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SOME INEQUALITIES CONCERNING THE CYCLIC AND RADIAL VARIATIONS OF A PLANE PATH-CURVE

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Let ψ be a path-curve in the Euclidean plane E_2, $z \in E_2$. Given real numbers $r > 0$ and α write $\mu_r^\psi(\alpha; z)$ for the number of points at which ψ meets the segment $S_r^\alpha(z) = \{\zeta; \zeta = z + \varrho \exp i\alpha, \ 0 < \varrho < r\}$. Then $\mu_r^\psi(\alpha; z)$ is (Lebesgue) measurable with respect to α and one may put $v_r^\psi(z) = \int_0^{2\pi} \mu_r^\psi(\alpha; z)\, d\alpha$. Further let $\nu^\psi(\varrho; z)$ stand for the number of points at which ψ meets the circle $\{\zeta; |\zeta - z| = \varrho\}$. $\nu^\psi(\varrho; z)$ being measurable with respect to ϱ one may introduce the integral $u_r^\psi(z) = \int_0^r \nu^\psi(\varrho; z)\, d\varrho$. Suppose now that β is a fixed real number and ψ is a path-curve through z not meeting $\bigcup_\alpha S_{2r}^\alpha(z)$, $\alpha \in (\beta - \delta, \ \beta + \delta) \cup (\beta + \pi - \delta, \ \beta + \pi + \delta)$, where $0 < \delta < \pi/2$. Then

(1) $$\sup_{0<\varrho<r} \varrho^{-1} u_\varrho^\psi(z) \leqq K[v_r^\psi(z) + \sup_{0<\varrho<r} v_{2\varrho}^\psi(z + \varrho \exp i\beta)]\,,$$

(2) $$\sup_{0<\varrho<r} v_r^\psi(z + \varrho \exp i\beta) \leqq M[v_{2r}^\psi(z) + \sup_{0<\varrho<2r} \varrho^{-1} u_\varrho^\psi(z)]$$

with constants K, M depending on δ only. These inequalities are useful in connection with investigations concerning the boundary behaviour of the logarithmic potential of the double distribution.

1

In this paragraph some auxiliary results concerning functions of a real variable are collected. They will be used in § 2 below for the proof of some inequalities implying (1) and (2). The term interval will be used to mean any non-void convex subset of the real line E_1. The variation of a (finite real-valued) function f on a compact interval K, to be denoted by var $[f; K]$, is defined as usual. If f is a function on an arbitrary interval J, we put var $[f; J] = \sup_K$ var $[f; K]$, K ranging over all compact intervals in J. For every set $G \subset J$ open in J put var $[f; G] = \sum_I$ var $[f; I]$, I ranging over all components of G. Letting, as usual, var $[f; M] = \inf_G$ var $[f; G]$, $G \supset M$, G open in J, we extend var f to a Carathéodory outer measure defined for any $M \subset J$, which, if restricted to the system of all var f – measurable subsets in J, represents a measure (cf., e.g., [4]). The integral $\int_M H \, \mathrm{d} \operatorname{var} f$ of a (real-valued, possibly infinite) function H is always to be interpreted as the (Lebesgue-Stieltjes) integral with respect to this measure. The following known theorem will be frequently used below.

1.1. *Let f be a continuous function of finite variation on the interval I and let F be a (possibly infinite) function on $f(I)$. For every $x \in E_1$ denote by $N_f(x)$ the number of points in $f^{-1}(x)$ $(0 \leqq N_f(x) \leqq +\infty)$. Then N_f is Lebesgue measurable on E_1 and*

$$\int_I F(f) \, \mathrm{d} \operatorname{var} f = \int_{f(I)} F(x) \, N_f(x) \, \mathrm{d}x \tag{3}$$

provided the integral on the left-hand side exists. (The integral on the right-hand side is the ordinary Lebesgue integral.)

A proof of this assertion in the case that I is compact can be found in [1]. It is easily seen that the theorem extends to the case described above.

We shall also need the following formulation of the Banach theorem (cf. [3], part V, § V. 1):

1.2. *Let f be a continuous function on the interval I. Then*

$$\operatorname{var} [f; I] = \int_{f(I)} N_f(x) \, \mathrm{d}x \left(= \int_{-\infty}^{\infty} N_f(x) \, \mathrm{d}x \right),$$

N_f having the same meaning as in 1·1.

1.3. Lemma. *Let f, h be continuous functions on $\langle a, b \rangle = \{x; a \leqq x \leqq b\}$ and suppose that $\operatorname{var} [f; \langle a, b \rangle] < +\infty$. Let $D = \{a = x_0 < \ldots < x_n = b\}$ be an arbitrary subdivision of $\langle a, b \rangle$ and suppose that $\xi_i \in \langle x_{i-1}, x_i \rangle$ $(1 \leqq i \leqq n)$. Then*

$$\sum_i h(\xi_i) \, |f(x_i) - f(x_{i-1})| \to \int_a^b h \, \mathrm{d} \operatorname{var} f$$

as $\max_i (x_i - x_{i-1}) \to 0$.

Proof. Let us agree to write $|D| = \max_i (x_i - x_{i-1})$. Put $s(a) = 0$, $s(x) = \operatorname{var} [f; \langle a, x \rangle]$, $a < x \leqq b$. Then s is non-decreasing and

$$\sum_i h(\xi_i) \, [s(x_i) - s(x_{i-1})] \to \int_a^b h \, \mathrm{d} \operatorname{var} f \text{ as } |D| \to 0 .$$

On the other hand, $s(x_i) - s(x_{i-1}) \geqq |f(x_i) - f(x_{i-1})|$ and

$$\left|\sum_i h(\xi_i) \, [s(x_i) - s(x_{i-1})] - \sum_i h(\xi_i) \, |f(x_i) - f(x_{i-1})|\right| \leqq$$
$$\leqq \max_{a \leqq x \leqq b} |h(x)| \, . \, \{\operatorname{var} [f; \langle a, b \rangle] - \sum_i |f(x_i) - f(x_{i-1})|\} \to 0$$

as $|D| \to 0$ (cf. [2], chap. VIII, theorem 2).

We shall say that f has locally finite variation on J provided $\operatorname{var}[f; K] < +\infty$ for every compact interval $K \subset J$.

1.4. Lemma. *Let f be a continuous function of locally finite variation on the interval J. Let F be a function on $f(J)$ and suppose that F possesses a continuous derivative on $f(J)$. Then*

$$\operatorname{var}[F(f); J] = \int_J |F'(f)| \, \mathrm{d} \operatorname{var} f . \tag{4}$$

Proof. Let $\langle a, b\rangle$ be an arbitrary compact interval contained in J ($a < b$). We shall prove that

$$\operatorname{var}[F(f); \langle a, b\rangle] = \int_a^b |F'(f)| \, \mathrm{d} \operatorname{var} f . \tag{5}$$

The rest of the proof is obvious and will be left to the reader. Consider an arbitrary subdivision $D = \{a = x_0 < \ldots < x_n = b\}$ of $\langle a, b\rangle$. Between $f(x_i)$ and $f(x_{i-1})$ such a point y_i can be found that $F(f(x_i)) - F(f(x_{i-1})) = F'(y_i)(f(x_i) - f(x_{i-1}))$. Since f is continuous we have a $\xi_i \in \langle x_{i-1}, x_i\rangle$ with $f(\xi_i) = y_i (1 \leqq i \leqq n)$. We have thus

$$\sum_i |F(f(x_i)) - F(f(x_{i-1}))| = \sum_i |F'(f(\xi_i))| \cdot |f(x_i) - f(x_{i-1})| \, .$$

Making $|D| \to 0$ we obtain on account of 1.3 the formula (5) (cf. also [2], chap. VIII, theorem 2).

1.5. Lemma. *Let f, g be continuous functions of locally finite variation on the interval I. If h is a continuous non-negative function on I then*

$$\int_I h \, \mathrm{d} \operatorname{var}(f \, . \, g) \leqq \int_I h|f| \, \mathrm{d} \operatorname{var} g + \int_I h|g| \, \mathrm{d} \operatorname{var} f . \tag{6}$$

Proof. It is sufficient to prove that, for any compact interval $\langle a, b\rangle \subset I$,

$$\operatorname{var}[f \, . \, g; \langle a, b\rangle] \leqq \int_a^b |f| \, \mathrm{d} \operatorname{var} g + \int_a^b |g| \, \mathrm{d} \operatorname{var} f . \tag{7}$$

Let $D = \{a = x_0 < \ldots < x_n = b\}$ be an arbitrary subdivision of $\langle a, b\rangle$. Then

$$\sum_i |f(x_i) \, g(x_i) - f(x_{i-1}) \, g(x_{i-1})| \leqq$$
$$\leqq \sum_i |f(x_i)| \cdot |g(x_i) - g(x_{i-1})| + \sum_i |g(x_{i-1})| \cdot |f(x_i) - f(x_{i-1})| \, .$$

Making $|D| \to 0$ we obtain the inequality (7) (cf. 1·3 and [2], chap. VIII, § 2, theorem 2).

1.6. Lemma. *Let f, g be continuous functions of finite variation on the interval I, $0 < k \leqq |f| \leqq K$, $|g| \leqq |f|$. If h is a continuous function of locally finite variation on I then*

$$\text{var}\,[\text{arccotg}\,(g + fh);\, I] \leqq L\{\text{var}\,[g; I] + \text{var}\,[f; I] + \text{var}\,[\text{arccotg}\, h; I]\} \tag{8}$$

with constant L depending on constants k, K only.

Proof. By 1·4 and 1·5 we obtain

$$\text{var}\,[\text{arccotg}\,(g + fh);\, I] = \int_I \frac{1}{1 + (g + fh)^2}\, \mathrm{d}\,\text{var}\,(g + fh) \leqq$$

$$\leqq \int_I \frac{1}{1 + (g + fh)^2}\, \mathrm{d}\,\text{var}\, g + \int_I \frac{|f|}{1 + (g + fh)^2}\, \mathrm{d}\,\text{var}\, h + \int_I \frac{|h|}{1 + (g + fh)^2}\, \mathrm{d}\,\text{var}\, f \leqq$$

$$\leqq \text{var}\,[g; I] + \int_I \frac{|f|\, \mathrm{d}\,\text{var}\, h}{1 + (|f| \, . \, |h| - |g|)^2} + \int_I \frac{|h|\, \mathrm{d}\,\text{var}\, f}{1 + (|f| \, . \, |h| - |g|)^2} .$$

We have

$$\frac{|f|}{1 + (|f| \, . \, |h| - |g|)^2} \leqq \frac{K}{1 + h^2} \cdot \frac{1 + h^2}{1 + (|f| \, . \, |h| - |g|)^2} \leqq K \, . \, k_1 \cdot \frac{1}{1 + h^2} ,$$

where we put

$$k_1 = \max\left[2, \sup_{y > 1} \frac{1 + y^2}{1 + k^2(y - 1)^2}\right].$$

Hence

$$\int_I \frac{|f|\, \mathrm{d}\,\text{var}\, h}{1 + (|f| \, . \, |h| - |g|)^2} \leqq K \, . \, k_1 \, . \int_I \frac{\mathrm{d}\,\text{var}\, h}{1 + h^2} = K \, . \, k_1\, \text{var}\,[\text{arccotg}\, h; I] .$$

In a similar way

$$\frac{|h|}{1 + (|f| \, . \, |h| - |g|)^2} \leqq \max\left[1, \sup_{y > 1} \frac{y}{1 + k^2(y - 1)^2}\right] \leqq k_1 ,$$

$$\int_I \frac{|h|\, \mathrm{d}\,\text{var}\, f}{1 + (|f| \, . \, |h| - |g|)^2} \leqq k_1\, \text{var}\,[f; I] .$$

We conclude that

$$\text{var}\,[\text{arccotg}\,(g + fh); I] \leqq \text{var}\,[g; I] + k_1\{\text{var}\,[f; I] + K\, \text{var}\,[\text{arccotg}\, h; I]\}$$

and (8) is established.

1.7. Lemma. *Let v be a non-negative integrable function on $\langle 0, q \rangle$. Then, for any $x \in \langle 0, q \rangle$,*

$$\int_0^q \frac{x}{\xi^2 + x^2}\, v(\xi)\, \mathrm{d}\xi \leqq \frac{\pi}{2} \sup_{0 < x < q} \frac{1}{x} \int_0^x v(\xi)\, \mathrm{d}\xi .$$

Proof. Put $F(0) = 0$, $F(x) = \int_0^x \nu(\xi)\, \mathrm{d}\xi$ $(0 < x \leqq q)$, $k = \sup_{0<x<q} x^{-1} F(x)$. Integrating by parts we obtain for any $x \in (0, q\rangle = \{x;\ 0 < x \leqq q\}$ the estimate

$$\int_0^q \frac{x}{\xi^2 + x^2} \nu(\xi)\, \mathrm{d}\xi = \frac{x}{q^2 + x^2} F(q) + 2x \int_0^q \frac{\xi F(\xi)}{(\xi^2 + x^2)^2} \mathrm{d}\xi \leqq$$

$$\leqq \frac{kqx}{q^2 + x^2} + 2kx \int_0^q \frac{\xi^2}{(\xi^2 + x^2)^2} \mathrm{d}\xi = k \operatorname{arctg} \frac{q}{x} < k \frac{\pi}{2}.$$

2

The term path will be used to denote a continuous complex-valued function defined on an interval. We shall suppose throughout that ψ is a fixed path on the interval J. Further we shall fix a point $z \in E_2$. For every $G \subset J$ and $x \in E_1$ we denote by $\mu^\psi(x; z, G)$ the number (possibly zero or infinite) of points in $\{t;\ t \in G,\ |\psi(t) - z| > 0,\ \psi(t) - z = |\psi(t) - z| \exp ix\}$.

2.1. Lemma. *Let $I \subset J$ be an interval, $|\psi(t) - z| > 0$ for every $t \in I$. Let ϑ_I be a real-valued continuous function on I with*

(9) $$\psi(t) - z = |\psi(t) - z| \exp i\vartheta_I(t), \quad t \in I.$$

Then $\mu^\psi(x; z, I)$ is Lebesgue measurable with respect to x and

$$\int_0^{2\pi} \mu^\psi(x; z, I)\, \mathrm{d}x = \operatorname{var}[\vartheta_I; I].$$

Proof. We shall write simply ϑ instead of ϑ_I. Let N_ϑ have the meaning described in 1·1. It is easily seen that

$$\mu^\psi(x; z, I) = \sum_{n=-\infty}^{\infty} N_\vartheta(x + 2n\pi).$$

N_ϑ being measurable the same is true about $\mu^\psi(x; \ldots)$ and we have by 1·2

$$\operatorname{var}[\vartheta; I] = \int_{-\infty}^{\infty} N_\vartheta(x)\, \mathrm{d}x = \int_0^{2\pi} \mu^\psi(x, \ldots)\, \mathrm{d}x.$$

2.2. Lemma. *Let G be open in J and denote by $\mathfrak{S}$ the system of all components of the set $\{t;\ t \in G,\ |\psi(t) - z| > 0\}$. For every $I \in \mathfrak{S}$ fix a continuous real-valued function ϑ_I on I with (9). Then $\mu^\psi(x; z, G)$ is measurable and*

$$\int_0^{2\pi} \mu^\psi(x; z, G)\, \mathrm{d}x = \sum_{I \in \mathfrak{S}} \operatorname{var}[\vartheta_I; I].$$

Proof. This assertion follows at once from 2·1 on account of the equality

$$\mu^\psi(x; z, G) = \sum_{I \in \mathfrak{S}} \mu^\psi(x; z, I).$$

2.3. Definition. *Let G be open in J. We define*

$$v^{\psi}(z; G) = \int_0^{2\pi} \mu^{\psi}(x; z, G)\,\mathrm{d}x\,.$$

2.4. Remark. The definition 2.3 is justified by 2.2. From geometric reasons the quantity $v^{\psi}(z; G)$ could be called the cyclic variation of $\psi \mid G$ with respect to z, while the function $\mu^{\psi}(x; z, G)$ could be called the cyclic indicatrix of $\psi \mid G$ with respect to z.

We shall write $v_r^{\psi}(z)$ instead of $v^{\psi}(z; G_r)$ where

$$G_r = \{t;\, t \in J,\, |\psi(t) - z| < r\}\,.$$

Given $G \subset J$ and $\varrho > 0$ we shall denote by

$$\nu^{\psi}(\varrho; z, G)\,(0 \leqq \nu^{\psi}(\varrho; z, G) \leqq +\infty)$$

the number of points in $\{t;\, t \in G,\, |\psi(t) - z| = \varrho\}$.

2.5. Lemma. *Let G be open in J and write $\mathfrak{S}$ for the system of all components of $\{t;\, t \in G,\, |\psi(t) - z| > 0\}$. Then $\nu^{\psi}(\varrho; z, G)$ is measurable with respect to ϱ and*

$$\int_0^{\infty} \nu^{\psi}(\varrho; z, G)\,\mathrm{d}\varrho = \sum_{I \in \mathfrak{S}} \mathrm{var}_t\,[|\psi(t) - z|;\, I]\,. \tag{10}$$

Proof. For every $I \in \mathfrak{S}$, $\nu^{\psi}(\varrho; z, I)$ is measurable and, on account of 1.2,

$$\int_0^{\infty} \nu^{\psi}(\varrho; z, I)\,\mathrm{d}\varrho = \mathrm{var}_t\,[|\psi(t) - z|;\, I]\,.$$

Noting that $\nu^{\psi}(\varrho; z, G) = \sum_{I \in \mathfrak{S}} \nu^{\psi}(\varrho; z, I)$ we obtain (10).

2.6. Definition. *Let G be open in J. We define*

$$u^{\psi}(z; G) = \int_0^{\infty} \nu^{\psi}(\varrho; z, G)\,\mathrm{d}\varrho\,.$$

2.7. Remark. This definition is justified by 2.5. The quantity $u^{\psi}(z; G)$ could be called the radial variation of $\psi \mid G$ with respect to z and the function $\nu^{\psi}(\varrho; z, G)$ could be called the radial indicatrix of $\psi \mid G$ with respect to z. We shall write $u_r^{\psi}(z)$ instead of $u^{\psi}(z; G_r)$ where $G_r = \{t;\, t \in J,\, |\psi(t) - z| < r\}$. Thus $u_r^{\psi}(z) = \int_0^r \nu^{\psi}(\varrho; z, J)\,\mathrm{d}\varrho$.

2.8. Theorem. *Let ψ be a path on J, $\beta \in E_1$, $z \in E_2$, $0 < \varrho \leqq r$, $\zeta = z + \varrho \exp i\beta$. Suppose that $z \pm x \exp i\alpha \notin \psi(J)$ whenever $0 < x \leqq r$, $|\alpha - \beta| < \delta$ $(0 < \delta < \pi/2)$ and put $G = \{t;\, t \in J,\, 0 < |\psi(t) - z| < r\}$. Then*

$$v^{\psi}(\zeta; G) \leqq M\{v_r^{\psi}(z) + \sup_{0 < x \leqq r} x^{-1} u_x^{\psi}(z)\} \tag{11}$$

with M depending on δ only.

Proof. We may suppose that $z = 0, \beta = 0$. Let $\mathfrak{S}$ be the system of all components of G. For every $I \in \mathfrak{S}$ and $\varrho \in \langle 0, r \rangle$ denote by $\vartheta_I(t, \varrho)$ a continuous function (of the variable t) on I with

$$(12) \qquad \psi(t) - \varrho = |\psi(t) - \varrho| \, . \exp i\vartheta_I(t, \varrho), \ t \in I \, .$$

We have then

$$(13) \qquad \sin \vartheta_I(t, \varrho) = \frac{\operatorname{Im} [\psi(t) - \varrho]}{|\psi(t) - \varrho|} = \frac{\operatorname{Im} \psi(t)}{|\psi(t) - \varrho|},$$

$$(14) \qquad \sin [\vartheta_I(t, \varrho) - \vartheta_I(t, 0)] = \frac{|\psi(t)|}{|\psi(t) - \varrho|} \, . \operatorname{Im} \frac{\psi(t) - \varrho}{\psi(t)} =$$

$$= \frac{|\psi(t)|}{|\psi(t) - \varrho|} \cdot \frac{\operatorname{Im} [|\psi(t)|^2 - \varrho\overline{\psi(t)}]}{|\psi(t)|^2} = \frac{\varrho \operatorname{Im} \psi(t)}{|\psi(t)| \, . \, |\psi(t) - \varrho|}$$

so that

$$(15) \qquad \frac{\sin \vartheta_I(t, \varrho)}{\sin [\vartheta_I(t, \varrho) - \vartheta_I(t, 0)]} = \frac{|\psi(t)|}{\varrho}, \quad t \in I, \ \varrho \in (0, r\rangle$$

(which, in fact, is the elementary sine theorem applied to the triangle $0, \varrho, \psi(t)$). Noting that

$$\psi(G) \cap \{\varrho \exp i\alpha; \ |\varrho| \leqq r, \ |\alpha| < \delta\} = \emptyset$$

we obtain

$$(16) \qquad |\sin \vartheta_I(t, 0)| = \frac{|\operatorname{Im} \psi(t)|}{|\psi(t)|} > \sin \delta, \ t \in G \, .$$

Fix now an $I \in \mathfrak{S}$. From (15) we conclude that

$$\frac{\varrho}{|\psi(t)|} = \cos \vartheta_I(t, 0) - \operatorname{cotg} \vartheta_I(t, \varrho) \, . \sin \vartheta_I(t, 0)$$

whence

$$\operatorname{cotg} \vartheta_I(t, \varrho) = \operatorname{cotg} \vartheta_I(t, 0) - \sin^{-1} \vartheta_I(t, 0) \, . \frac{\varrho}{|\psi(t)|}, \quad t \in I \, .$$

Defining the function $\tilde{\vartheta}$ on I by

$$\tilde{\vartheta}(t) = \operatorname{arccotg} \left\{ \operatorname{cotg} \vartheta_I(t, 0) - \sin^{-1} \vartheta_I(t, 0) \cdot \frac{\varrho}{|\psi(t)|} \right\}, \quad t \in I \, ,$$

we observe easily that the difference $\vartheta_I(t, \varrho) - \tilde{\vartheta}(t)$ must reduce to a constant on I. Hence

$$(17) \qquad \operatorname{var}_t [\vartheta_I(t, \varrho); I] = \operatorname{var} [\tilde{\vartheta}(t); I] \, .$$

Our aim being to prove (11) we may clearly suppose that $v_r^\psi(z) + \sup_{0<x\leq r} x^{-1} . u_x^\psi(z) < < +\infty$. In particular,

$$v_r^\psi(z) = \sum_{I\in\mathfrak{S}} \operatorname{var}\,[\vartheta_I(t,0); I] < +\infty\,,$$

$$u_r^\psi(z) = \sum_{I\in\mathfrak{S}} \operatorname{var}\,[|\psi(t)|; I] = \int_0^\infty v^\psi(\xi; 0, G)\,\mathrm{d}\xi < +\infty$$

(cf. 2·2 and 2·5). Next we use 1·6 concluding that

$$\text{(18)}\qquad \operatorname{var}\,[\tilde{\vartheta}; I] \leq L\Big\{\operatorname{var}\,[\operatorname{cotg}\vartheta_I(t,0); I] +$$

$$+ \operatorname{var}\,[\sin^{-1}\vartheta_I(t,0); I] + \operatorname{var}_t\left[\operatorname{arccotg}\frac{\varrho}{|\psi(t)|}\,; I\right]\Big\}$$

with L depending on δ only. Applying 1·4 we obtain (cf. also (16))

$$\text{(19)}\qquad \operatorname{var}\,[\operatorname{cotg}\ \vartheta_I(t,0); I] \leq \sin^{-2}\delta\,.\operatorname{var}\,[\vartheta_I(t,0); I]\,,$$

$$\text{(20)}\qquad \operatorname{var}\,[\sin^{-1}\vartheta_I(t,0); I] \leq \sin^{-2}\delta\,.\operatorname{var}\,[\vartheta_I(t,0); I]\,,$$

$$\text{(21)}\qquad \operatorname{var}_t\left[\operatorname{arccotg}\frac{\varrho}{|\psi(t)|}\,; I\right] = \int_I \frac{\varrho}{\varrho^2 + |\psi|^2}\,\mathrm{d}\operatorname{var}|\psi|\,.$$

From (17)–(21) we derive (cf. 2·2)

$$\text{(22)}\qquad v^\psi(\varrho; G) = \sum_{I\in\mathfrak{S}} \operatorname{var}_t\,[\vartheta_I(t,\varrho); I] \leq$$

$$\leq 2L\sin^{-2}\delta\left\{v_r^\psi(0) + \frac{1}{2}\int_G \frac{\varrho}{\varrho^2+|\psi|^2}\,\mathrm{d}\operatorname{var}|\psi|\right\}.$$

By 1·1 and 1·7 it follows

$$\int_G \frac{\varrho}{\varrho^2+|\psi|^2}\,\mathrm{d}\operatorname{var}|\psi| = \int_0^r \frac{\varrho}{\varrho^2+\xi^2}\,v^\psi(\xi; 0, G)\,\mathrm{d}\xi \leq \frac{\pi}{2}\sup_{0<x<r} x^{-1}\int_0^x v^\psi(\xi; 0, G)\,\mathrm{d}\xi =$$

$$= \text{(cf. 2·6 and 2·7)} = \frac{\pi}{2}\sup_{0<x<r} x^{-1}u_x^\psi(0)$$

which together with (22) gives

$$v^\psi(\varrho; G) \leq 2L\sin^{-2}\delta\left\{v_r^\psi(0) + \frac{\pi}{4}\sup_{0<x<r} x^{-1}u_x^\psi(0)\right\}.$$

We see that it is sufficient to put $M = 2L\sin^{-2}\delta$ to satisfy (11).

2.9. Remark. Observing that $\{t; |\psi(t) - \zeta| < \frac{1}{2}r\} \subset \{t; |\psi(t) - z| < r\}$ whenever $\zeta = z + \varrho\exp i\beta$ with $\varrho \in (0, r/2)$ we obtain as a corollary of 2·8 the inequality

$$\sup_{0<\varrho<\frac{1}{2}r} v_{r/2}^\psi(z + \varrho\exp i\beta) \leq M[v_r^\psi(z) + \sup_{0<x<r} x^{-1}u_x^\psi(z)]$$

(compare (2)).

2.10. Theorem. *Let ψ be a path on J, $z \in E_2$, $r > 0$, $\beta \in E_1$, $\zeta = z + r \exp i\beta$. Suppose that $z \pm \varrho \exp i\alpha \notin \psi(J)$ whenever $0 < \varrho < r$, $|\alpha - \beta| < \delta$ $(0 < \delta < \pi/2)$ and put $G = \{t; t \in J, 0 < |\psi(t) - z| < r\}$. Then*

$$r^{-1} u_r^\psi(z) \leqq K[v_r^\psi(z) + v^\psi(\zeta; G)] \tag{23}$$

with K depending on δ only.

Proof. We suppose again that $z = 0$, $\beta = 0$. Let $\mathfrak{S}$, $\vartheta_I(t, \varrho)$ $(\varrho \in \langle 0, r \rangle)$ have the same meaning as in the proof of 2·8. We may assume that

$$v_r^\psi(0) = \sum_{I \in \mathfrak{S}} \operatorname{var} [\vartheta_I(t, 0); I] < +\infty ,$$

$$v^\psi(\zeta; G) = \sum_{I \in \mathfrak{S}} \operatorname{var}_t [\vartheta_I(t, r); I] < +\infty .$$

Fix now an $I \in \mathfrak{S}$ and write $\vartheta_I(t, r) = g(t)$, $\vartheta_I(t, r) - \vartheta_I(t, 0) = f(t)$, $\sin g = \tilde{g}$, $F = \sin^{-1}$, $F(f) = \tilde{f}$. Then $\operatorname{var} \tilde{f}\tilde{g} \leqq \sup |\tilde{f}| \operatorname{var} \tilde{g} + \sup |\tilde{g}| \operatorname{var} \tilde{f}$. Clearly, $\sup |\tilde{g}| \leqq 1$; (14) and (16) imply $|\tilde{f}| < \sin^{-1}\delta . (|\psi| + r)/r \leqq 2 \sin^{-1} \delta$. Further we have by 1·4 $\operatorname{var} \tilde{f} \leqq \sup |F'(f)| \operatorname{var} f \leqq 4 \sin^{-2} \delta \operatorname{var} f$, $\operatorname{var} \tilde{g} \leqq \operatorname{var} g$. Consequently, $\operatorname{var} \tilde{f}\tilde{g} \leqq 2 \sin^{-2} \delta (\operatorname{var} g + 2 \operatorname{var} f)$. Hence it follows on account of (15) that $r^{-1} \operatorname{var}[|\psi(t)|; I] = \operatorname{var}_t [\{\sin \vartheta_I(t, r)\}/\{\sin [\vartheta_I(t, r) - \vartheta_I(t, 0)]\}; I] \leqq 2 \sin^{-2} \delta . . \{3 \operatorname{var}_t [\vartheta_I(t, r); I] + \operatorname{var}_t [\vartheta_I(t, 0); I]\}$ for every $I \in \mathfrak{S}$. On account of 2·5 (cf. also 2·6 and 2·7) and 2·2 (cf. also 2·3 and 2·4) we obtain $r^{-1} . u_r^\psi(0) = \sum_{I \in \mathfrak{S}} r^{-1} . . \operatorname{var}_t [|\psi(t)|; I] \leqq 2 \sin^{-2} \delta \{3 \sum_{I \in \mathfrak{S}} \operatorname{var}_t [\vartheta_I(t, r); I] + \sum_{I \in \mathfrak{S}} \operatorname{var} [\vartheta_I(t, 0); I]\} = 2 \sin^{-2} \delta . \{3 v^\psi(r; G) + v_r^\psi(0)\}$. We see that it is sufficient to put $K = 6 \sin^{-2} \delta$ to satisfy (23).

2.11. Remark. Given $\zeta = z + \varrho \exp i\beta$ with $0 < \varrho < \frac{1}{2} r$ then $G_\varrho = \{t; |\psi(t) - z| < < \varrho\} \subset \{t; |\psi(t) - \zeta| < 2\varrho\}$ and, consequently, $v^\psi(\zeta; G_\varrho) \leqq v_{2\varrho}^\psi(\zeta)$. Hence it follows on account of 2·10 $\sup_{0 < \varrho < r/2} \varrho^{-1} u_\varrho^\psi(z) \leqq K[v_{r/2}^\psi(z) + \sup_{0 < \varrho < r/2} v_{2\varrho}^\psi(z + \varrho \exp i\beta)]$ (compare (1)).

2.12. Remark. The inequalities (1), (2) make it possible to establish simple necessary and sufficient conditions for the existence of non-tangential limits of the logarithmic potential of a continuous double distribution. Such conditions were announced in [5] where, however, in theorem 1 the assumption that the path-curve φ be rectifiable is to be completed.

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Резюме

НЕКОТОРЫЕ НЕРАВЕНСТВА ОТНОСИТЕЛЬНО ЦИКЛИЧЕСКОЙ И РАДИАЛЬНОЙ ВАРИАЦИИ ПЛОСКОГО ПУТИ

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Путем разумеется непрерывное отображение одномерного интервала J в евклидову плоскость E_2. Если ψ — путь, $z \in E_2$ и $r > 0$, α — действительные числа, то обозначим через $\mu_r^\psi(\alpha; z)$ $(0 \leqq \mu_r^\psi(\alpha; z) \leqq +\infty)$ число точек, в которых ψ пересекается с открытым отрезком $\{\zeta; \zeta = z + \varrho \exp i\alpha,\ 0 < \varrho < r\} = S_r^\alpha(z)$. Так как функция $\mu_r^\psi(\alpha; z)$ переменного α измерима (по Лебегу), то можно палагать по определению $v_r^\psi(z) = \int_0^{2\pi} \mu_r^\psi(\alpha; z)\,\mathrm{d}\alpha$. Аналогично обозначим через $\nu^\psi(\varrho; z)$ число пересечений ψ с окружностью $\{\zeta; |\zeta - z| = \varrho\}$ и положим (что возможно вследствие измеримости $\nu^\psi(\varrho; z)$ относительно переменного ϱ) $u_r^\psi(z) = \int_0^r \nu^\psi(\varrho; z)\,\mathrm{d}\varrho$. Если ψ не имеет общих точек с множеством $\bigcup_\alpha S_{2r}^\alpha(z)$, где $\alpha \in (\beta - \delta,\ \beta + \delta) \cup (\beta + \pi - \delta,\ \beta + \pi + \delta)$, $0 < \delta < \pi/2$, то для каждого ϱ, $0 < \varrho < r$, справедливы неравенства

$$\varrho^{-1} u_\varrho^\psi(z) \leqq K[v_r^\psi(z) + v_{2\varrho}^\psi(z + \varrho \exp i\beta)]\,,$$

$$v_r^\psi(z + \varrho \exp i\beta) \leqq M[v_{2r}^\psi(z) + \sup_{0<x<2r} x^{-1} u_x^\psi(z)]\,,$$

где константы K, M зависят только от δ. Эти неравенства находят применение в исследованиях граничного поведения логарифмического потенциала двойного слоя.