A NOTE ON PERIMETER AND MEASURE

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Several sufficient conditions are given for a compact set of finite perimeter to be of measure zero.

1. Introductory remark. Simple examples may be given of an open set $G \subset E_m$ such that both G and $\bar{G}$ are of finite perimeter and $F = \bar{G} - G$ has positive volume. Moreover, for $m \geqq 3$, G may be assumed connected and uniformly locally connected (cf. section 6.3). Such a situation cannot occur if certain topological restrictions are imposed on F or on G. In the relatively simple case $m = 2$, F is of (plane) measure zero provided G is a domain or a uniformly locally connected open set and F has finite perimeter. If F is a simple closed curve and G is its complementary domain then F has measure zero whenever G is of finite perimeter (for the perimeter of G coincides with the length of F).[1]) More complicated situations arise if $m \geqq 3$. In the well-known example of A. S. BESICOVITCH of a topological sphere F in E_3 of finite Lebesgue area and of positive volume (as constructed in [2]) the bounded complementary domain G of F has finite perimeter. However, this is no longer true about $\bar{G}$. Generally, if F is a closed surface in E_3, G one of its complementary domains and if both G and $\bar{G}$ have finite perimeter, then F is of (3-dimensional) measure zero. (As W. H. FLEMING noticed in [6], remark on p. 437, this was pointed out by H. FEDERER; the same result was announced in [8] and proved in [9].) Similar conclusion remains in force if only $F = \bar{G} - G$ is assumed to be of finite perimeter. The present note deals with conditions which, imposed on a closed set F (in E_3 or E_2) of finite perimeter and on an open set G disjoint with F and "close" to F, imply that F has measure zero.

2. Notation. Given an open set $G \subset E_m$ and an integer i with $0 \leqq i < m$, we shall denote by $\mathscr{A}_i G$ the set of all x in $\bar{G}$ (= closure of G) with the following property: To any neighbourhood $U_0(x)$ of x (in E_m) there can be assigned a neighbourhood $U_1(x) \subset U_0(x)$ of x such that every i-cycle (with integer coefficients)[2]) in $G \cap U_1(x)$

[1]) Cf. J. MAŘÍK [12].

[2]) To be interpreted in the sense of § 3, chap. XIV of P. S. ALEKSANDROFF's (П. С. Александров) monograph [1] (cf. also chap. XV, 0 : 1).

bounds in $G \cap U_0(x)$. (For a general study of analoguous properties, the reader is referred to R. L. WILDER's monograph [13].)

2.1. Lemma. *For every open set $G \subset E_m$ and every integer $i \in \langle 0, m)$ the set $\mathscr{A}_i G$ is an $F_{\sigma\delta}$.*

Proof. Fix G and i. Write $U(x, r)$ for $\{y; y \in E_m, |x - y| < r\}$. Given positive integers $n < k$, denote by H_{nk} the set of all $x \in \bar{G}$ for which the following condition is satisfied: For every $\varepsilon > 0$ and every i-cycle z^i in $G \cap U(x, 1/k)$ there is an $(i + 1)$-chain c^{i+1} in $G \cap U(x, 1/n + \varepsilon)$ bounded by z^i. H_{nk} is closed. To see this it is sufficient to observe that, given $\varepsilon > 0$ and an i-cycle z^i with $\overline{z^i} \subset U(x, 1/k) \cap G$[3]) we have $U(y, 1/n + \frac{1}{2}\varepsilon) \subset U(x, 1/n + \varepsilon)$ and $\overline{z^i} \subset U(y, 1/k)$ for every y sufficiently close to x. Since, clearly, $\mathscr{A}_i G = \bigcap_{n=1}^{\infty} \bigcup_{k=n+1}^{\infty} H_{nk}$, we see that $\mathscr{A}_i G$ is an $F_{\sigma\delta}$.

3. Notation. Fix a positive integer m. Given an integer $i \in \langle 1, m \rangle$, we denote by $\mathfrak{G}_i$ the system of all Lebesgue measurable subsets A in E_m for which there exists a finite signed Borel measure Φ_i^A over the boundary $\mathscr{B}A$ of A, such that

$$(1) \qquad \int_A \frac{\partial \varphi(x)}{\partial x_i} \, dx = \int_{\mathscr{B}A} \varphi(x) \, d\Phi_i^A$$

for every infinitely differentiable function φ with compact support. Let $\|A\|_i$ stand for the total variation of Φ_i^A on $\mathscr{B}A$ and put $\|A\|_i = +\infty$ for every Lebesgue measurable $A \subset E_m$ which does not belong to $\mathfrak{G}_i$. We have thus

$$\|A\|_i = \sup_\varphi \int_A \frac{\partial \varphi(x)}{\partial x_i} \, dx ,$$

φ ranging over the class of all infinitely differentiable functions φ with compact support for which $\max_x |\varphi(x)| \leq 1$. We shall denote by $\tilde{\mathfrak{G}}_i$ the system of all Lebesgue measurable A such that $A \cap K \in \mathfrak{G}_i$ for every cube $K \subset E_m$. (Thus $\tilde{\mathfrak{G}}_i$ coincides with the system of all Lebesgue measurable A for which there exists a locally finite signed Borel measure Φ_i^A over $\mathscr{B}A$ such that (1) holds whenever φ is an infinitely differentiable function with compact support on E_m.) Further, put

$$\mathfrak{G} = \bigcap_{i=1}^{m} \mathfrak{G}_i , \quad \tilde{\mathfrak{G}} = \bigcap_{i=1}^{m} \tilde{\mathfrak{G}}_i .$$

Defining (for Lebesgue measurable $A \subset E_m$)

$$\|A\| = \sup_v \int_A \operatorname{div} v(x) \, dx ,$$

v ranging over the class of all (m-dimensional) infinitely differentiable vector-valued

[3]) Cf. [1] (chap. XV, 0 : 1) for notation.

functions v with compact support for which $\max_x |v(x)| \leqq 1$, we see that $\mathfrak{S}$ is the system of all Lebesgue measurable $A \subset E_m$ such that $\|A\| < +\infty$. $\mathfrak{S}$ is an algebra. $\|E_m - A\| = \|A\|$ for every Lebesgue measurable $A \subset E_m$. Given a monotone sequence $\{A_n\}_{n=1}^{\infty}$ of elements of $\mathfrak{S}$ with $\sup_n \|A_n\| < +\infty$, we have $\lim_n A_n \in \mathfrak{S}$. $\|A\|$ will be termed the perimeter of A. (For bounded A the notation $\|A\|_i$, $\|A\|$ was introduced by J. MAŘÍK in [11]. Another equivalent definition of perimeter for Borel subsets in E_m was given by E. DE GIORGI in [3]; *cf.* also [4] and H. FEDERER [5]. The reader is referred to [7] for further bibliography on the subject.)

4. We shall collect here several known results to be used later. Suppose there is given a set $M \subset E_1$. A point $a \in E_1$ will be termed an *eM*-point provided both $(E_1 - M) \cap I$ and $M \cap I$ have positive outer linear measure for every open interval $I \subset E_1$ containing a. The number (possibly zero or infinite) of all *eM*-points will be denoted by $\varepsilon(M)$. Further we shall use the following notation. Given positive integers $i \leqq m$, a subset A in E_m and a point $x = [x_1, \ldots, x_{m-1}] \in E_{m-1}$, we write A_x^i for the set of all $\zeta \in E_1$ with $[x_1, \ldots, x_{i-1}, \zeta, x_i, \ldots, x_{m-1}] \in A$. The following assertion is known ([7]; *cf.* also J. MAŘÍK [11] and chap. 7 of K. KRICKEBERG [10]).

4.1. *Let A be a Lebesgue measurable subset in E_m. Then $\varepsilon(A_x^i)$, considered as a function of the variable x on E_{m-1}, is Lebesgue measurable and*

$$\|A\|_i = \int_{E_{m-1}} \varepsilon(A_x)\, \mathrm{d}x\,.$$

Write $A^\zeta = \{x;\, x \in E_{m-1}, \zeta \in A_x^m\}$ $(A \subset E_m, \zeta \in E_1)$. Using Fubini's theorem, we obtain from 4.1 the following assertion:

4.2. *Let A be a Lebesgue measurable subset in E_m, i a positive integer with $i < m$. Then $\|A^\zeta\|_i$ [4]) is a Lebesgue measurable function of the variable ζ on E_1 and*

$$\|A\|_i = \int_{E_1} \|A^\zeta\|_i\, \mathrm{d}\zeta$$

(*cf.* also W. H. FLEMING [6], p. 455, and K. KRICKEBERG [10], p. 125). Hence it follows, in particular, that $A^\zeta \in \mathfrak{S}_i$ (in E_{m-1}; $i < m$) for almost every $\zeta \in E_1$ provided $A \in \mathfrak{S}_i$ (in E_m).

Given a set $A \subset E_m$, we denote by A^* the set of all $[x_1, \ldots, x_{m-1}, \zeta] = [x, \zeta]$ $(x \in E_{m-1}, \zeta \in E_1)$ for which ζ belongs to the interior of A_x^m (in E_1). Further we write L_k for the k-dimensional Lebesgue measure.

4.3. *Let $A \subset E_m$ be a closed set, $A \in \mathfrak{S}_m$. Then A^* is an F_σ-set and* $\mathsf{L}_m(A - A^*) = 0$. (*Cf.* [7].)

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[4]) Here $\| \ldots \|_i$ is considered in E_{m-1}.

5. Theorem. *Let F be a locally compact subset in E_3 and suppose that $F \in \tilde{\mathfrak{G}}_i \cap \cap \tilde{\mathfrak{G}}_j$, where $1 \leqq i \neq j \leqq 3$. Further suppose that G is an open subset in E_3 with $G \cap F = \emptyset$, $\mathscr{A}_0 G \supset F$. Then either $\mathsf{L}_3 F = 0$ or $\mathsf{L}_3(F - \mathscr{A}_1 G) > 0$.*

Proof. Let $i = 2, j = 3$. We may clearly assume that F is compact and $F \in \mathfrak{G}_2 \cap \cap \mathfrak{G}_3$. Suppose, if possible, that

$$\mathsf{L}_3(F - \mathscr{A}_1 G) = 0 \quad \text{and} \quad \mathsf{L}_3 F > 0 .$$

Then a $\zeta \in E_1$ can be chosen such that

$$\mathsf{L}_2 F^\zeta > 0 , \quad \|F^\zeta\|_2 < +\infty , \quad \mathsf{L}_2(F - F^*)^\zeta = 0 = \mathsf{L}_2(F - \mathscr{A}_1 G)^\zeta .$$

Write $B = F^\zeta$. Let B_n stand for the set of all $[\xi, \eta] \in E_2$ with

$$\{\xi\} \times \left\langle \eta - \frac{1}{n}, \eta + \frac{1}{n} \right\rangle \subset B ,$$

so that $B^* = \bigcup_{n=1}^{\infty} B_n$. Clearly, every B_n is closed. We have $\mathsf{L}_2(B - B^*) = 0$ because $\|B\|_2 < +\infty$. Taking into account that $\mathsf{L}_2 B > 0$, we fix a positive integer n with $\mathsf{L}_2 B_n > 0$, so that

$$\mathsf{L}_2(B_n \cap (F^*)^\zeta \cap (\mathscr{A}_1 G)^\zeta) > 0 .$$

Consequently, we have a

$$[\xi_0, \eta_0] \in (F^*)^\zeta \cap (\mathscr{A}_1 G)^\zeta \cap B_n$$

such that $[\xi_0, \eta_0]$ is a point of density of B_n. Fix a sequence $\{[\xi_k, \eta_k]\}_{k=1}^{\infty}$ of points in B_n such that

$$\lim_{k \to \infty} [\xi_k, \eta_k] = [\xi_0, \eta_0] , \quad \xi_{2l-1} < \xi_0 , \quad \xi_{2l} > \xi_0 \quad (l = 1, 2, \ldots) .$$

Further fix an $\varepsilon > 0$ such that the segment

$$E = \{\xi^0\} \times \{\eta^0\} \times \langle \zeta - \varepsilon, \zeta + \varepsilon \rangle$$

is completely contained in F and write U_0 for the (open) sphere of center $[\xi_0, \eta_0, \zeta]$ and radius ε. Let U_1 be a sphere of center $[\xi_0, \eta_0, \zeta]$ and radius $\delta < \varepsilon$ such that any 1-cycle in $G \cap U_1$ bounds in $G \cap U_0$. Put $q = \min(1/n, \frac{1}{2}\delta)$ and write S_1, S_2 for the sphere of radius $\frac{1}{4}q$ and of center

$$[\xi_0, \eta_0 - \tfrac{1}{2}q, \zeta] , \quad [\xi_0, \eta_0 + \tfrac{1}{2}q, \zeta]$$

respectively. Clearly, $S_1 \cup S_2 \subset U_1 - E$. Further, write $\hat{S}_h$ for the sphere concentric with S_h $(h = 1, 2)$ and of radius $\varepsilon_1 < \frac{1}{4}q$ small enough to secure that any 0-cycle in $G \cap \hat{S}_h$ bounds in $G \cap S_h$. Put

$$H_k = \{\xi_k\} \times \langle \eta_k - \tfrac{1}{2}q, \ \eta_k + \tfrac{1}{2}q \rangle \times \{\zeta\}$$

and fix a p such that

$$[\xi_p, \eta_p - \tfrac{1}{2}q, \zeta] \in \hat{S}_1 \ni [\xi_{p+1}, \eta_{p+1} - \tfrac{1}{2}q, \zeta] ,$$
$$[\xi_p, \eta_p + \tfrac{1}{2}q, \zeta] \in \hat{S}_2 \ni [\xi_{p+1}, \eta_{p+1} + \tfrac{1}{2}q, \zeta] .$$

Let O_p and O_{p+1} be convex neighbourhoods of H_p and H_{p+1} respectively, such that $O_p \cup O_{p+1} \subset U_1 - E$. Since $H_p \subset \mathscr{A}_0 G$, we can fix an $\alpha > 0$ such that any 0-cycle $o_1 - o_2$ bounds in $O_p \cap G$ provided o_1, o_2 are points in G with

$$\alpha > \varrho(o_1, o_2) + \varrho(o_2, H_p) .$$ [5]

Suppose now that H_p is naturally ordered from $[\xi_p, \eta_p - \frac{1}{2}q, \zeta] = u_{p0}$ to $[\xi_p, \eta_p + \frac{1}{2}q, \zeta]$, and decompose H_p into segments of length not exceeding $\frac{1}{4}\alpha$ by means of points

$$u_{p0} < u_{p1} < \ldots < u_{ps} = [\xi_p, \eta_p + \tfrac{1}{2}q, \zeta] .$$

Let us associate with every u_{pj} a point $o_{pj} \in G$ such that

$$\varrho(u_{pj}, o_{pj}) < \tfrac{1}{4}\alpha , \quad o_{p0} \in S_1 , \quad o_{ps} \in S_2 .$$

We have thus

$$\varrho(o_{p,j-1}, o_{pj}) + \varrho(o_{pj}, H_p) < \alpha$$

and, consequently, the 0-cycle $o_{pj} - o_{p,j-1}$ bounds a 1-chain c^1_{pj} $(1 \leqq j \leqq s)$ in $G \cap O^p$. In a similar way we fix points $o_{p+1,j} \in O_{p+1} \cap G$ $(0 \leqq j \leqq t)$ such that $o_{p+1,0} \in \hat{S}_2$, $o_{p+1,t} \in \hat{S}_1$ and such that the 0-cycle $o_{p+1,j} - o_{p+1,j-1}$ bounds a 1-chain $c^1_{p+1,j}(1 \leqq j \leqq t)$ in $G \cap O_{p+1}$. Since $o_{p+1,0} \in S_2 \cap G \ni o_{ps}$, the 0-cycle $o_{p+1,0} - o_{ps}$ bounds a 1-chain c^1_2 in $S_2 \cap G$. Similarly, the 0-cycle $o_{p0} - o_{p+1,t}$ bounds a 1-chain c^1_1 in $S_1 \cap G$. Put

$$z^1 = c^1_{p1} + \ldots + c^1_{ps} + c^1_2 + c^1_{p+1,1} + \ldots + c^1_{p+1,t} + c^1_1 .$$

Clearly, z^1 is a 1-cycle in

$$(O_1 \cup S_2 \cup O_2 \cup S_1) \cap G \subset U_1 \cap G .$$

Thus z^1 should bound in $G \cap U_0 \subset U_0 - E$. Put

$$v_1 = [\xi_{p+1}, \eta_{p+1} - \tfrac{1}{2}q, \zeta] , \quad v_2 = [\xi_{p+1}, \eta_{p+1} + \tfrac{1}{2}q, \zeta]$$

and let $\tilde{c}^1_1, \tilde{c}^1_2, \tilde{c}^1_3, \tilde{c}^1_4$ be 1-chains corresponding naturally to the oriented segments $\overrightarrow{u_{p0}u_{ps}}, \overrightarrow{u_{ps}v_2}, \overrightarrow{v_2v_1}, \overrightarrow{v_1u_{p0}}$ respectively. It is easily seen that z^1 is homologous to $\tilde{c}^1_1 + \tilde{c}^1_2 + \tilde{c}^1_3 + \tilde{c}^1_4 = \tilde{z}_1$ in

$$O_1 \cup S_2 \cup O_2 \cup S_1 \subset U_0 - E .$$

Since $[\xi_0, \eta_0]$ belongs to the interior of the parallelogram with vertices

$$[\xi_p, \eta_p - \tfrac{1}{2}q] , \quad [\xi_p, \eta_p + \tfrac{1}{2}q] , \quad [\xi_{p+1}, \eta_{p+1} + \tfrac{1}{2}q] , \quad [\xi_{p+1}, \eta_{p+1} - \tfrac{1}{2}q] ,$$

the straight line $\{\xi_0\} \times \{\eta_0\} \times E_1 = P$ prevents $\tilde{z}^1$ from bounding. Consequently, not even z^1 can bound in $U_0 - E \subset E_3 - P$, which is a contradiction.

6. Corollaries. A set $S \subset E_3$ will be termed a simple surface provided every point $x \in S$ has a neighbourhood in S which is homeomorphic with E_2. If, moreover, S is a continuum, then S will be termed a simple closed surface. It follows from known

[5]) We write ϱ for the Euclidean distance function.

theorems in topology and from the above theorem that the following assertions are true.

6.1. Corollary. *Let S be a simple surface in E_3 and suppose that $S \in \tilde{\mathfrak{G}}_i \cap \tilde{\mathfrak{G}}_j$, where $1 \leqq i + j \leqq 3$. Then $\mathsf{L}_3 S = 0$.*

6.2. Corollary. *Let S be a simple closed surface in E_3 and let G_1, G_2 be its complementary domains. If both G_1 and G_2 belong to $\mathfrak{G}_i \cap \mathfrak{G}_j$ $(1 \leqq i + j \leqq 3)$,* then $\mathsf{L}_3 S = 0$.

Remark. A result slightly more general than 6.2 was proved in [9].

6.3. Example. Consider a closed cube K in E_3. For every $n > 1$ divide K into 2^{3n} equal cubes K_i^n $(1 \leqq i \leqq 2^{3n})$ and denote by D_n the union of the edges of all the cubes K_i^n that are interior to K. Further, fix a descending sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ of positive real numbers with $\lim_k \varepsilon_k = 0$ and put

$$U_k = \{x;\, x \in E_3,\, \varrho(x, D_k) < \varepsilon_k\}\,,$$

$$G_n = \bigcup_{k=1}^{n} U_k\,, \quad G = \bigcup_{n=1}^{\infty} G_n\,, \quad F = K - G\,.$$

If ε_k tends rapidly to zero as $k \to \infty$, then $\mathsf{L}_3 F > 0$; moreover, one can achieve $\sup_n \|G_n\| < +\infty$. Then G (as a limit of a non-descending sequence of sets having uniformly bounded perimeters) belongs to $\mathfrak{G}$ and the same is true about $\bar{G} = K$ and $F = K - G$. G is easily seen to be connected and uniformly locally connected, so that $F \subset \mathscr{A}_0 G$.

7. Theorem. *Let F be a locally compact subset in E_2, $F \in \tilde{\mathfrak{G}}_i$* $(i = 1$ or $2)$. *Further, let $G \subset E_2$ be an open set, $G \cap F = \emptyset$. Then either $\mathsf{L}_2 F = 0$ or $\mathsf{L}_2(F - \mathscr{A}_0 G) > 0$.*

Proof. We may clearly assume that F is compact and $F \in \mathfrak{G}_2$. Suppose, if possible, that $\mathsf{L}_2 F > 0$ and $\mathsf{L}_2(F - \mathscr{A}_0 G) = 0$. Write F_n for the set of all $[\xi, \eta]$ with

$$\{\xi\} \times \langle \eta - 1/n, \eta + 1/n \rangle \subset F\,,$$

so that $\bigcup_{n-1}^{\infty} F_n = F^*$. Every F_n is closed. Since $\mathsf{L}_2 F > 0$ and $\mathsf{L}_2(F - F^*) = 0$, we have $\mathsf{L}_2 F_n > 0$ for suitably chosen n. Consequently, $\mathsf{L}_2(F_n \cap \mathscr{A}_0 G) > 0$. Let

$$[\xi_0, \eta_0] = o \in F_n \cap \mathscr{A}_0 G$$

be a point of density of the set $F_n \cap \mathscr{A}_0 G$. We have then a sequence $\{[\xi_k, \eta_k]\}_{k=1}^{\infty}$ of points in $F_n \cap \mathscr{A}_0 G \subset \bar{G}$ tending to o as $k \to \infty$ and such that $\xi_{2j-1} < \xi_0$, $\xi_{2j} > \xi_0$ $(j = 1, 2, \ldots)$. Let us associate with every $[\xi_k, \eta_k]$ a point $[\tilde{\xi}_k, \tilde{\eta}_k] = o_k \in G$ such that $\lim_k o_k = o$, $\tilde{\xi}_{2j-1} < \xi_0$, $\tilde{\xi}_{2j} > \xi_0$ $(j = 1, 2, \ldots)$. Write U for the open circular disc of center o and of radius $1/n$. We have then a k_0 such that $o_k \in U$ whenever $k > k_0$. Since $U - G \supset U \cap F_n \supset \{[\xi, \eta];\, \xi = \xi_0\} \cap U$, we see that the 0-cycle $o_k - o_{k+1}$ $(k > k_0)$ cannot bound in $U \cap G$. This is in contradiction with $o \in \mathscr{A}_0 G$, $\lim_k o_k = o$.

8. Definition. Write $e^1 = [1, 0]$, $e^2 = [0, 1]$. A point $o \in E_2$ will be termed a c-point of $A \subset E_2$ provided there is a $\delta > 0$ such that $o + \alpha e^1 \in A \ni o + \alpha e^2$ whenever $|\alpha| \leqq \delta$. The set of all c-points of A will be denoted by A^+.

9. Lemma. *Let F be a locally compact subset in E_2, $F \in \tilde{\mathfrak{G}}$. Then $\mathsf{L}_2(F - F^+) = 0$.*

Proof. We may clearly suppose that F is compact and $F \in \mathfrak{G}$. Write $F_2^* = F^*$ for the set of all $[\xi, \eta] \in F$ for which η belongs to the interior of F_ξ^2 (with respect to E_1). Similarly, let F_1^* be the set of all $[\xi, \eta] \in F$ for which F_η^1 contains ξ in its interior. Since $F \in \mathfrak{G}_2$, we have $\mathsf{L}_2(F - F_2^*) = 0$. In exactly the same way, $F \in \mathfrak{G}_1$ implies $\mathsf{L}_2(F - F_1^*) = 0$. Consequently,

$$\mathsf{L}_2(F - (F_1^* \cap F_2^*)) = 0 .$$

Now it is sufficient to observe that $F^+ = F_1^* \cap F_2^*$.

Remark. An analogous assertion may also be proved for subsets in E_m with $m > 2$.

10. Theorem. *Let F be a locally compact subset in E_2, $F \in \tilde{\mathfrak{G}}$. Further, let G be a domain in E_2, $G \cap F = \emptyset$. Then either $\mathsf{L}_2 F = 0$ or $\mathsf{L}_2(F - \bar{G}) > 0$.*

Proof. We may assume that F is compact and $F \in \mathfrak{G}$. Suppose that

$$\mathsf{L}_2 F > 0 \quad \text{and} \quad \mathsf{L}_2(F - \bar{G}) = 0 . \tag{2}$$

Write F_+^n for the set of all $[\xi, \eta] = o \in F$ for which the set

$$C_n(o) = (\langle \xi - 1/n, \xi + 1/n \rangle \times \{\eta\}) \cup (\{\xi\} \times \langle \eta - 1/n, \eta + 1/n \rangle)$$

is completely contained in F. Clearly, every F_+^n is closed and $\bigcup_{n=1}^{\infty} F_+^n = F^+$. We have thus $\mathsf{L}_2 F_+^n > 0$ for suitably chosen n. Consequently, $\mathsf{L}_2(F_+^n \cap \bar{G}) > 0$ (compare (2)). Fix a point

$$o_0 = [\xi_0, \eta_0] \in F_+^n \cap \bar{G}$$

such that o_0 is a point of density of the set F_+^n. Put

$$Q_1 = \{[\xi, \eta];\ \xi > \xi_0,\ \eta > \eta_0\} ,$$
$$Q_3 = \{[\xi, \eta];\ \xi < \xi_0,\ \eta < \eta_0\} .$$

Then there is a sequence $\{[\xi_k, \eta_k]\}_{k=1}^{\infty}$ of points in F_+^n such that

$$[\xi_{2j-1}, \eta_{2j-1}] = o_{2j-1} \in Q_3 , \quad [\xi_{2j}, \eta_{2j}] = o_{2j} \in Q_1 \quad (j = 1, 2, \ldots)$$

and $\lim_k o_k = o_0$. Write

$$P_h = (\xi_{2h-1}, \xi_{2h}) \times (\eta_{2h-1}, \eta_{2h}) \quad (h = 1, 2, \ldots) .$$

We have then a h_0 such that the boundary of P_h is completely contained in

$$C_n(o_{2h-1}) \cup C_n(o_{2h}) \subset F \subset E_2 - G$$

whenever $h > h_0$. Since $o \in P_h \cap \bar{G}$, we conclude that $P_h \cap G \neq \emptyset$ $(h = 1, 2, \ldots)$. Noting that the diameter of P_h tends to zero as $h \to +\infty$ we see that G cannot be connected. Thus we have a contradiction.

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Резюме

ЗАМЕТКА О ПЕРИМЕТРЕ И МЕРЕ

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Если G — открытое множество в E_m, то символом $\mathscr{A}_i G$ $(0 \leqq i < m)$ обозначим множество всех точек x замыкания $\bar{G}$ множества G, обладающих следующим свойством: Для каждой окрестности $U_0(x)$ точки x существует такая окрестность $U_1(x) \subset U_0(x)$ точки x, что каждый целочисленный i-мерный цикл в $G \cap U_1(x)$ гомологичен нулю в $G \cap U_0(x)$ (соотв. топологические понятия надо понимать в смысле монографии П. С. Александрова [1], § 3, гл. XIV; см. тоже гл. XV, 0 : 1).

Символом $\mathfrak{G}_k$ $(1 \leqq k \leqq m)$ обозначим систему всех измеримых (по Лебегу) множеств $A \subset E_m$, для которых конечно число $\sup\limits_{\varphi} \int_A (\partial\varphi(x)/\partial x_k)\, \mathrm{d}x$; здесь верхняя грань берётся по отношению ко всем бесконечно дифференцируемым функциям φ, обращающимся в нуль вне некоторого компактного множества и удовлетворяющим условию $\max\limits_{x} |\varphi(x)| \leqq 1$. Пусть $\tilde{\mathfrak{G}}_k$ — система всех $A \subset E_m$,

для которых $A \cap K \in \mathfrak{S}_k$ для каждого m-мерного куба K. Символом L_m обозначим меру Лебега в пространстве E_m.

Теорема. *Пусть F — локально компактное множество в E_3, $1 \leqq i < j \leqq 3$, $F \in \tilde{\mathfrak{S}}_i \cap \tilde{\mathfrak{S}}_j$. Пусть, далее, G — открытое множество в E_3, $G \cap F = \emptyset$, $\mathscr{A}_0 G \supset F$. Тогда $\mathsf{L}_3 F = 0$ или $\mathsf{L}_3(F - \mathscr{A}_1 G) > 0$.*

Теорема. *Пусть F — локально компактное множество в E_2, $F \in \tilde{\mathfrak{S}}_i$ ($i = 1$ или 2). Пусть, далее, G — открытое множество в E_2, $G \cap F = \emptyset$. Тогда $\mathsf{L}_2 F = 0$ или $\mathsf{L}_2(F - \mathscr{A}_0 G) > 0$.*

Теорема. *Пусть F — локально компактное множество в E_2, $F \in \tilde{\mathfrak{S}}_1 \cap \tilde{\mathfrak{S}}_2$. Пусть, далее, G — область в E_2, $G \cap F = \emptyset$. Тогда $\mathsf{L}_2 F = 0$ или $\mathsf{L}_2(F - \bar{G}) > 0$.*