheta_u; \langle a, b\rangle] - \operatorname{var}[\vartheta_v; \langle a, b\rangle]| \leqq$$
$$\leqq \operatorname{var}[\vartheta_u - \vartheta_v; \langle a, b\rangle] \leqq |u - v| . \varrho^{-1}(u) . \varrho^{-1}(v) . \operatorname{var}[\psi; \langle a, b\rangle]$$

and v^ψ is locally Lipschitzian on $E_2 - K$.

Remark. Theorems 1.11 and 1.12 together with further results concerning v^ψ were announced in [6].

2

We shall assume throughout that ψ is a path on $\langle a, b\rangle$ with $\operatorname{var}[\psi; \langle a, b\rangle] < +\infty$, $K = \psi(\langle a, b\rangle)$. If $z \notin K$ then $\vartheta_z(t)$ will stand for a single-valued continuous argument of $\psi(t) - z$ on $\langle a, b\rangle$. Further we put

$$a(z) = \Delta \arg[\psi - z; \langle a, b\rangle] \quad (= \vartheta_z(b) - \vartheta_z(a)).$$

2.1. Definition. *Given $z \notin K$ we define for every (real-valued) continuous function f on $\langle a, b\rangle$*

$$w_\psi(z;f) = \int_a^b f(t)\,\mathrm{d}\vartheta_z(t) \quad \left(= \operatorname{Im}\int_a^b \frac{f(t)}{\psi(t)-z}\,\mathrm{d}\psi(t)\right).$$

If F is a continuous function on K we put

$$W_\psi(z; F) = w_\psi(z;f)\,, \quad \textit{where} \quad f(t) = F(\psi(t))\,, \quad a \leqq t \leqq b\,.$$

2.2. Remark. Clearly, this definition does not depend on the choice of the argument $\vartheta_z(t)$. It follows easily from 1.12 that $w_\psi(z;f)$ is a continuous function of the variable z on $E_2 - K$ and $\lim\limits_{|z|\to+\infty} w_\psi(z;f) = 0$.

2.3. Notation. As in the introduction, we shall write $C(M)$ for the Banach space of all (real-valued) bounded continuous functions on $M \subset E_2$, $G = E_2 - K$. For every $F \in C(E_2)$ we put

$$UF(z) = W_\psi(z; F) - F(z)\,a(z)\,, \quad z \in G\,.$$

UF is continuous on G and $UF(z) \to 0$ as $|z| \to +\infty$.

2.4. Theorem. *Suppose that $+\infty > \sup\limits_\zeta N_\psi(\zeta)$, $\zeta \in K$. If v^ψ is bounded on a set dense in K then, for every $F \in C(E_2)$, UF is bounded and uniformly continuous on G and the operator*

$$U : F \to UF$$

from $C(E_2)$ into $C(G)$ is bounded.

Proof. By 1.12 and 1.11, v^ψ is bounded on E_2. Put $k = \sup\limits_z v^\psi(z)$, $z \in E_2$. Given $z \in G$ and $F \in C(E_2)$, $\|F\| \leqq 1$, we have

$$|a(z)| = |\vartheta_z(b) - \vartheta_z(a)| \leqq \operatorname{var}[\vartheta_z; \langle a, b\rangle] = v^\psi(z) \leqq k\,,$$

$$|W_\psi(z; F)| = \left|\int_a^b F(\psi(t))\,\mathrm{d}\vartheta_z(t)\right| \leqq \operatorname{var}[\vartheta_z; \langle a, b\rangle] \leqq k\,,$$

so that $|U\,F(z)| \leqq 2k$. We see that $UF \in C(G)$ and $\|U\| \leqq 2k$.

To prove that UF is uniformly continuous on G ($F \in C(E_2)$) it is sufficient to show that the sequence

$$\{U\,F(z_n)\}_{n=1}^\infty$$

si convergent provided $z_n \in G$, $z_n \to \zeta \in K$ ($n \to \infty$). Let us fix such a sequence $\{z_n\}_{n=1}^\infty$. If F reduces to a constant γ on K then $W_\psi(z; F) = \gamma\,.\,(\vartheta_z(b) - \vartheta_z(a)) = \gamma\,.\,a(z)$ ($z \in G$) and $UF(z_n) = (\gamma - F(z_n))\,.\,a(z_n) \to 0$ as $n \to \infty$ (note that $|a(z_n)| \leqq$

$\leqq v^{\psi}(z_n) \leqq k$). We may therefore suppose that $F(\zeta) = 0$. Assuming this we shall show that, for any $\varepsilon > 0$, such a $s(\varepsilon)$ and n_0 can be assigned that

$$n > n_0 \Rightarrow |U F(z_n) - s(\varepsilon)| < \varepsilon . \tag{14}$$

Given $\varepsilon > 0$ we can find disjoint intervals $\langle a_j, b_j \rangle \subset \langle a, b \rangle - \psi^{-1}(\zeta)$ $(1 \leqq j \leqq p;$ $p \leqq N_\psi(\zeta) + 1)$ such that $k|F(\psi(t))| < \frac{1}{3}\varepsilon$ whenever $t \in \langle a, b \rangle - \bigcup_{j=1}^{p} \langle a_j, b_j \rangle = R$. For every j,

$$\left| \int_{a_j}^{b_j} F(\psi(t)) \, d\vartheta_{z_n}(t) - \int_{a_j}^{b_j} F(\psi(t)) \, d\vartheta_\zeta(t) \right| \leqq$$

$$\leqq \|F\| \operatorname{var} [\vartheta_{z_n} - \vartheta_\zeta; \langle a_j, b_j \rangle] \to 0 \text{ as } n \to \infty \text{ (cf. 1.12)} .$$

We have thus a n_1 with

$$n > n_1 \Rightarrow \left| \sum_{j=1}^{p} \int_{a_j}^{b_j} F(\psi(t)) \, d\vartheta_{z_n}(t) - \sum_{j=1}^{p} \int_{a_j}^{b_j} F(\psi(t)) \, d\vartheta_\zeta(t) \right| < \tfrac{1}{3}\varepsilon .$$

Since $|F(z) \, a(z)| \leqq k|F(z)|$ and $F(z_n) \to 0$ as $n \to \infty$ we can fix a n_2 with

$$n > n_2 \Rightarrow |F(z_n) \, a(z_n)| < \tfrac{1}{3}\varepsilon .$$

Put

$$s(\varepsilon) = \sum_{j=1}^{p} \int_{a_j}^{b_j} F(\psi(t)) \, d\vartheta_\zeta(t) , \quad b_0 = a , \quad a_{p+1} = b , \quad n_0 = \max(n_1, n_2) .$$

We have then for every $n > n_0$

$$|UF(z_n) - s(\varepsilon)| \leqq \left| \sum_{j=1}^{p} \int_{a_j}^{b_j} F(\psi(t)) \, d\vartheta_{z_n}(t) - \sum_{j=1}^{p} \int_{a_j}^{b_j} F(\psi(t)) \, d\vartheta_\zeta(t) \right| +$$

$$+ \sum_{j=0}^{p} \left| \int_{b_j}^{a_{j+1}} F(\psi(t)) \, d\vartheta_{z_n}(t) \right| + |F(z_n) \, a(z_n)| < \tfrac{1}{3}\varepsilon + \max_{t \in R} |F(\psi(t))| \, .$$

$$. \operatorname{var}_t [\vartheta_{z_n}(t); \langle a, b \rangle] + \tfrac{1}{3}\varepsilon \leqq \tfrac{2}{3}\varepsilon + \max_{t \in R} |F(\psi(t))| \, k \leqq \varepsilon$$

and (14) is verified.

The proof of the following simple lemma may be left to the reader.

2.5. Lemma. *Let ϑ be a continuous function of bounded variation on $\langle a, b \rangle$ and let Q be a finite subset in $\langle a, b \rangle$. Write $Z(Q)$ for the subspace of all $f \in C(\langle 0, 1 \rangle)$ vanishing on Q. Then*

$$\operatorname{var} [\vartheta; \langle a, b \rangle] = \sup_f \int_a^b f \, d\vartheta , \quad f \in Z(Q) , \quad \|f\| \leqq 1 .$$

2.6. Remark. We say that the path ψ(on $\langle a, b\rangle$) is simple provided $\psi(t_1) \neq \psi(t_2)$ whenever $0 < |t_1 - t_2| < b - a$, $t_1, t_2 \in \langle a, b\rangle$.

2.7. Theorem. *Let ψ be simple, $M \subset G$. Suppose that UF is bounded on M whenever $F \in C(E_2)$. Then v^ψ is bounded on $\overline{M}$ (= the closure of M).*

Proof. We know from 1.12 that v^ψ is lower-semicontinuous on E_2, continuous on G and $v^\psi(z) \to 0$ as $|z| \to \infty$. In order to verify

$$+\infty > \sup_z v^\psi(z)\,, \quad z \in \overline{M}\,,$$

it is therefore sufficient to prove that

$$\sup_n v^\psi(z_n) < +\infty \tag{15}$$

for every sequence of points $z_n \in M$ with $\lim_{n\to\infty} z_n = \zeta \in K$. Fix such a sequence and put $Q = \{a, b\} \cup \psi^{-1}(\zeta)$. With any $f \in Z(Q)$ (cf. 2.5 for notation) we can associate an $F \in C(E_2)$ such that

$$F(\psi(t)) = f(t) \ (a \leqq t \leqq b)\,, \quad F(z_n) = 0 \ (n = 1, 2, \ldots)\,.$$

Clearly, $UF(z_n) = \int_a^b f \,\mathrm{d}\vartheta_{z_n}$. UF being bounded, so must be the sequence $UF(z_n)$. We see that the sequence

$$\left\{\int_a^b f \,\mathrm{d}\vartheta_{z_n}\right\}_{n=1}^{\infty}$$

is bounded for every $f \in Z(Q)$. On account of the Banach-Steinhaus theorem and 2.5 we conclude that

$$\sup_n v^\psi(z_n) = \sup_n \operatorname{var}\left[\vartheta_{z_n}; \langle a, b\rangle\right] < +\infty$$

and (15) is checked.

Similar and even easier arguments leed to the following theorem, where ψ need not be simple.

2.8. Proposition. *Suppose that $w^\psi(z; f)$ is bounded on $M \subset G$ whenever $f \in C(\langle a, b\rangle)$. Then*

$$+\infty > \sup_z v^\psi(z)\,, \quad z \in \overline{M}\,. \tag{16}$$

Proof may be omitted.

2.9. Notation. Let ψ^j be a simple path on $\langle a_j, b_j\rangle$,

$$K_j = \psi^j(\langle a_j, b_j\rangle)\ (1 \leqq j \leqq p)\,, \quad K = \bigcup_{j=1}^{p} K_j\,.$$

Suppose that

$$1 \leqq j < k \leqq p \Rightarrow K_j \cap K_k = \emptyset\,.$$

For every $\alpha \in E_1$ denote by $\mu^K(z; \alpha)$ the number of points in $K \cap \{z + r \exp i\alpha; r > 0\}$. It is easily seen that

$$\mu^K(z, \alpha) = \sum_{j=1}^{p} \mu^{\psi^j}(z; \alpha)$$

provided the half-line $\{z + r \exp i\alpha; r > 0\}$ does not meet $\bigcup_{j=1}^{p} \{\psi^j(a_j), \psi^j(b_j)\}$. In particular, $\mu^K(z; \alpha)$ is Lebesgue measurable with respect to α (cf. 1.2) and we may put

$$v^K(z) = \int_0^{2\pi} \mu^K(z; \alpha) \, d\alpha .$$

Clearly, $v^K(z) = \sum_{j=1}^{p} v^{\psi^j}(z)$.

2.10. Theorem. *Let us keep the notation introduced in 2.9. Further suppose that* $\operatorname{var}[\psi^j; \langle a_j, b_j \rangle] < +\infty$ *and* ψ^j *is closed (i.e.* $\psi^j(a_j) = \psi^j(b_j)$*),* $1 \leqq j \leqq p$*. Let D be a domain (= open and connected set) disjoint with K and containing K in its closure. For every* $F \in C(K)$ *put*

$$WF(z) = \sum_{j=1}^{p} W_{\psi^j}(z; F), \quad z \in E_2 - K .$$

Then WF is continuous on $E_2 - K$, $WF(z) \to 0$ *as* $|z| \to +\infty$ *and the following conditions* (a)–(c) *are equivalent to each other*:

(a) *For every* $F \in C(K)$, *WF is uniformly continuous on D (or, which is the same, WF can be extended from D to a continuous function on* $\bar{D}$*).*

(b) $WF \in C(D)$ whenever $F \in C(K)$.

(c) $+\infty > \sup_{\zeta} v^K(\zeta), \ \zeta \in K.$

If (c) *takes place, then the operator*

$$W : F \to WF$$

from $C(K)$ *into* $C(D)$ *is bounded.*

Proof. Let us agree to write simply $W_{\psi^j}(z; F)$, WF instead of $W_{\psi^j}(z; F|_K)$, $WF|_K$ provided $F \in C(E_2)$. Put

$$a_j(z) = \Delta \arg [\psi^j - z; \langle a_j, b_j \rangle] ,$$

$$U_j F(z) = W_{\psi^j}(z; F) - a_j(z) F(z) , \quad z \in E_2 - K , \quad F \in C(E_2) .$$

Thus

$$WF(z) = \sum_{j=1}^{p} (U_j F(z) + a_j(z) F(z)) , \quad z \in E_2 - K , \quad F \in C(E_2) .$$

Noting that every a_j reduces to a constant on D (which is equal to zero if D is unbounded) and using 2.4 we easily verify that (a), (b) hold and the operator

$$W : F \to WF$$

from $C(K)$ into $C(D)$ is bounded provided (c) takes place, because every $F \in C(K)$ can be extended to an $\tilde{F} \in C(E_2)$ with $\|\tilde{F}\|_{E_2} = \|F\|_K$.

Conversely, let (a) take place. Fix a j $(1 \leqq j \leqq p)$ and an $\varepsilon > 0$ such that

$$\varepsilon < \operatorname{dist}(K_j, K_k) = \inf\{|z - \zeta|;\ z \in K_j,\ \zeta \in K_k\}$$

whenever $1 \leqq k \neq j \leqq p$. Further put $M_j = \{z;\ z \in D,\ \operatorname{dist}(z, K_j) < \varepsilon\}$. For every $k \neq j$, $U_k F$ is continuous on $\overline{M}_j \subset E_2 - K_k$ $(F \in C(E_2))$ and, consequently, uniformly continuous on M_j. Hence

$$U_j F = WF - a_j F - \sum_{k \neq j} (U_k F + a_k F)$$

must be uniformly continuous on M_j whenever $F \in C(E_2)$ and, by theorems 2.7 and 1.11, $+\infty > \sup_z v^{\psi_j}(z)$, $z \in E_2$. We see that

$$(a) \Rightarrow (c).$$

Similar reasonings show that

$$(b) \Rightarrow (c).$$

2.11. Remark. Main results of this article together with further theorems concerning the logarithmic potential of the double distribution were announced in [4]. Quite recently some related problems for the Eucleidean 3-space were treated by Ю. Д. Бураго, В. Г. Мазья and В. Д. Сапожникова in [1]. They consider a closed surface $\Gamma \subset E_3$ (= the Euclidean 3-space) and denote by Ω its bounded complementary domain. For every $P \in E_3$ they introduce a quantity denoted by v_P^0 which – in the case of a plane curve Γ – corresponds to our „cyclic variation" $v^\Gamma(P)$. They assume that

$$(*) \qquad \infty > \sup_P v_P^0, \quad P \in E_3$$

and then define the potential $W(P)$ of the double distribution (with a continuous density spread on Γ) and investigate ist behaviour. In particular, they announce a theorem showing that (*) is a sufficient condition for the existence of limits

$$W_i(S) = \lim_{P \to S} W(P), \quad P \in \Omega \quad \text{and} \quad W_e(S) = \lim_{P \to S} W(P), \quad P \in E_3 - \overline{\Omega}$$

for every $S \in \Gamma$. This is a result related to the theorem 4 announced in [4] and the theorem 2.4 proved in the present paper.

2.12. Remark. All paths considered in the present paper were supposed to be rectifiable. This is not necessary because every path ψ with $\sup_{\zeta} N_\psi(\zeta) < +\infty$ (cf. 1.7 for notation) is rectifiable provided $v^\psi(z_k) < +\infty$ for at least 3 points z_1, z_2, z_3 which are not situated on a single straight-line. (Cf. [6].) Since, by theorem 1.11, v^ψ is bounded on the whole plane whenever it is bounded on the curve itself, we see that the assumption concerning rectifiability of the paths considered might be dropped in 2.4 and, partly, in 2.10.

2.13. Remark. Suppose that Γ is a simple closed curve and D is its bounded complementary domain. For simplicity, let Γ be oriented in the counterclockwise sense. We have proved that

$$(**) \qquad \sup_{\zeta \in \Gamma} v^\Gamma(\zeta) < +\infty$$

is a necessary and sufficient condition that, for any continuous function F on Γ, the corresponding potential

$$W F(z) = \operatorname{Im} \int_\Gamma \frac{F(\zeta)}{\zeta - z} \, d\zeta$$

be extendable to a continuous function on $D \cup \Gamma$. Assume (**) and consider the operator

$$T : F \to TF$$

on the Banach space $C(\Gamma)$ (cf. 2.3) defined by

$$T F(\zeta) = \pi F(\zeta) - \lim_{\substack{z \to \zeta \\ z \in D}} W F(z), \quad z \in \Gamma .$$

This operator is bounded. The rôle played by T in connection with the Dirichlet problem is well known (cf. [11], n°s 81 and 91). If Γ is a curve of bounded rotation in the sense of Radon then the Fredholm radius of T is given by the Radon theorem ([11], n° 91); in particular, T is completely continuous provided there are no angular points on Γ. It is interesting to observe that Radon's theorem on the Fredholm radius of T ceases to hold for curves Γ submitted to (**) only. Indeed, example can be given of a curve Γ without angular points fulfilling (**) such that the corresponding operator T is not completely continuous. An expression for the Fredholm radius of T which is valid for any curve Γ submitted to (**) only was given in [4], theorem 5.

2.14. Remark. Investigation of the logarithmic potential of the double distribution is closely connected with investigation of Cauchy's type integrals which find many applications. Interested reader is referred to N. I. Muschelišvili's monograph [8] and the surveys in "Математика в СССР за сорок лет 1917–1957" for the bibliography on the subject.

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Резюме

О ЛОГАРИФМИЧЕСКОМ ПОТЕНЦИАЛЕ ДВОЙНОГО СЛОЯ

ЙОСЕФ КРАЛ (Josef Král), Прага

Путем на $\langle a, b\rangle$ подразумевается непрерывное отображение отрезка $\langle a, b\rangle$ в евклидову плоскость E_2 (которая отождествляется с множеством комплексных чисел). Пусть ψ – путь на $\langle a, b\rangle$, $\psi(\langle a, b\rangle) = K$. Для $\zeta \in K$ обозначим через $N_\psi(\zeta)$ число точек множества $\psi^{-1}(\zeta)$ $(0 \leqq N_\psi(\zeta) \leqq +\infty)$. Для $z \in E_2$ и действительного числа α обозначим через $\mu^\psi(z, \alpha)$ число точек множества $\{t;\ a \leqq t \leqq b,\ \psi(t) \neq z,\ \psi(t) - z = |\psi(t) - z| \,.\, \exp i\alpha\}$ $(0 \leqq \mu^\psi(z, \alpha) \leqq +\infty)$. Так как функция $\mu^\psi(z, \alpha)$ измерима относительно α, то можно полагать $v^\psi(z) = \int_0^{2\pi} \mu^\psi(z, \alpha)\, d\alpha$.

Сначала доказывается

Теорема. *Если* $\sup_{\zeta\in K} (N_\psi(\zeta) + v^\psi(\zeta)) < +\infty$, *то* $\sup_{z\in E_2} v^\psi(z) < +\infty$.

Для $M \subset E_2$ обозначим через $C(M)$ пространство всех действительных ограниченных непрерывных функций F на M с нормой $\|F\| = \sup_{z\in M} |F(z)|$. Предпола-

гая путь ψ спрямляемым, положим для каждой функции $F \in C(E_2)$ и каждой точки $z \in G = E_2 - K$

$$UF(z) = \operatorname{Im} \int_{\psi} \frac{F(\zeta)}{\zeta - z} \, d\zeta - F(z) \, a(z) ,$$

где $a(z)$ обозначает приращение непрерывно изменяющегося аргумента комплексного числа $\zeta - z$ при перемещении ζ вдоль пути ψ.

На основе предшествующей теоремы получается

Теорема. *Пусть* $\sup_{\zeta \in K} N_{\psi}(\zeta) < +\infty$. *Если функция* $v^{\psi}(\zeta)$ *переменного* ζ *ограничена на множестве, плотном в* K, *то для каждой функции* $F \in C(E_2)$ *функция* UF *равномерно непрерывна на* G *и оператор*

$$U : F \to UF ,$$

действующий из $C(E_2)$ *в* $C(G)$, *ограничен.*

Наоборот, если для каждой функции $F \in C(E_2)$ *функция* UF *ограничена на множестве* $M \subset G$, *то функция* $v^{\psi}(z)$ *переменного* z *ограничена на замыкании* $\overline{M}$ *множества* M.

Предположим теперь, что ψ^j – простой замкнутый путь на $\langle a_j, b_j \rangle$ (это значит, что $\psi^j(u) = \psi^j(v)$ верно для $u < v$ тогда и только тогда, если $u = a_j$ и $v = b_j$), $K_j = \psi(\langle a_j, b_j \rangle)$ $(j = 1, \ldots, p)$ и

$$1 \leqq j < k \leqq p \Rightarrow K_j \cap K_k = \emptyset .$$

Положим $K = \bigcup_{j=1}^{p} K_j$, $v^K(z) = \sum_{j=1}^{p} v^{\psi^j}(z)$. Пусть, далее, D-область, $D \cap K = \emptyset$, $K \subset \overline{D}$. Считая пути ψ^j $(j = 1, \ldots, p)$ спрямляемыми, полагаем для $F \in C(K)$ и $z \in D$

$$WF(z) = \sum_{j=1}^{p} \operatorname{Im} \int_{\psi^j} \frac{F(\zeta)}{\zeta - z} \, d\zeta .$$

Из предшествующих теорем вытекает

Теорема. *Для того, чтобы для каждой функции* $F \in C(K)$ *функция* WF *оказалась равномерно непрерывной на* D *(или, что то же самое,* WF *продолжалась до непрерывной функции на* $D \cup K$*) необходимо и достаточно, чтобы*

$$\text{(A)} \qquad \sup_{\zeta \in K} v^K(\zeta) < +\infty .$$

Если имеет место (A), *то оператор*

$$W\colon \ F \to WF ,$$

действующий из $C(K)$ *в* $C(D)$, *ограничен.*