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ON SEMICHAINED REFINEMENTS OF CHAINS IN EQUIVALENCE LATTICE

(*A note to a theorem of O. Borůvka*)

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Prof. O. BORŮVKA discovered in 1959 a theorem in which he generalized his previous results on a certain isomorphism between Zassenhaus refinements of two chains in an equivalence lattice. This paper contains a new proof of this theorem and its further generalization.

Introduction. Using the methods of lattice theory and of the theory of set decompositions, we generalize further the theorem discovered by Prof. O. BORŮVKA on semichained refinements of two chains in the lattice of equivalence relations of a given set; [1], pp. 65—68. Using the terminology of [1] we may express the theorem of Prof. Borůvka as follows:

To a given modular pair of decompositions series there exists a pair of their refinements which are semichained by the same basis.

We shall complete the original existence proof from [1] with the direct construction. Lattice theoretic terminology will be used; note, however, that it differs from the terminology of set decomposition theory introduced in [1]. As our starting point we shall use a certain lemma ([3], pp. 86–87), in which all the essentials needed of the theory of set decompositions are concentrated. The constructive character of our investigation makes possible its application to the theory of scientific classifications ([1], pp. 72–73). We are also led to a lattice theoretic problem following from the study of different types of Zassenhaus refinements of two chains in the given lattice.

Preliminaries from lattice theory. Let $\mathbf{L}$ be a lattice with partial ordering $\leqq$ and let $\wedge$, $\vee$ denote the lattice operations (meet and join). If $a \geqq b$ are elements of $\mathbf{L}$, then the *lattice quotient* a/b is defined as the sublattice consisting of the elements $x \in \mathbf{L}$ with $a \geqq x \geqq b$. Next we define some notions (only for the need of this paper): The ordered triad of elements $a_1 \geqq a_2$; b of $\mathbf{L}$ will be called *modular* and denoted by $\mathbf{M}(a_1, a_2, b)$, if $a_1 \wedge (a_2 \vee b) = a_2 \vee (a_1 \wedge b)$. The ordered quadruple of elements $a_1 \geqq a_2$; $b_1 \geqq b_2$ of $\mathbf{L}$ will be called *modular* and denoted by $\mathbf{M}(a_1, a_2, b_1, b_2)$, if

$\mathbf{M}(a_1 \wedge b_1, a_2 \wedge b_1, b_2)$. If $a_1 \geqq a_2$; $b_1 \geqq b_2$ are elements of $\mathbf{L}$, we call the quotients a_1/a_2, b_1/b_2 as *similar* if a so-called *middle quotient* $c_1/c_2 \subset \mathbf{L}$ exists such that $a_1 = a_2 \vee c_1$, $b_1 = b_2 \vee c_1$, $c_2 = a_2 \wedge c_1 = b_2 \wedge c_1$. By a *chain* (with repetitions) between elements $a_0 \geqq a_n$ in $\mathbf{L}$ we shall understand a finite sequence of elements of $\mathbf{L}$, for which the induced partial ordering of the lattice $\mathbf{L}$ is the ordering of the sequence; such a chain consists of members $a_0 \geqq a_1 \geqq \ldots \geqq a_n$, or $\{a_\alpha\}_{\alpha=0,\ldots,n}$. If there are given chains

$$\mathbf{A} = \{a_i\}_{i=0,\ldots,r},$$
$$\mathbf{B} = \{b_j\}_{j=0,\ldots,s}, \quad b_0 = a_0, \; b_s = a_r,$$

in $\mathbf{L}$, then we define their *Zassenhaus refinements*

$$\mathbf{A}^* = \{a_{k,j}\}_{k=0,\ldots,r-1;\, j=0,\ldots,s},$$
$$\mathbf{B}^* = \{b_{l,i}\}_{l=0,\ldots,s-1;\, i=0,\ldots,k},$$

with members of the form $a_{k,j} = a_{k+1} \vee (a_k \wedge b_j)$, $b_{l,i} = b_{l+1} \vee (b_l \wedge a_i)$ where the indexes i, j, k, l here and hence forth are

$$i = 0, \ldots, r; \; j = 0, \ldots, s; \; k = 0, \ldots, r-1; \; l = 0, \ldots, s-1\,.$$

We term the chains $\mathbf{A}$, $\mathbf{B}$ in $\mathbf{L}$ *similar*, if there exists a one-to-one mapping $\mathbf{A} \to \mathbf{B}$: $a_i \to b_{f(i)}$ such that a_k/a_{k+1}, $b_{f(k)}/b_{f(k)+1}$ are similar quotients for every k.

Theorem 1. *For the refinements* $\mathbf{A}^*$, $\mathbf{B}^*$ *of given chains* $\mathbf{A}$, $\mathbf{B}$ *in* $\mathbf{L}$ *there holds*:

a) $a_{k,l} \wedge b_{l,k} = a_k \wedge b_l$ *for every* k, l;

b) $\mathbf{M}(a_{k,l}, a_{k,l+1}, b_{l,k})$, $\mathbf{M}(b_{l,k}, b_{l,k+1}, a_{k,l})$ *for every* k, l;

c) *if* $\mathbf{A}^*$, $\mathbf{B}^*$ *are similar by the mapping* $a_{k,l} \to b_{l,k}$*) *with the middle quotients* $a_k \wedge b_l/e_{k,l}$, *then* $a_{k,l} = a_{k,l+1} \wedge b_{l,k+1}$ *for every* k, l;

d) *setting* $c_{k,l} = (a_k \wedge b_{l+1}) \vee (b_l \wedge a_{k+1})$, $d_{k,l} = a_k \wedge b_l \wedge (a_{k+1} \vee b_{l+1})$ *for all* k, l, *then from the condition*

(I) $\quad \mathbf{M}(a_k, a_{k+1}, b_l, b_{l+1})$, $\mathbf{M}(b_l, b_{l+1}, a_k, a_{k+1})$ *for all* k, l

there follows $a_{k,l+1} \wedge b_{l,k+1} = c_{k,l}$ *for all* k, l;

e) *from the condition*

(II) $\quad d_{k,l} = a_{k,l+1} \wedge a_k \wedge b_l = b_{l,k+1} \wedge b_l \wedge a_k$ *for all* k, l

there follows $a_{k,l+1} \wedge b_{l,k+1} = d_{k,l}$ *for all* k, l;

f) *the condition*

(II*) $\quad \mathbf{M}(a_k, a_{k+1}, b_j)$, $\mathbf{M}(b_l, b_{l+1}, a_i)$ *for all* i, j, k, l

implies condition (II).

Proof. a) It is necessary to decide whether

(1) $$(a_{k+1} \vee (a_k \wedge b_l)) \wedge (b_{l+1} \vee (b_l \wedge a_k)) = a_k \wedge b_l\,.$$

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*) $a_{k,s} = a_{k+1,0} = a_{k+1}$, $b_{l,r} = b_{l+1,0} = b_{l+1}$.

Now $a_{k,l} \wedge b_{l,k} \geqq a_k \wedge b_l$, because $a_{k,l} \geqq a_k \wedge b_l$ and also $b_{l,k} \geqq a_k \wedge b_l$; also $a_{k,l} \leqq a_k$, $b_{l,k} \leqq b_l$, so that $a_{k,l} \wedge b_{l,k} \leqq a_k \wedge b_l$; thus finally $a_{k,l} \wedge b_{l,k} = a_k \wedge b_l$.

b) We shall prove the relation $\mathbf{M}(a_{k,l}, a_{k,l+1}, b_{l,k})$; the second relation may be proved similarly. Obviously $a_{k+1} \vee (a_k \wedge b_{l+1}) \vee (a_k \wedge b_l) = a_{k+1} \vee (a_k \wedge b_l)$, so that $a_{k,l+1} \vee (a_k \wedge b_l) = a_{k,l}$. Thus by $a_{k,l} \wedge b_{l,k} = a_k \wedge b_l$ and $a_{k,l} \leqq a_{k,l+1} \vee \vee b_{l,k}$ (the last relation is obtained as follows: $a_{k+1} \vee (a_k \wedge b_l) \leqq (a_{k+1} \vee (a_k \wedge \wedge b_{l+1})) \vee (b_{l+1} \vee (b_l \wedge a_k)) = a_{k+1} \vee (a_k \wedge b_l) \vee b_{l+1})$, we have $a_{k,l+1} \vee \vee (a_{k,l} \wedge b_{l,k}) = a_{k,l} \wedge (a_{k,l+1} \vee b_{l,k})$, that is $\mathbf{M}(a_{k,l}, a_{k,l+1}, b_{l,k})$.

c) Obviously $e_{k,l} = a_{k,l+1} \wedge b_{l,k} = a_{k,l} \wedge b_{l,k+1}$, and thus also $e_{k,l} = (a_{k,l+1} \wedge \wedge b_{l,k}) \wedge (a_{k,l} \wedge b_{l,k+1}) = a_{k,l+1} \wedge b_{l,k+1}$.

d) We shall consider

$$(2_{1,2}) \quad (a_{k+1} \vee (a_k \wedge b_{l+1})) \wedge (b_{l+1} \vee (b_l \wedge a_{k+1})) = \begin{cases} (a_k \wedge b_{l+1}) \vee (a_{k+1} \wedge b_l) \\ (a_k \wedge b_l) \wedge (a_{k+1} \vee b_{l+1}) . \end{cases}$$

If (I) holds, then $a_k \wedge b_l \wedge ((a_{k+1} \wedge b_l) \vee b_{l+1}) = (a_{k+1} \wedge b_l) \vee (a_k \wedge b_{l+1})$, $b_l \wedge a_k \wedge ((b_{l+1} \wedge a_k) \vee a_{k+1}) = (b_{l+1} \wedge a_k) \vee (b_l \wedge a_{k+1})$. Then the left sides are equal and similarly as in c), we obtain $c_{k,l} = a_k \wedge b_l \wedge b_{l,k+1} \wedge a_{k,l+1} = b_{l,k+1} \wedge \wedge a_{k,l+1}$. Thus (2_1) holds.

e) Let the condition (II) hold, that is $d_{k,l} = a_{k,l+1} \wedge a_k \wedge b_l = b_{l,k+1} \wedge b_l \wedge a_k$. Then certainly $d_{k,l} = a_{k,l+1} \wedge b_{l,k+1} \wedge a_k \wedge b_l$ and again as in the preceding proof $d_{k,l} = a_{k,l+1} \wedge b_{l,k+1}$. Thus (2_2) holds.

f) From the conditions $\mathbf{M}(a_k, a_{k+1}, b_{l+1})$, $\mathbf{M}(b_l, b_{l+1}, a_{k+1})$ it follows that $(a_{k+1} \vee \vee (a_k \wedge b_{l+1})) \wedge a_k \wedge b_l = a_k \wedge (a_{k+1} \vee b_{l+1}) \wedge a_k \wedge b_l = a_k \wedge b_l \wedge (a_{k+1} \vee \vee b_{l+1})$. Similarly for $b_{l,k+1} \wedge b_l \wedge a_k$.

It may be easily shown that (II) does not imply (II*).

Theorem 2. *The refinements* $\mathbf{A}^*$, $\mathbf{B}^*$ *of given chains* $\mathbf{A}$, $\mathbf{B}$ *in* $\mathbf{L}$ *are similar by the mapping* $a_{k,l} \to b_{l,k}$ *with middle quotients* $a_k \wedge b_l/c_{k,l}$, $a_k \wedge b_l/d_{k,l}$ *respectively, if and only if the conditions* (I), *respectively* (II), *hold. If we replace* (II) *by* (II*) *then it is necessary to replace "if and only if" by "if".*

Proof. The quotients $a_{k,l}/a_{k,l+1}$, $b_{l,k}/b_{l,k+1}$ are similar with the middle quotient $a_k \wedge b_l/c_{k,l}$ when and only when the following equations hold

$$(3) \qquad a_{k+1} \vee (a_k \wedge b_l) = (a_k \wedge b_l) \vee a_{k+1} \vee (a_k \wedge b_{l+1}) ,$$

$$(4) \qquad (a_k \wedge b_{l+1}) \vee (a_{k+1} \vee b_l) = (a_{k+1} \vee a_k \wedge b_{l+1}) \wedge (a_k \wedge b_l) ,$$

$$(5) \qquad b_{l+1} \vee (b_l \wedge a_k) = (b_l \wedge a_k) \vee b_{l+1} \vee (b_l \wedge a_{k+1}) ,$$

$$(6) \qquad (b_l \wedge a_{k+1}) \vee (b_{l+1} \wedge a_k) = (b_{l+1} \vee (b_l \wedge a_{k+1})) \wedge (b_l \wedge a_k) .$$

The equations (3) and (5) are fulfilled trivially, because $a_k \wedge b_l \geqq a_k \wedge b_{l+1}$, $b_l \wedge \wedge a_k \geqq b_l \wedge a_{k+1}$. The equations (4), (6) are fulfilled when and only when $\mathbf{M}(b_l \wedge a_k, b_{l+1} \wedge a_k, a_{k+1})$, $\mathbf{M}(a_k \wedge b_l, a_{k+1} \wedge b_l, b_{l+1})$, that is if (I) holds.

The equations (3), (5) together with the equations

$$(7) \qquad a_k \wedge b_l \wedge (a_{k+1} \vee b_{l+1}) = (a_{k+1} \vee (a_k \wedge b_{l+1})) \wedge a_k \wedge b_l\,,$$

$$(8) \qquad b_l \wedge a_k \wedge (b_{l+1} \vee a_{k+1}) = b_{l+1} \vee (b_l \wedge a_{k+1}) \wedge b_l \wedge a_k\,,$$

characterize the second type of the similarity. As we know, (3) and (5) hold trivially, whereas (7) and (8) directly express the condition (II).

From (II*) there follows at once the validity of (7) and (8); *cf.* also theorem 15. As to the first part of theorem 2, *cf.* [4], theorem 7,1, p. 18.

Theorem 3. *The refinements* $\mathbf{A}^*$, $\mathbf{B}^*$ *of given chains* $\mathbf{A}$, $\mathbf{B}$ *in* $\mathbf{L}$ *are similar by the mapping* $a_{k,l} \to b_{l,k}$ *with middle quotients* $a_{k,l} \wedge b_{l,k}/a_{k,l+1} \wedge b_{l,k+1}$, *if and only if the following condition is satisfied*:

$$\text{(III)} \qquad a_{k,l+1} \wedge (a_k \wedge b_l) = b_{l,k+1} \wedge (b_l \wedge a_k) \quad \textit{for every } k, l\,.$$

Proof. According to theorem 1a, $a_{k,l} \wedge b_{l,k} = a_k \wedge b_l$ and the considered similarity is characterized by the equations (3), (5); these are valid trivially as in the proof of theorem 2 *et seq.*, since $a_{k,l+1} \wedge a_k \wedge b_l = b_{l,k+1} \wedge b_l \wedge a_k = a_{k,l+1} \wedge b_{l,k+1}$. If (III) holds, then the last part of the preceding equations follows directly from theorem 1c.

Preliminaries from decomposition theory. Let $\mathbf{E}(S)$, or shortly $\mathbf{E}$, be the lattice of all equivalence relations on the given set S, which can be naturally derived from the lattice of all binary relations in S (isomorphic to the lattice of all subsets of $S \times S$ partially ordered by set inclusion). If $a \in \mathbf{E}$, then let S/a be the symbol for the set of all corresponding a-blocks. Relations $a, b \in \mathbf{E}$ will be called *associable* if every a-block contained in an arbitrary $(a \vee b)$-block, overlaps every b-block contained in the same $(a \vee b)$-block; symbolically $a \square b$; cf. [3], p. 72.

Relations $a, b \in \mathbf{E}$ are called *totally associable* when $a \square b'$, $a' \square b$ for every $a' \geqq a$, $b' \geqq b$ in $\mathbf{E}$; symbolically $a \boxdot b$; *cf.* [3], p. 89.

It is well known that $a \square b$ implies $\mathbf{M}(a', a, b)$ for all $a' \geqq a$ of $\mathbf{E}$; however, the inversion of this implication is not correct, *cf.* [1], pp. 28–29.

If S' is a non-void subset of S and $a \in \mathbf{E}(S)$ then we shall consider the one-to-one mapping of the set S'_a, consisting of all a-blocks (elements of S/a) overlapping S', onto the set $S'_{(a)}$ of intersection of these a-blocks with S'; this one-to-one mapping $S'_a \to S'_{(a)}$: $A \to A \cap S'$ for every $A \in S/a$, $A \cap S' \neq \emptyset$, we extend to a mapping $S/a \to S'_{(a)} \cup \{\emptyset\} : A \to A \cap S'$ for every $A \in S/a$, which we call the *semicontractions*. If $S/a = S'_a$ we speak about a *contraction*; *cf.* [3], pp. 86–87. If $a \geqq b$ are relations of $\mathbf{E}(S)$ and if we reduce the effect of the relation b to an (arbitrary) a-block, we obtain the relation $a//b$, the so-called *relation-quotient* of relations a, b; of course $a//b \in \mathbf{E}(A)$ for (arbitrary) $A \in S/a$. If $a \geqq b$; $c \geqq d$ are relations of $\mathbf{E}(S)$ with $a \geqq c$, then we say that the quotient $c//d$ is deduced from $a//b$ by *semicontraction* or *contraction* if every $A \in S/a$ is transferred into any $C \in S/c$, $C \subset A$ by some semicontraction or contraction $A/a//b \to C/c//d \cup \{\emptyset\}$. If $a_1 \geqq a_2$, $b_1 \geqq b_2$ are the relations of $\mathbf{E}(S)$, if a_1/a_2, b_1/b_2 are similar with middle quotient c_1/c_2, and if $c_1//c_2$ is deduced by some (semi)contraction from $a_1//a_2$ and by some (semi) contraction from $b_1//b_2$, then we call $a_1//a_2$, $b_1//b_2$ *(semi)chained*. The chains $\mathbf{A}$, $\mathbf{B}$ of $\mathbf{E}(S)$ are called *semi(chained)* if there exists a one-to-one mapping $\mathbf{A} \to \mathbf{B} : a_i \to b_{f(i)}$ such that $a_k//a_{k+1}$, $b_{f(k)}//b_{f(k)+1}$ are (semi)chained for every k.

Semichained and chained refinements. The fundamental properties of the relations $a \wedge b$, $a \vee b$ ($a, b \in \mathbf{E}(S)$) are described in [3], pp. 67–72.

Lemma. *Let $a, b \in \mathbf{E}(S)$. Then $b//a \wedge b$ can be deduced from $a \vee b//a$ by some semicontraction; and especially by a contraction, if and only if $a \square b$.*

Proof. For $\mathbf{B} \in S/b$ we define the mapping $\mathbf{B}_a \to \mathbf{B}_{(a)} : C \to C \cap \mathbf{B}$ for all $C \in S/a$, $C \cap \mathbf{B} \neq \emptyset$. The set-union $\mathbf{s}(\mathbf{B}_a)$ of all a-blocks which are elements of $\mathbf{B}_a$ is contained in a set $\mathbf{M} \in S/a \vee b$. Using this, there can be easily constructed the required semicontraction $\mathbf{D}/a \vee b//a \to \mathbf{B}/b//a \vee b \cup \{\emptyset\}$ of the quotient $a \vee b//a$ into $b//a \wedge b$. From the definition of associability follows that $\mathbf{s}(\mathbf{B}_a) = \mathbf{D}$ exactly then if $a \square b$.

Theorem 4. *Let relations $a_1 \geqq a_2$, $b_1 \geqq b_2$ of $\mathbf{E}(S)$ be given such that a_1/a_2, b_1/b_2 are similar with middle quotient c_1/c_2. Then $a//a_2$, $b_1//b_2$ are semichained and the chaining becomes direct if $a_2 \square c_1$, $b_2 \square c_1$.*

Proof. We apply the lemma to both pairs $a_1//a_2$, $c_1//c_2$; $b_1//b_2$, $c_1//c_2$. Thus $c_1//c_2$ can be obtained by a certain semicontraction from $a_1//a_2$ and by a certain semicontraction from $b_1//b_2$, so that $a_1//a_2$, $b_1//b_2$ are semichained quotients. According to the second part of the lemma, both the preceding semicontractions become contractions (and the semichaining of $a_1//a_2$, $b_1//b_2$ becomes a chaining) if and only if $a_2 \square c_1$, $b_2 \square c_1$.

Theorem 5. *If $\mathbf{A}$, $\mathbf{B}$ are chains of $\mathbf{E}(S)$ which satisfy condition* III *(or I, II, II* respectively), then $\mathbf{A}^*$, $\mathbf{B}^*$ are semichained by the mapping $a_{k,l} \to b_{l,k}$; the semichaining becomes a chaining if and only if*

(IV) $\quad a_{k+1} \vee (a_k \wedge b_{l+1}) \square a_k \wedge b_l\,, \quad b_{l+1} \vee (b_l \wedge a_{k+1}) \square b_l \wedge a_k$ *for all k, l.*

The theorem is a consequence of theorems 2, 3, 4 and the definition of (semi)chained pairs of chains.

Borůvka's theorem cited in the introduction coincides with theorem 5 for the case of condition (II*) and semicontraction. In [3], pp. 88–95, it is proved that condition (IV) may be strengthened to any one of the following

(V) $\quad a_{k+1} \square a_k \wedge b_j\,, \quad b_{l+1} \square b_l \wedge a_i$ for all i, j, k, l;

(VI) $\quad a_i \square b_j$ for all i, j,

(VII) $\quad a_i \boxed{\square} b_j$ for all i, j.

If the given chains $\mathbf{A}$, $\mathbf{B}$ of $\mathbf{E}(S)$ satisfy condition (V) or (VI), then so do $\mathbf{A}^*$, $\mathbf{B}^*$.

A similar problem may be formulated for conditions (I), (II), (II*), (III), (VII). It is natural to ask for necessary and sufficient conditions for the validity of the following implication:

*If chains **A**, **B** of some lattice **L** satisfy the condition*

(VIII) *for every α, β from prescribed index sets J_α, J_β there holds a relation*

$$\mathbf{r}(a_\alpha, b_\beta)\,,$$

*then the refinements **A***, ***B*** satisfy the same condition.*

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Резюме

ПОЛУСПУТАННЫЕ УПЛОТНЕНИЯ ЦЕПЕЙ ЭКВИВАЛЕНТНОЙ СТРУКТУРЫ

ВАЦЛАВ ГАВЕЛ (Václav Havel), Брно

Использованы условия для того,чтобы уплотнения Зассенгайза двух конечных цепей некоторой структуры были снизу просто подобны и с междуквоциентами наперед заданой формы.

Эти условия приведены в доказательстве теоремы существования реляционного полуизомогфизма между уплотнениями Зассенгайза двух конечных цепей структуры эквивалентностей данного множества. Теорема представляет собой некоторое обобшение более ранних результатов [1], [3].