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ON THE PSEUDODIMENSION OF ORDERED SETS

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In this paper the pseudodimension and the α-pseudodimension of ordered sets are defined. The relation between the pseudodimension and the dimension is determined. The economy of the representation of ordered set and the α-economy are defined. Finally, the α-pseudodimension of many ordered sets for various α is obtained and the α-pseudodimension is compared with the α-dimension.

1. INTRODUCTION

In literature concerning the theory of ordered sets[1]), there often appear papers concerning the dimension of ordered sets. The dimension of an ordered set was defined in 1941 by B. DUSHNIK and E. W. MILLER in the following manner (see [4] p. 601): Let G be a non-void set ordered by a binary relation ϱ. The dimension of G is the smallest cardinal of a system of the linear extensions[2]) $\{\tau_i\}$ of the relation ϱ which realize ϱ in the sense that ϱ is the meet of all τ_i, $\varrho = \bigcap_i \tau_i$. It is clear that this definition may be formulated in the following slightly different way:

Let G be a non-void ordered set and let $\{L_i\}$ be a system of linearly ordered sets. Let f_i be a single valued mapping defined on G having a single valued inverse and such that $f_i(G) \subseteqq L_i$.

If, for any two elements x, y in G, $x \leqq y$ holds if and only if $f_i(x) \leqq f_i(y)$ in L_i for every i, we say that the system of mappings $\{f_i\}$ realizes the ordering of the set G and the dimension of the set G is defined as the smallest cardinal of such a system of mappings.

This definition was used by H. KOMM. Komm defined the α-dimension of the ordered set G as the smallest cardinal of a system of the $1-1$ mappings of the set G into a linearly ordered set L of a type α which realizes the ordering of the set G (see [5] p. 511).

Now in the definition of dimension it is easy to omit the assumption that the

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[1]) A (partially) ordered set is a set with a nonsymmetric transitive binary relation.

[2]) A linear extension of the ordering ϱ is a linear ordering τ such that $\varrho \subseteqq \tau$ (see [1]).

mappings $\{f_i\}$ are $1-1$. Hence let $\{f_i\}$ be a system of mappings of ordered set G into the linearly ordered sets L_i. If $x \leqq y$ in G is equivalent to $f_i(x) \leqq f_i(y)$ in L_i for every i, we say that the system of mappings $\{f_i\}$ realizes the ordering of the set G. The smallest cardinal of such a system we shall call the pseudodimension of the set G and denote by pdim G. If all the sets L_i are of the same type α, we shall speak about the α-pseudodimension of the set G (α-pdim G). From this definition there follows pdim $G \leqq$ dim G, α-pdim $G \leqq \alpha$-dim G. In the paper we shall prove that always pdim $G =$ dim G, and that this relation does not hold for the α-pseudodimension. From Komm's definition there follows that α-dim G need not exist. An evident necessary condition for the existence of α-dim G is for instance the relation card $\alpha \geqq$ card G. On the other hand, α-pdim G exists always when α is not equal to 0 nor 1. But even when α-dim G exists, the relation α-pdim $G = \alpha$-dim G need not hold. Further, in the paper the economy of representation and α-representation of ordered set is defined. It is shown that this concept is in close relation with the pseudodimension and the α-pseudodimension.

The α-pseudimension for $\alpha = \mathbf{2}$, $\alpha = \omega$, $\alpha = \lambda$ of many specific ordered sets is obtained. It is also shown that α-pdim $G < \alpha$-dim G is possible.

The notation used in the paper is usually the customary. Identity of ordered sets will be denoted by $=$, isomorphism by $\cong$.

Let us define now the lexicographic sum (see [13]) and the cardinal product. Let N be a non-void ordered set and $\{M_\alpha \mid \alpha \in N\}$ a system of disjoint non-void ordered sets. The lexicographic sum of the sets M_α over the set N is a set of all pairs $[x, y]$, where $x \in N$, $y \in M_x$ ordered in the following way: $[x_1, y_1] \leqq [x_2, y_2]$ if and only if $x_1 < x_2$ or $x_1 = x_2$ and $y_1 \leqq y_2$.[3]) Let K be a non-void set and let $\{L_\kappa \mid \kappa \in K\}$ be a system of disjoint ordered sets. By the cardinal product $\prod_{\kappa \in K} L_\kappa$ of the sets L_κ over the set K we shall mean the set of all single-valued functions defined on K and such that $f(\kappa) \in L_\kappa$ for every κ, and ordered as follows: $f \leqq g$ in $\prod_{\kappa \in K} L_\kappa \Leftrightarrow f(\kappa) \leqq g(\kappa)$ for every $\kappa \in K$. If K is a two-point set, this definition agrees with Birkhoff's definition of a cardinal product.

We shall often call a linearly ordered set a chain, the set in which every two distinct elements are incomparable, i.e. which is ordered by the relation $=$, will be called an antichain.

By the symbol n we shall mean on the one hand an antichain containing n elements, on the other hand the type of this antichain, i.e. a cardinal number; analogously by $\mathbf{n}$ we shall mean a chain containing n elements, and also the type of this chain. We remark that throughout this paper every set in which an order is not explicitly given will be considered to be an antichain.

Finally, note these further references: S. Ginsburg ([6]), V. Sedmak ([7]), T. Hiraguchi ([9], [10], [11]).

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[3]) This set is denoted by $\sum_{\alpha \in N} M_\alpha$.

Definition 2.1. Let G be a non-void ordered set, K a set. To every $\kappa \in K$ let there be assigned a linearly ordered set L_κ and an isotone mapping f_κ of G into L_κ such that $x \leqq y$ in G if and only if $f_\kappa(x) \leqq f_\kappa(y)$ for every $\kappa \in K$. Then we shall say that the system $\{L_\kappa, f_\kappa \mid \kappa \in K\}$ is a realizer of the ordering of G. By the cardinal of the realizer we mean the cardinal card K. If all the sets L_κ are of type α, we shall call the corresponding realizer $\{L_\kappa, f_\kappa \mid \kappa \in K\}$ an α-realizer.

Theorem 2.1. *Let G be an ordered set, K a set. To every $\kappa \in K$ let there be assigned a chain L_κ. Then the following statements are equivalent:*

(A) *G is isomorphic to a subset of the cardinal product $\prod_{\kappa \in K} L_\kappa$: $G \cong G' \subseteqq \prod_{\kappa \in K} L_\kappa$.*

(B) *For every $\kappa \in K$ there exists a mapping f_κ of the set G into L_κ such that $\{L_\kappa, f_\kappa \mid \kappa \in K\}$ is a realizer of G.*

Proof. 1. Let (A) hold and let φ be the isomorphism of G onto $G' \subseteqq \prod_{\kappa \in K} L_\kappa$. Thus φ assigns to every $x \in G$ a function f defined on K such that $f(\kappa) \in L_\kappa$. Fix κ; then to every $x \in G$ there is assigned just one value $f(\kappa) \in L_\kappa$, i.e. there is defined a mapping of G into L_κ which we denote by f_κ. We shall show that the system $\{L_\kappa, f_\kappa \mid \kappa \in K\}$ is a realizer of G. First, let $x \leqq y$ in G; then, if $\varphi(x) = f$, $\varphi(y) = g$, the relation $f(\kappa) \leqq g(\kappa)$ holds for every $\kappa \in K$, hence $f_\kappa(x) \leqq f_\kappa(y)$ and hence f_κ is isotone for every $\kappa \in K$. If $f_\kappa(x) \leqq f_\kappa(y)$ holds for every $\kappa \in K$, then according to the definition of the function f_κ, the set $\{f_\kappa(x) \mid \kappa \in K\}$ is the set of values of a certain function f defined on K and belonging to $\prod_{\kappa \in K} L_\kappa$; analogously $\{f_\kappa(y) \mid \kappa \in K\}$ is the set of values of a function g belonging to $\prod_{\kappa \in K} L_\kappa$ and hence $f(\kappa) \leqq g(\kappa)$ holds for every $\kappa \in K$, i.e. $f \leqq g$. As $\varphi(x) = f$, $\varphi(y) = g$ and φ is an isomorphism, this implies $x \leqq y$. Hence $\{L_\kappa, f_\kappa \mid \kappa \in K\}$ is indeed a realizer of the set G.

2. Let (B) hold. Form the cardinal product $\prod_{\kappa \in K} L_\kappa$ and to every element $x \in G$ assign an element $\varphi(x) = f$ of this cardinal product defined by $f(\kappa) = f_\kappa(x)$. In this manner there is defined a mapping φ of G onto a subset $G' \subseteqq \prod_{\kappa \in K} L_\kappa$ and we shall show that this mapping is an isomorphism. First, it is clear that φ is $1-1$. Now if $x \leqq y$ in G, then $f_\kappa(x) \leqq f_\kappa(y)$ for every $\kappa \in K$ because $\{L_\kappa, f_\kappa \mid \kappa \in K\}$ is a realizer of G so that if we denote $\varphi(x) = f$, $\varphi(y) = g$, then $f(\kappa) \leqq g(\kappa)$ for every $\kappa \in K$, i.e. $f \leqq g$ in $\prod_{\kappa \in K} L_\kappa$. Secondly let $f \leqq g$, i.e. $f(\kappa) \leqq g(\kappa)$ for every $\kappa \in K$; then, if we denote $\varphi^{-1}(f) = x$, $\varphi^{-1}(g) = y$, we have $f(\kappa) = f_\kappa(x)$, $g(\kappa) = f_\kappa(y)$ for every κ so that $f_\kappa(x) \leqq f_\kappa(y)$ holds for every $\kappa \in K$ and hence $x \leqq y$ in G. Thus φ is indeed an isomorphism.

Theorem 2.2. *Let G be an ordered set, K a set, L a chain of a type α. Then the following statements are equivalent:*

(A) *G is isomorphic to a subset of the cardinal power L^K : $G \cong G' \subseteq L^K$.*

(B) *To every $\kappa \in K$ there exists a chain L_κ of type α and a mapping f_κ of set G into L_κ such that $\{L_\kappa, f_\kappa \mid \kappa \in K\}$ is an α-realizer of G.*

Proof. This theorem is an immediate consequence of theorem 2.1.

Lemma 2.1. *Let A, B be chains of type α, β such that $A \cong A_1 \subseteq B$. Let G be an ordered set, K a set. If $G \cong G_1 \subseteq A^K$, then $G \cong G_2 \subseteq B^K$.*

Proof. If the assumptions of lemma are true, then clearly $A^K \cong A_1^K$. Let φ be an isomorphism of A^K onto A_1^K. Then $G \cong G_1 \cong \varphi(G_1) = G_2 \subseteq B^K$.

Theorem 2.3. *Let G be an ordered set, let L be any chain with at least two elements. Then there exists a set K such that $G \cong G' \subseteq L^K$.*

Proof. Let R be a chain containing just two elements. According to [2] there exists a set K such that $G \cong G' \subseteq R^K$. Now our statement is a consequence of lemma 2.1.

Theorem 2.4. *Let G be an ordered set, K a set. For every $\kappa \in K$ let there exist a chain L_κ such that $G \cong G_1 \subseteq \prod_{\kappa \in K} L_\kappa$. Then there exists a chain L such that $G \cong$ $\cong G_2 \subseteq L^K$.*

Proof. Let B be any chain of cardinal card K, let φ be any $1-1$ mapping of B onto K, and put $L = \sum_{\beta \in B} L_{\varphi(\beta)}$. Clearly $L_\kappa \subseteq L$ for every $\kappa \in K$. Hence $\prod_{\kappa \in K} L_\kappa \cong L' \subseteq$ $\subseteq L^K$. If φ is an isomorphism of $\prod_{\kappa \in K} L_\kappa$ onto L', then $G \cong G_1 \cong \varphi(G_1) = G_2 \subseteq$ $\subseteq L' \subseteq L^K$.

Corollary 2.1. *Let G be an ordered set, let α be a type of any chain which contains at least two elements. Then there exists at least one α-realizer (and hence at least one realizer) of the set G.*

Proof. This is a consequence of theorems 2.3, 2.2.

3. THE PSEUDODIMENSION OF AN ORDERED SET

Definition 3.1. Let G be an ordered set. The minimum of cardinals of all realizers (α-realizers) of the set G is called the pseudodimension (α-pseudodimension) of the set G and is denoted by pdim G (α-pdim G).

Theorem 3.1. *Let G be an ordered set, α a type of any chain which contains at least two elements. Then G has α-pseudodimension and thus also pseudodimension. There holds* $\text{pdim } G = \min_\alpha \alpha\text{-pdim } G$.

Proof. This follows from corollary 2.1. and theorem 2.4.

Theorem 3.2. *Let G be an ordered set. Then* $\text{pdim } G = \dim G$.

Proof. By the definition of pseudodimension, we have $\text{pdim } G \leq \dim G$. Thus it is sufficient to show that dim $G \leqq$ pdim G. Thus, let pdim $G = m$. This means that there exists a realizer $\{L_\kappa, f_\kappa \mid \kappa \in K\}$ of the set G of cardinal m. Let $M_\kappa \subseteq L_\kappa$ be the set of all elements y such that there exists at least one $x \in G$ with $f_\kappa(x) = y$. Then f_κ is an isotone mapping of G onto M_κ, and M_κ is a linearly ordered set. Let $y \in M_\kappa$ be any element. Let M_κ^y denote the set which is a linear extension of the set $f_\kappa^{-1}(y)$. Form the lexicographic sum $\sum_{y \in M_\kappa} M_\kappa^y = N_\kappa$. This set is linearly ordered and it is clear that it is a linear extension of the set G. Denote by g_κ the identity mapping of G onto N_κ. Then g_κ is isotone mapping of G onto N_κ and this mapping is $1-1$. Now show that the mappings $\{g_\kappa \mid \kappa \in K\}$ realize the ordering of the set G.

Let $g_\kappa(x) \leqq g_\kappa(y)$ for every $\kappa \in K$. Then it is clear from the definition of the mapping g_κ that $f_\kappa(x) \leqq f_\kappa(y)$ for every $\kappa \in K$ and thus $x \leqq y$ in G. Hence $\{g_\kappa \mid \kappa \in K\}$ indeed realize the ordering of the set G and thus dim $G \leqq m$.

Theorem 3.3. *Let G be an ordered set, m a cardinal. Then the following statements are equivalent:*

(A) pdim $G \leqq m$.

(B) *There exists a set K of cardinality m, and for every $\kappa \in K$ a chain L_κ such that G is isomorphic to a subset of the cardinal product* $\prod_{\kappa \in K} L_\kappa$.

Proof. This theorem follows from theorem 2.1.

Corollary 3.1. *Let G be an ordered set, m a cardinal. Then the following statements are equivalent:*

(A) pdim $G \leqq m$.

(B) *There exists a chain L and a set K of cardinality m such that G is isomorphic to a subset of L^K.*

Proof. This follows from theorems 2.1 and 2.4.

Theorem 3.3α. *Let G be an ordered set, K set of cardinality m, α the type of a chain L with at least two elements. Then the following statements are equivalent:*

(A) α-pdim $G \leqq m$.

(B) *G is isomorphic to a subset of L^K.*

Proof. Our theorem is an immediate consequence of theorem 3.3.

Theorem 3.4. *Let m be any cardinal, K a set of cardinality m, α the type of any chain L with at least two elements. Then* α-pdim $L^K = m$.

Proof. Denote $L^K = G$. According to theorem 3.3α we have α-pdim $G \leqq m$. Let B be a chain which contains just two elements. Then $m =$ dim B^K[4]) $\leqq$ dim $L^K =$ $=$ dim $G =$ pdim $G \leqq$ α-pdim G.

Corollary 3.2. *For every cardinal m and every type α of a chain containing at least two elements there exists an ordered set G such that* α-pdim $G = m$.

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[4]) see [5], Theorem 2.1. It is evident that the ordered sets $P(a)$ — the system of all subsets of any set of cardinality a ordered by set inclusion — and $\mathbf{2}^a$ are isomorphic.

Theorem 3.5. *Let A be a chain of type α, B a chain of type β. Let $A \cong A_1 \subseteq B$. Let G be any ordered set. Then* α-pdim $G \geqq \beta$-pdim G.

Proof. Under the assumptions of our theorem $A^K \cong A_1^K \subseteq B^K$ holds for every set K. Our statement is now an immediate consequence of theorem 3.3α.

Corollary 3.3. *Let α be the type of a chain containing at least two elements. Then* **2**-pdim $G \geqq \alpha$-pdim G.

Theorem 3.6. *Let L be a chain containing at least two elements, let G be an infinite ordered set. Let* card $\alpha \leqq \alpha$-pdim G. *Then* **2**-pdim $G = \alpha$-pdim G.

Proof. According to corollary 3.3 we have **2**-pdim $G \geqq \alpha$-pdim G. Now $G \cong G_1 \subseteq L^K$, where card $K = \alpha$-pdim G. According to [2], $L \cong L_1 \subseteq B^M$ where B is a chain containing just two elements and M is a set with card M = card L. Now, using the assumption card $\alpha \leqq \alpha$-pdim G, we have $L^K \cong L_1^K \subseteq (B^M)^K = B^{MK} = B^K$. Hence $G \cong G_2 \subseteq B^K$ and we have **2**-pdim $G \leqq$ card $K = \alpha$-pdim G.

Corollary 3.4. *Let G be an infinite ordered set, let α be the type of a chain L containing at least two elements and whose separability of ordering is $m \leqq$*α-pdim G. *Then* **2**-pdim $G = \alpha$-pdim G.

Proof. From the assumptions it follows that $L \cong L_1 \subseteq B^M$ where B is a chain containing just two elements and M is a set with card $M = m$ (see [3], theorem 2). Now the statement may be proved analogously as theorem 3.6.

4. THE ECONOMY OF A REPRESENTATION OF AN ORDERED SET

Definition 4.1. Let G be an ordered set. Further, let M be a set and for every $m \in M$ let L_m be a chain containing at least two elements. Let $F \subseteq \prod_{m \in M} L_m$, let r be an isomorphism of the set G onto F. Then we shall say that $[M, F, r]$ is a representation of the set G. By the economy of this representation we mean the cardinal card M and we define $ek\{G; [M, F, r]\}$ = card M.

Theorem 4.1. *Let G be an ordered set, $[M, F, r]$ a representation of this set. Then* pdim $G \leqq$ card M.

Proof. This follows from theorem 3.3.

Corollary 4.1. *Let G be an ordered set. Then* $\min_{[M,F,r]} ek\{G; [M, F, r]\}$ = pdim G = dim G.

Proof. This statement follows from theorems 4.1., 2.1., 3.2.

Definition 4.2. Let L be a chain of type α containing at least two elements. Let G be an ordered set. Further, let M be a set, $F \subseteq L^M$, let r be an isomorphism of G onto F. Then we shall say that $[M, F, r]$ is an α-representation of the set G. By the

α-economy of this α-representation we mean the cardinal card M and we put α-$ek\{G; [M, F, r]\}$ = card M.

Theorem 4.2. *Let G be an ordered set, $[M, F, r]$ an α-representation of this set. Then* α-pdim $G \leqq$ card M.

Proof. This follows from theorem 3.3α.

Corollary 4.2. *Let G be an ordered set, let α be the type of a chain containing at least two elements. Then* $\min\limits_{[M,F,r]} \alpha\text{-}ek\{G; [M, F, r]\} = \alpha\text{-pdim } G$.

Proof. This statement is a consequence of theorems 4.2., 2.2.

5. EXAMPLES

Theorem 5.1. *Let G be a finite antichain such that card $G = m$. Let n be a positive integer such that* $\binom{n-1}{[\frac{1}{2}(n-1)]} < m \leqq \binom{n}{[\frac{1}{2}n]}$. *Then* **2**-pdim $G = n$.

Proof. Our statement will be proved if we show that the cardinal power $\mathbf{2}^n$ contains an antichain of cardinality $\binom{n}{[\frac{1}{2}n]}$ and that it contains no antichain with greater cardinal. Consider the set of all sequences $\{x_k\}$; $k = 1, \ldots, n$, where $x_k = 0$ or 1, and which contains $[\frac{1}{2}n]$ 0's and n-$[\frac{1}{2}n]$ 1's. This set contains clearly $\binom{n}{[\frac{1}{2}n]}$ elements and each two elements of this set are incomparable. In [12] it is proved that the cardinal power $\mathbf{2}^n$ contains no antichain with cardinal greater than $\binom{n}{[\frac{1}{2}n]}$.

Theorem 5.2. *Let G be a chain,* card $G = m$. *Then*

a) *if m is finite, then* **2**-pdim $G = m - 1$.

b) *if m is transfinite, then* **2**-pdim $G = n$, *where n is the separability of ordering of the set G.*

Proof. a) is a consequence of the obvious fact that the cardinal power $\mathbf{2}^n$ contains a chain with $n + 1$ elements:

$$\begin{gathered}[0, 0, \ldots, 0]\\ [0, 0, \ldots, 1]\\ \vdots\\ [1, 1, \ldots, 1]\end{gathered}$$

and it contains no chain with more elements.

b) Is a consequence of the corollary of theorem 2 in [3].

Let us denote by $P(a)$ the set of all subsets of a set of cardinality a ordered by set inclusion. H. KOMM proved in [5] that dim $P(a) = a$. We shall prove:

Theorem 5.3. **2**-pdim $P(a) = a$.

Proof. $P(a) \cong B^M$ where B is a chain containing two elements and M is a set such that card $M = a$. From this we obtain **2**-pdim $P(a) =$ **2**-pdim $B^M = a$.

Our theorem can be also proved in the following manner:

Let M be a set of cardinality a. We shall construct to every $x \in M$ a function f_x in the following manner: for $A \subseteq M$ we put $f_x(A) = 0$ if $x \bar{\in} A$ and $f_x(A) = 1$ if $x \in A$. The system of all these functions has cardinality a and, moreover, it realizes $P(a)$. Indeed, let $A \subseteq M$, $B \subseteq M$, $A \subsetneqq B$. Then $f_x(A) \leqq f_x(B)$ for every $x \in M$; also there exists an element $y \in B$, $y \bar{\in} A$ such that $f_y(A) = 0 < 1 = f_y(B)$. Let $A \parallel B$; then there exist elements $x, y \in M$, $x \in A$, $x \bar{\in} B$, $y \bar{\in} A$, $y \in B$ such that $f_y(A) = 0 < < 1 = f_y(B)$, $f_x(A) = 1 > 0 = f_x(B)$. Hence **2**-pdim $P(a) \leqq a$. On the other hand **2**-pdim $P(a) \geqq$ pdim $P(a) =$ dim $P(a) = a$. Hence **2**-pdim $P(a) = a$.

Corollary 5.1. *Let α be the type of a chain containing at least two elements. Then* α-pdim $P(a) = a$.

Proof. This is a consequence of theorem 5.3 and corollary 3.3.

Let E_k denote the set of all sequences $[x_1, x_2, \ldots, x_k]$, where $x_i (1 \leqq i \leqq k)$ is any real number. H. Komm defined two orders $P'(E_k)$ and $P(E_k)$ in the following manner: $[x_1, x_2, \ldots x_k] \leqq [y_1, y_2, \ldots y_k]$ in $P'(E_k)$ if and only if $x_i \leqq y_i$ for every i; and $[x_1, x_2, \ldots x_k] < [y_1, y_2, \ldots y_k]$ in $P(E_k)$ if and only if $x_i < y_i$ for every i. If we denote the set of all real numbers ordered in the natural way by E, then evidently $P'(E_k)$ is the same as the cardinal power E^k.

Theorem 5.4. **2**-pdim $P(E_k) = 2^{\aleph_0}$.

Proof. As card $P(E_k) = 2^{\aleph_0}$, we have **2**-pdim $P(E_k) \leqq 2^{\aleph_0}$. On the other hand we shall prove (corollary 6.4) λ-pdim $P(E_k) = 2^{\aleph_0}$; this implies **2**-pdim $P(E_k) \geqq \lambda$-pdim . . $P(E_k) = 2^{\aleph_0}$.

Theorem 5.5. **2**-pdim $P'(E_k) = \aleph_0$.

Proof. The set E has separability of ordering $\aleph_0$. Thus $E \cong E_1 \subseteq \mathbf{2}^{\aleph_0}$. From this we have $P'(E_k) = E^k \cong E_1^k \subseteq (\mathbf{2}^{\aleph_0})^k = \mathbf{2}^{\aleph_0 k} = \mathbf{2}^{\aleph_0}$. Hence **2**-pdim $P'(E_k) \leqq \aleph_0$. But **2**-pdim $P'(E_k)$ is certainly infinite. Hence **2**-pdim $P'(E_k) = \aleph_0$.

6. THE COMPARISON OF THE α-DIMENSION WITH α-PSEUDODIMENSION

From the definition it follows that α-pdim $G \leqq \alpha$-dim G, if α-dim G exists. We now shall show that there exists an ordered set G such that α-pdim $G < \alpha$-dim G.

It is clear that if α is a finite ordinal then α-dim G exists if and only if card $G \leqq \leqq$ card α; in this case α-dim $G = \alpha$-pdim $G =$ dim G. We shall therefore first consider the ω-dimension and ω-pseudodimension.

Define two ordered sets: Let N denote the set of all positive integers ordered by magnitude. Let N_k be the set of all sequences $[x_1, x_2, \ldots, x_k]$ where $x_i \in N$, $1 \leqq i \leqq k$. We define $P'(N_k) = N^k$, $P(N_k)$ in a manner similar to $P'(E_k)$, $P(E_k)$ respectively.

Theorem 6.1. *ω-dim N^2 exists.*

Proof. As ω is a homogeneous order type (see [5]), it is sufficient to show that N^2 has an ω-extension. Let us decompose the set of all positive integers into countably many classes $\{W_n\}$ in such a manner that each W_n is infinite. Let $W_n = \{a_1^n, a_2^n, a_3^n, \ldots\}$, where $a_i^n < a_j^n$ for $i < j$. Let us assume that for $m < n$, $a_i^m < a_i^n$ holds for every i. Such a decomposition can always be constructed, for instance in the following manner:

$$W_1 = \{1, 2, 4, \ldots, 2^n, \ldots\}, \qquad W_2 = \{3, 9, 27, \ldots, 3^n, \ldots\},$$
$$W_3 = \{5, 25, 125, \ldots, 5^n, \ldots\}, \quad W_4 = \{6, 36, 216, \ldots, 6^n, \ldots\}.$$

Now to every element $[x, y] \in N^2$ we assign the element a_y^x. This mapping is evidently one-one and it is an ω-extension, because $[x_1, y_1] \leqq [x_2, y_2]$ in N^2 implies $x_1 \leqq x_2$ and $y_1 \leqq y_2$ and this implies $a_{y_1}^{x_1} \leqq a_{y_2}^{x_2}$ according to our construction.

Corollary 6.1. *ω-dim N^k exists for every positive integer k.*

Proof. We shall prove our theorem by induction. For $k = 2$ our statement holds. Assume it holds for $k - 1$, so that there exists an ω-extension φ of the set N^{k-1}. Construct the same decomposition as in the proof of theorem 6.1 and assign to every element $[x_1, x_2, \ldots, x_{k-1}, x_k] \in N^k$ the element $a_{x_k}^{\varphi([x_1, x_2, \ldots, x_{k-1}])}$. This mapping is evidently an ω-extension so that ω-dim N^k exists.

Corollary 6.2. *ω-dim $P(N_k)$ exists for every positive integer k.*

Proof. The ω-extension of N^k constructed in the proof of corollary 6.1 is evidently the ω-extension of $P(N_k)$.

Theorem 6.2. *If there exists in any ordered set G an ascending chain C of type ω and an element a such that $a \parallel x$ holds for every $x \in C$, then G has no finite ω-dimension.*

Proof. Assume the contrary, ω-dim $G = r < \aleph_0$. Let $\{f_i, N; i = 1, 2, \ldots, r\}$ be the ω-realizer of G in Komm's sense. Denote $C = \{x_n\}$, where $x_n < x_{n+1}$, for $n = 1, 2, \ldots$. Then $f_i(x_n) < f_i(x_{n+1})$ holds for every i and n. Choose any i. As f_i is an ω-extension, there exists an index p_i such that $f_i(x_{p_i}) < f_i(a) < f_i(x_{p_i+1})$ and hence $f_i(a) < f_i(x_n)$ for every $n > p_i$. Let $p = \max_i \{p_i\}$. Then $f_i(a) < f_i(x_n)$ for $n > p$ and for every i, which implies $a < x_n$. This contradicts our assumption and thus the theorem is proved.

Theorem 6.3. *If in the ordered set G there exists an infinite antichain, then ω-pdim G is infinite.*

Proof. If ω-pdim $G = k < \aleph_0$, then G is isomorphic to a subset of the cardinal power N^k. We shall show (lemma 6.1) that the cardinal power N^k contains no infinite antichain, which will contradict our assumption.

Lemma 6.1. *If $k < \aleph_0$, then the cardinal power N^k contains no infinite antichain.*

Proof. We shall prove our statement by induction. Let $k = 2$. Assume that N^2 contains an infinite antichain $C = \{[x_n, y_n]\}$; $n = 1, 2, \ldots$. Note that the elements $[x, y]$, $[x, z]$ are always comparable in N^2 so that all x_n are distinct and we may assume $x_1 < x_2 < \ldots$. But then it is clear that the elements y must satisfy the relation $y_1 > y_2 > y_3 \ldots$ so that the elements y determine an infinite descending chain in N. This is impossible and, therefore, N^2 does not contain any infinite antichain.

Now, let us assume that our theorem holds for $k - 1$ so that N^{k-1} does not contain any infinite antichain. Suppose that N^k contains an infinite antichain R. Denote the projection of the set R into N^{k-1} by R'. R' is certainly infinite. Suppose that R' is finite, i.e.

$$R' = \{[x_1^{i_1}, x_2^{i_2}, \ldots, x_{k-1}^{i_{k-1}}]\}, \quad \text{where} \quad 1 \leqq i_1 \leqq n_1,\ 1 \leqq i_2 \leqq n_2, \ldots, \\ 1 \leqq i_{k-1} \leqq n_{k-1}.$$

Then $R = \{[x_1^{i_1}, x_2^{i_2}, \ldots, x_{k-1}^{i_{k-1}}, x_k^{i_k}]\}$ where the set of indexes i_k is infinite. From this we obtain that there exists at least one system of indexes $i_1^0, i_2^0, \ldots, i_{k-1}^0$ such that $[x_1^{i_1 0}, x_2^{i_2 0}, \ldots, x_{k-1}^{i0_{k-1}}, x_k^{i_k}] \in R$ holds for infinitely many i_k. But all these elements are comparable in N^k and this contradicts the assumption that R is an antichain. Hence R' is infinite. The assumption that N^{k-1} contains no infinite antichain implies that every infinite subset of R' contains at least two comparable elements, so that according to [4] (Theorem 5.24) R' contains an infinite chain $C = \{[x_1^n, x_2^n, \ldots, x_{k-1}^n]\}$; $n = 1, 2, \ldots$. As N^{k-1} satisfies the descending chain condition, this chain is necessarily ascending and we may suppose $[x_1^1, x_2^1, \ldots, x_{k-1}^1] < [x_1^2, x_2^2, \ldots, x_{k-1}^2] < \ldots$. This chain determines a certain subset in R: $\{[x_1^n, x_2^n, \ldots, x_{k-1}^n, x_k^n]\}$ and it is clear that $x_k^1 > x_k^2 > \ldots$ so that we obtain an infinite descending chain in N. This, however, is impossible and the theorem is proved.

Theorem 6.4. ω-dim $N^k = \aleph_0$ *for any positive integer* $k \geqq 2$.

Proof. N^k is a countable set, hence ω-dim $N^k \leqq \aleph_0$. Now, put $x_n = [n + 1, 1, 1, \ldots, 1]$ for $n = 1, 2, \ldots, a = [1, 2, 1, \ldots, 1]$. Then the set $\{x_n\}$ is a chain of type ω and $a \parallel x_n$ for every n. Hence according to theorem 6.2. N^k has no finite ω-dimension.

Theorem 6.5. ω-pdim $N^k = k$ *for every* k.

Proof. The theorem is a consequence of theorem 3.4.

Theorem 6.6. ω-pdim $P(N_k) = \aleph_0$ *for any positive integer* $k \geqq 2$.

Proof. $P(N_k)$ is a countable set, therefore ω-pdim $P(N_k) \leqq \aleph_0$. Now, put $x_n = [n, 1, 1, \ldots, 1]$; $n = 1, 2, \ldots$. Then the set $\{x_n\}$ forms an infinite antichain. Hence according to theorem 6.3. $P(N_k)$ does not have finite ω-pseudodimension.

Corollary 6.3. ω-dim $P(N_k) = \aleph_0$.

Proof. This fact follows from the inequalities

$$\aleph_0 = \omega\text{-pdim } P(N_k) \leqq \omega\text{-dim } P(N_k) \leqq \aleph_0 .$$

Let us now consider the λ-pseudodimension. We shall determine λ-pdim $P'(E_k)$, λ-pdim $P(E_k)$. It is known ([5]) that both λ-dim $P'(E_k)$ and λ-dim $P(E_k)$ exist and λ-dim $P'(E_k) = \aleph_0$, λ-dim $P(E_k) = 2^{\aleph_0}$.

Theorem 6.7. λ-pdim $P'(E_k) = k$ *for every* k.

Proof. This follows from theorem 3.4.

Theorem 6.8. λ-pdim $P(E_2) = 2^{\aleph_0}$.

Proof. As card $P(E_2) = 2^{\aleph_0}$, we have λ-pdim $P(E_2) \leqq 2^{\aleph_0}$. Suppose that λ-pdim $P(E_2) < 2^{\aleph_0}$ so that $P(E_2)$ is realized by the set $\{f_\kappa, \mid \kappa \in K, \text{card } K < 2^{\aleph_0}\}$ of mappings into a set L of order type λ. Consider in $P(E_2)$ the set of all points $[x, 0]$ and $[x, 1]$. For any x_0 we have $[x_0, 0] \parallel [x_0, 1]$. As card $K < 2^{\aleph_0}$, there exists a $\kappa_0 \in K$ such that $f_{\kappa_0}([x, 1]) < f_{\kappa_0}([x, 0])$ for a non-denumerable set of x's *i.e.* for $x \in C$, where card $C > \aleph_0$. Now, if $x, y \in C$, $x < y$, then $[x, 0] < [y, 1]$ in $P(E_2)$ so that $f_{\kappa_0}([x, 1]) < f_{\kappa_0}([x, 0]) \leqq f_{\kappa_0}([y, 1]) < f_{\kappa_0}([y, 0])$. If we denote $f_{\kappa_0}([x, 1]) = a_x$ $(\in L)$, $f_{\kappa_0}([x, 0]) = b_x$ $(\in L)$, we have $a_x < b_x \leqq a_y < b_y$ for $x, y \in C$, $x < y$. From this it follows that we may construct in L a non-denumerable set of non-overlapping intervals. But this is impossible, because L has the order type λ and hence each set of non-overlapping intervals in L is denumerable. Thus, λ-pdim $P(E_2) = 2^{\aleph_0}$.

Corollary 6.4. λ-pdim $P(E_k) = 2^{\aleph_0}$ *for every finite* $k \geqq 2$.

Proof. As card $P(E_k) = 2^{\aleph_0}$, we have λ-pdim $P(E_k) \leqq 2^{\aleph_0}$. On the other hand λ-pdim $P(E_k) \geqq \lambda$-pdim $P(E_2) = 2^{\aleph_0}$.

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Резюме

О ПСЕВДОРАЗМЕРНОСТИ УПОРЯДОЧЕННЫХ МНОЖЕСТВ

ВИТЕЗСЛАВ НОВАК (Vítězslav Novák), Брно

В статье определяется псевдоразмерность и α-псевдоразмерность упорядоченного множества следующим образом: Пусть G — непустое упорядоченное множество. Псевдоразмерностью множества G ($\operatorname{pdim} G$) мы разумеем наименьшую мощность системы изотонных отображений $\{f_i\}$ G в линейно упорядоченные множества L_i такой, что имеет место $x \leqq y \Leftrightarrow f_i(x) \leqq f_i(y)$ для всех i. Если все множества L_i того же типа α, мы говорим о α-псевдоразмерности множества G (α-$\operatorname{pdim} G$).

В статье доказывается: $\operatorname{pdim} G = \dim G$ для всякого упорядоченного множества G (теорема 3.2), $\operatorname{pdim} G = \min\limits_{\alpha} \alpha\text{-}\operatorname{pdim} G$ (теорема 3.1).

Далее доказывается: Теорема 3.3. Для упорядоченного множества G эквивалентны следующие утверждения: (A) $\operatorname{pdim} G \leqq m$. (B) Существует множество K мощности m, и для всякого $\kappa \in K$ цепь L_κ такова, что $G \cong G' \subseteq \prod\limits_{\kappa \in K} L_\kappa$.

Теорема 3.3α: Для упорядоченного множества G эквивалентны следующие утверждения: (A) α-$\operatorname{pdim} G \leqq m$. (B) Существует множество K мощности m и цепь L типа α такая, что $G \cong G' \subseteq L^K$.

Далее, определяется экономия представления упорядоченного множества. Если r — изоморфизм упорядоченного множества G на множество $F \subseteq \prod\limits_{m \in M} L_m$ ($F \subseteq L^M$), где L_m — цепи (L — цепь типа α), то тройка $[M, F, r]$ называется представлением (α-представлением) множества G. Мощность $\operatorname{card} M$ называется экономией (α-экономией) этого представления. В статье доказывается:

$$\min_{[M,F,r]} ek\{G;\ [M, F, r]\} = \operatorname{pdim} G = \dim G \quad (\text{следствие } 4.1)\,,$$

$$\min_{[M,F,r]} \alpha - ek\{G;\ [M, F, r]\} = \alpha\text{-}\operatorname{pdim} G \quad (\text{следствие } 4.2)\,.$$

Наконец, в статье приведен ряд примеров в которых доказывается, что существуют такие упорядоченные множества G, для которых α-$\operatorname{pdim} G <$ $< \alpha$-$\dim G$.