Jiří Jelínek; Josef Král

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NOTE ON SEQUENCES OF INTEGRABLE FUNCTIONS

JIŘÍ JELÍNEK and JOSEF KRÁL, Praha

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Some theorems are proved concerning the integrability of $\liminf_{n\to\infty} f_n^+$, $\liminf_{n\to\infty} |f_n|$ for sequences of integrable functions f_n ($n = 1, 2, \ldots$) which may assume both positive and negative values.

Notation. The terms number, function, measure always mean a real number, function, measure (finite or infinite), respectively. X is a fixed non-void set, $\mathbf{S}$ is a σ-algebra of its subsets, μ is a measure on $\mathbf{S}$ (so that $(X, \mathbf{S}, \mu)$ represents a measure space – cf. [1]). μ is always assumed σ-finite.[1]) If α is a real number we write, as usual, $\alpha^+ = \max(\alpha, 0)$, $\alpha^- = (-\alpha)^+$. The meaning of the symbols f^+, f^-, where f is a function on X, is obvious.

Let now f_n ($n = 1, 2, \ldots$) be functions on X. We shall say that the sequence $\{f_n\}_{n=1}^\infty$ is uniformly lower semiconvergent on $Y \subset X$ if an integer $n_0(\varepsilon)$ can be associated with any $\varepsilon > 0$ such that for $f = \liminf_{n\to\infty} f_n$ the following implications are true:

$$(n > n_0(\varepsilon),\ x \in Y,\ f(x) < +\infty) \Rightarrow f_n(x) + \varepsilon > f(x)\,,$$
$$(n > n_0(\varepsilon),\ x \in Y,\ f(x) = +\infty) \Rightarrow f_n(x) > 1/\varepsilon\,.$$

The following generalization of Egoroff's theorem will be needed in the sequel.

Lemma 1. *Suppose that* $\mu(X) < +\infty$, and let $\{f_n\}_{n=1}^\infty$ *be a sequence of* $\mathbf{S}$*-measurable functions on* X. *Then there exists, for every* $\delta > 0$, *a set* $Z \in \mathbf{S}$ *such that* $\mu(Z) < \delta$ *and that* $\{f_n\}_{n=1}^\infty$ *is uniformly lower semiconvergent on* $X - Z$.

Proof is similar to that of Egoroff's theorem (cf. [3], p. 18; [4], p. 249). Put $f = \liminf_{n\to\infty} f_n$,

$$X_{km} = \left\{x; f(x) < +\infty,\ f_m(x) + \frac{1}{k} > f(x)\right\} \cup \{x;\ f(x) = +\infty,\ f_m(x) > k\}\,,$$
$$Y_{kn} = \bigcap_{m=n}^{\infty} X_{km}\,.$$

[1]) I.e. $X = \bigcup_{n=1}^{\infty} X_n$, where $X_n \in \mathbf{S}$, $\mu(X_n) < +\infty$ ($n = 1, 2, \ldots$).

Then

$$Y_{k1} \subset Y_{k2} \subset \ldots, \quad \bigcup_{n=1}^{\infty} Y_{kn} = X .$$

Hence it follows that, to every k, a positive integer n_k can be assigned with $\mu(X - Y_{kn_k}) < 2^{-k} . \delta$. Writing

$$Z = \bigcup_{k=1}^{\infty} (X - Y_{kn_k}),$$

we have

$$\mu(Z) \leqq \sum_{k=1}^{\infty} \mu(X - Y_{kn_k}) < \sum_{k=1}^{\infty} 2^{-k}\delta = \delta ,$$

and the sequence $\{f_n\}_{n=1}^{\infty}$ is easily seen to be uniformly lower semiconvergent on $X - Z = \bigcap_{k=1}^{\infty} Y_{kn_k}$.

Lemma 2. *Let f be a non-negative* **S**-*measurable function on X. Then, for every real number $c < \int_X f \, \mathrm{d}\mu$, there exists a $\delta > 0$ such that*

$$(T \in \mathbf{S},\ \mu(T) < \delta) \Rightarrow \int_{X-T} f \, \mathrm{d}\mu > c .$$

Proof. One can choose a non-negative μ-integrable function h on X such that

$$h \leqq f, \quad \int_X h \, \mathrm{d}\mu > c .$$

Making use of absolute continuity of the indefinite integral $\int h \, \mathrm{d}\mu$, fix a $\delta > 0$ such that

$$(T \in \mathbf{S},\ \mu(T) < \delta) \Rightarrow \int_T h \, \mathrm{d}\mu < \int_X h \, \mathrm{d}\mu - c .$$

We have then

$$\int_{X-T} f \, \mathrm{d}\mu \geqq \int_{X-T} h \, \mathrm{d}\mu = \int_X h \, \mathrm{d}\mu - \int_T h \, \mathrm{d}\mu > c$$

whenever $T \in \mathbf{S}$, $\mu(T) < \delta$.

Proposition 1. *Let $\mu(X) < +\infty$ and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of μ-integrable functions on X. Suppose that $\sup_n \int_M f_n \, \mathrm{d}\mu < +\infty$ for every $M \in$* **S**. *Then $\liminf_{n \to \infty} f_n^+$ is μ-integrable on X.*

Remark 1. In the preceding proposition, the sequence $\{\int_X f_n^+ \, \mathrm{d}\mu\}_{n=1}^{\infty}$ need not be bounded (cf. example 1 below), so that the conclusion of this proposition cannot be simply deduced from Fatou's lemma.

Proof of proposition 1. Put $f = \liminf_{n\to\infty} f_n$, $F = \{x; f(x) = +\infty\}$. Thus $f^+ = \liminf_{n\to\infty} f_n^+$. First prove

(1) $$\mu(F) = 0\,.$$

Suppose, if possible, that $\mu(F) = \alpha > 0$. Applying lemma 1 we conclude that there exists a set $Z \in \mathbf{S}$ such that $\mu(Z) < \frac{1}{2}\alpha$ and that the sequence $\{f_n\}_{n=1}^{\infty}$ is uniformly lower semiconvergent on $F - Z$. (Here Egoroff's theorem could also be used instead of lemma 1.) Hence it follows for $c_n = \inf_{x\in F-Z} f_n(x)$ that

(2) $$\lim_{n\to\infty} c_n = +\infty\,.$$

Since

$$\int_{F-Z} f_n \,\mathrm{d}\mu \geqq c_n\, \mu(F - Z) \geqq c_n[\mu(F) - \mu(Z)] > c_n \cdot \tfrac{1}{2}\alpha\,,$$

we conclude from (2) that

(3) $$\lim_{n\to\infty} \int_{F-Z} f_n \,\mathrm{d}\mu = +\infty\,,$$

which contradicts the assumptions of our proposition. Thus (1) is proved.

Next prove that the equality

(4) $$\int_X f^+ \,\mathrm{d}\mu = +\infty$$

also violates the assumptions of our proposition. Using (4), we shall show that there exist a sequence of mutually disjoint sets $M_k \in \mathbf{S}$ $(k = 1, 2, \ldots)$ and a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that, for every positive integer k, the following relations are fulfilled:

(5) $$\lim_{n\to\infty} \int_{M_1\cup\ldots\cup M_k} f_n^- \,\mathrm{d}\mu = 0\,,$$

(6) $$\int_{N_k} f^+ \,\mathrm{d}\mu = +\infty\,, \quad \text{where} \quad N_k = X - (M_1 \cup \ldots \cup M_k)\,,$$

(7) $$\int_{M_{k+1}} f_{n_{k+1}} \,\mathrm{d}\mu > k + 1 + \int_{M_1\cup\ldots\cup M_k} f_{n_{k+1}}^- \,\mathrm{d}\mu\,,$$

(8) $$1 \leqq i \leqq k \Rightarrow \int_{M_{k+1}} |f_{n_i}| \,\mathrm{d}\mu < 2^{-k-1}\,.$$

On defining $M = \bigcup_{k=1}^{\infty} M_k$ we obtain, on account of (7), (8),

$$\int_M f_{n_{k+1}}\,\mathrm{d}\mu = \int_{M_1\cup\ldots\cup M_k} f_{n_{k+1}}\,\mathrm{d}\mu + \int_{M_{k+1}} f_{n_{k+1}}\,\mathrm{d}\mu + \sum_{p>k+1}\int_{M_p} f_{n_{k+1}}\,\mathrm{d}\mu \geqq$$
$$\geqq \int_{M_1\cup\ldots\cup M_k} f^+_{n_{k+1}}\,\mathrm{d}\mu + \left(\int_{M_{k+1}} f_{n_{k+1}}\,\mathrm{d}\mu - \int_{M_1\cup\ldots\cup M_k} f^-_{n_{k+1}}\,\mathrm{d}\mu\right) -$$
$$- \sum_{p>k+1}\int_{M_p} |f_{n_{k+1}}|\,\mathrm{d}\mu > 0 + (k+1) - \sum_{p>k+1} 2^{-p} > k\,.$$

Hence it follows that $\limsup_{n\to\infty} \int_M f_n\,\mathrm{d}\mu = +\infty$ which is a contradiction.[2])

Put $M_1 = \emptyset$ and $n_1 = 1$. Suppose now that for a fixed integer $k \geqq 1$ there are given sets $M_1, \ldots, M_k$ and integers $n_1 < \ldots < n_k$ such that (5), (6) hold and that $M_i \cap M_j = \emptyset$ whenever $1 \leqq i \neq j \leqq k$. (This is the case for $k = 1$.) We shall show that a set $M_{k+1} \subset N_k = X - (M_1 \cup \ldots \cup M_k)$ can be chosen in such a manner that (5), (6) remain valid with k replaced by $k + 1$, and that (8), and – for sufficiently large $n_{k+1} > n_k$ – also (7), hold. Put

$$a_k = \sup_n \int_{M_1\cup\ldots\cup M_k} f_n^-\,\mathrm{d}\mu\,.$$

Clearly $0 \leqq a_k < +\infty$ (see (5)). Further, fix a $\beta > 0$ such that

$$(9) \qquad (1 \leqq i \leqq k,\ Y \in \mathbf{S},\ \mu(Y) < \beta) \Rightarrow \int_Y |f_{n_i}|\,\mathrm{d}\mu < 2^{-k-1}\,;$$

this is possible since $f_{n_1}, \ldots, f_{n_k}$ are μ-integrable on X. Writing $F_m = \{x; f(x) > m\}$ and using (1), we find a positive integer m_k with $\mu(F_{m_k}) < \beta$. By (6) we have

$$(10) \qquad \int_{N_k \cap F_{m_k}} f^+\,\mathrm{d}\mu = +\infty\,,$$

because f^+ is bounded and, consequently, μ-integrable on $N_k - F_{m_k}$. Using lemma 2 we fix a real number $\delta > 0$ such that

$$(11) \qquad (T \in \mathbf{S},\ \mu(T) < \delta) \Rightarrow \int_{N_k \cap F_{m_k} - T} f^+\,\mathrm{d}\mu > a_k + k + 1\,.$$

Applying lemma 1 we choose a $Z \in \mathbf{S}$, $Z \subset N_k \cap F_{m_k}$ such that

$$(12) \qquad \mu(Z) < \delta$$

and such that the sequence $\{f_n\}_{n=1}^{\infty}$ is uniformly lower semiconvergent on $N_k \cap F_{m_k}$ –

[2]) In [2], p. 158, this method of proof is called „Methode des gleitenden Buckels“.

$- Z$. Put $M_{k+1} = F_{m_k} \cap N_k - Z$. Since $f > m_k$ on M_{k+1}, we have a positive integer p such that $f_n > 0$ on M_{k+1} whenever $n > p$. Consequently,

$$n > p \Rightarrow \int_{M_1 \cup \ldots \cup M_{k+1}} f_n^- \, d\mu = \int_{M_1 \cup \ldots \cup M_k} f_n^- \, d\mu + \int_{M_{k+1}} f_n^- \, d\mu = \int_{M_1 \cup \ldots \cup M_k} f_n^- \, d\mu$$

(note that $M_{k+1} \subset N_k$, so that $M_1, \ldots, M_{k+1}$ are disjoint) and (5) remains true with k replaced by $k + 1$. Since the sequence $\{\int_{M_{k+1}} f_n \, d\mu\}_{n=1}^{\infty}$ is bounded we conclude by Fatou's lemma that f is μ-integrable on M_{k+1} and that

$$\liminf_{n \to \infty} \int_{M_{k+1}} f_n \, d\mu \geqq \int_{M_{k+1}} f \, d\mu = \int_{M_{k+1}} f^+ \, d\mu > a_k + k + 1$$

(cf. (11), (12)). Therefore we can fix a positive integer $n_{k+1} > n_k$ such that $\int_{M_{k+1}} f_{n_{k+1}} \, . \, d\mu > a_k + k + 1$; this implies (7). According to (10) we have

$$\int_Z f^+ \, d\mu = \int_{N_k \cap F_{mk}} f^+ \, d\mu - \int_{M_{k+1}} f^+ \, d\mu = +\infty \, .$$

Since $Z \subset N_k \cap (X - M_{k+1}) = N_{k+1}$ we see that also (6) remains valid with k replaced by $k + 1$. In view of (9) we have (8). The proof is complete.

Example 1. Denote by X the set of all real numbers x with $0 < x < 1$. Further, let $\mathbf{S}$ be the system of all Lebesgue measurable subsets of X and let μ be the Lebesgue measure. Define

$$f_n(x) = 4^n \quad \text{for} \quad 0 < x \leqq 2^{-n} \, , \quad f_n(x) = -4^n \quad \text{for} \quad 2^{-n} < x < 1 \, .$$

Given a set $M \in \mathbf{S}$ we have

$$\mu(M \cap (0, 2^{-n}\rangle) \leqq \mu(M \cap (2^{-n}, 1))$$

and, consequently, $\int_M f_n \, d\mu \leqq 0$ for every sufficiently large n. On the other hand,

$$\lim_{n \to \infty} \int_X f_n^+ \, d\mu = \lim_{n \to \infty} 4^n \, . \, 2^{-n} = +\infty \, .$$

We see that in proposition 1, the sequence $\{\int_X f_n^+ \, d\mu\}_{n=1}^{\infty}$ need not be bounded.

Example 2. In proposition 1 the assumption $\mu(X) < +\infty$ cannot be omitted even if we require $\{f_n\}_{n=1}^{\infty}$ to be convergent and $\{\int_M f_n \, d\mu\}_{n=1}^{\infty}$ to be bounded from above whenever $M \in \mathbf{S}$, $\mu(M) < +\infty$. To see this denote by X, $\mathbf{S}$ the set of all finite real numbers and the system of all Lebesgue measurable subsets of X respectively. Further define

$$f_n(x) = 1 \text{ for } -n < x < n \, , \quad f_n(x) = 0 \text{ for } n \leqq |x| \, .$$

Then, clearly, $\int_M f_n \, d\mu \leqq \mu(M) \, (n = 1, 2, \ldots)$ for every $M \in \mathbf{S}$ and $\int_X \liminf_{n \to \infty} f_n^+ \, d\mu = = +\infty$.

On the other hand, the following theorem is true.

Theorem 1. *Let* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of* μ*-integrable functions on* X *and suppose that the sequence* $\{\int_M f_n \, \mathrm{d}\mu\}_{n=1}^{\infty}$ *is bounded from above whenever* $M \in \mathbf{S}$. *Then* $\liminf_{n\to\infty} f_n^+$ *is* μ*-integrable on* X.

Proof. Put $f = \liminf_{n\to\infty} f_n$ and suppose that

$$\int_X f^+ \, \mathrm{d}\mu = +\infty \, . \tag{13}$$

Let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of sets $Y_n \in \mathbf{S}$, $\mu(Y_n) < +\infty$ $(n = 1, 2, \ldots)$ such that $Y_1 \subset Y_2 \subset \ldots$, $\bigcup_{n=1}^{\infty} Y_n = X$ and put $Z_n = Y_{n+1} - Y_n$ $(n = 1, 2, \ldots)$. Noting that, by proposition 1, f^+ is μ-integrable on every Y_n and that $\lim_{n\to\infty} \int_{Y_n} f^+ \, \mathrm{d}\mu = +\infty$ (compare (13)), clearly we may suppose that

$$\int_{Z_n} f^+ \, \mathrm{d}\mu > n \, , \quad n = 1, 2, \ldots \tag{14}$$

(this can always be achieved by passing to a subsequence of $\{Y_n\}_{n=1}^{\infty}$ if necessary). We shall prove that there exist a sequence $\{M_k\}_{k=1}^{\infty}$ of disjoint sets $M_k \in \mathbf{S}$ and a sequence of positive integers $n_1 < n_2 < \ldots$ such that, for every positive integer k, the following relations (15)–(20) hold:

(15) $\{f_n\}_{n=1}^{\infty}$ is uniformly lower semiconvergent on $Q_k = M_1 \cup \ldots \cup M_k$,

$$\Big(\bigcup_{n=p}^{\infty} Z_n\Big) \cap Q_k = \emptyset \text{ for sufficiently large } p \, , \tag{16}$$

$$\inf_{x \in Q_k} f(x) > 0 \, , \tag{17}$$

$$1 \leqq i \leqq k \Rightarrow \int_{M_{k+1}} |f_{n_i}| \, \mathrm{d}\mu < 2^{-k-1} \, , \tag{18}$$

$$x \in Q_k \Rightarrow f_{n_{k+1}}(x) \geqq 0 \, , \tag{19}$$

$$\int_{M_{k+1}} f_{n_{k+1}} \, \mathrm{d}\mu > k + 1 \, . \tag{20}$$

Defining then $M = \bigcup_{k=1}^{\infty} M_k$, we obtain from (18)–(20)

$$\int_M f_{n_{k+1}} \, \mathrm{d}\mu = \int_{Q_k} f_{n_{k+1}} \, \mathrm{d}\mu + \int_{M_{k+1}} f_{n_{k+1}} \, \mathrm{d}\mu + \sum_{p>k+1} \int_{M_p} f_{n_{k+1}} \, \mathrm{d}\mu >$$
$$> 0 + k + 1 - \sum_{p>k+1} 2^{-p} > k \quad (k = 1, 2, \ldots) \, ,$$

so that $\limsup_{n\to\infty} \int_M f_n \, \mathrm{d}\mu = +\infty$. This contradicts the assumption of our theorem.

Put $M_1 = \emptyset$, $n_1 = 1$ and suppose that, to a given positive integer k, disjoint sets $M_1, \ldots, M_k \in \mathbf{S}$ and integers $n_1 < \ldots < n_k$ have been assigned such that (15)–(17) hold. We shall show that a set $M_{k+1} \in \mathbf{S}$, $M_{k+1} \subset X - Q_k$ can be chosen such that (15)–(17) remain valid with k replaced by $k + 1$, and that (18)–(20) are true for suitable $n_{k+1} > n_k$. Fix an integer $p > k + 1$ with $(\bigcup_{n=p}^{\infty} Z_n) \cap Q_k = \emptyset$ (compare (16)). Since $\sum_{i=1}^{k} |f_{n_i}|$ is μ-integrable on X and $Z_m \cap Z_n = \emptyset$ for $m \neq n$, we can take p large enough to secure

$$(21) \qquad \sum_{i=1}^{k} \int_{Z_p} |f_{n_i}| \,\mathrm{d}\mu < 2^{-k-1} .$$

Write $U_n = \{x; x \in Z_p, f(x) > 1/n\}$. Clearly,

$$(22) \qquad U_n \cap Q_k = \emptyset$$

for every positive integer n. Since $U_1 \subset U_2 \subset \ldots$, $\bigcup_{n=1}^{\infty} U_n = \{x; x \in Z_p, f(x) > 0\}$, there is a positive integer r with

$$(23) \qquad n > r \Rightarrow \int_{U_n} f^+ \,\mathrm{d}\mu > k + 1$$

(cf. (14)). Fix now an integer $m > r$. Lemma 2 yields a $\delta > 0$ with

$$(24) \qquad (T \in \mathbf{S},\ \mu(T) < \delta) \Rightarrow \int_{U_m - T} f^+ \,\mathrm{d}\mu > k + 1 .$$

Applying lemma 1 we obtain a set $Z \in \mathbf{S}$, $\mu(Z) < \delta$ such that $\{f_n\}_{n=1}^{\infty}$ is uniformly lower semiconvergent on $U_m - Z = M_{k+1}$. By (22), M_{k+1} is disjoint with Q_k. According to (15), $\{f_n\}_{n=1}^{\infty}$ is also uniformly lower semiconvergent on $Q_k \cup M_{k+1} = Q_{k+1}$. From $M_{k+1} \subset U_m$ and from (17) it follows that

$$\inf_{x \in Q_{k+1}} f(x) > 0 .$$

Hence we obtain for sufficiently large s

$$(25) \qquad (n > s,\ x \in Q_{k+1}) \Rightarrow f_n(x) > 0 .$$

Using Fatou's lemma we obtain on account of (24) that

$$\liminf_{n \to \infty} \int_{M_{k+1}} f_n \,\mathrm{d}\mu \geqq \int_{M_{k+1}} f \,\mathrm{d}\mu = \int_{M_{k+1}} f^+ \,\mathrm{d}\mu > k + 1 ,$$

so that

$$(26) \qquad n > t \Rightarrow \int_{M_{k+1}} f_n \,\mathrm{d}\mu > k + 1$$

for suitable t. Fixing now an integer $n_{k+1} > \max(n_k, s, t)$ we see that (19), (20) are true. The inclusion $M_{k+1} \subset Z_p$ together with (21) yields (18). Since $M_{k+1} \cap (\bigcup_{n=p+1}^{\infty} Z_n) = \emptyset$, we see that (16) holds with k replaced by $k+1$. Since the same is known about (15), (17), the proof is complete.

Lemma 3. *Let $\delta > 0$, $\mu(X) > \frac{1}{2}\delta$ and suppose that $\mu(A) \leqq \delta$ for every μ-atom $A \in \mathbf{S}$.*[3]) *Then there exists a $B \in \mathbf{S}$ such that $\frac{1}{2}\delta < \mu(B) \leqq \delta$.*

Proof. Put $\sigma = \sup\{\mu(C);\ C \in \mathbf{S},\ \mu(C) \leqq \delta\}$ and suppose, if possible, that $\sigma \leqq \frac{1}{2}\delta$. Then there exist $C_n \in \mathbf{S}$ with $\sigma - 1/n < \mu(C_n) \leqq \sigma$ $(n = 1, 2, \ldots)$. Note that $\mu(B_j) \leqq \sigma$ $(j = 1, 2, \ldots)$ imply $\mu(B_1 \cup B_2) \leqq \delta$ and, consequently, $\mu(B_1 \cup B_2) \leqq \sigma$. Hence it follows easily that $\mu(\bigcup_{k=1}^{n} C_k) \leqq \sigma$ for every n; thus for $C = \bigcup_{k=1}^{\infty} C_k$ we have that $\mu(C) = \sigma$. Let $\mathfrak{B}$ be the system of all $B \in \mathbf{S}$ with $B \cap C = \emptyset$, $\mu(B) > 0$. Clearly $X - C \in \mathfrak{B}$. Put $\iota = \inf\{\mu(B);\ B \in \mathfrak{B}\}$. Observe that

$$(D \in \mathbf{S},\ \mu(D) > \sigma) \Rightarrow \mu(D) > \delta\,.$$

Hence we conclude that $\mu(B) > \delta$ for every $B \in \mathfrak{B}$; indeed, $\mu(B \cup C) > \sigma$ and, consequently,

$$\mu(B) + \mu(C) = \mu(B \cup C) > \delta\,, \quad \mu(B) > \delta - \sigma \geqq \sigma\,, \quad \mu(B) > \delta\,.$$

We see that $\iota \geqq \delta$ and that $\mathfrak{B}$ does not contain any μ-atom. If $\iota = \infty$ then $X - C$ would be a μ-atom. Thus $\iota < \infty$, and we can fix a $B \in \mathfrak{B}$ with $\mu(B) < \iota + \delta$. Since B is not a μ-atom, there are $B_i \in \mathbf{S}$ $(i = 1, 2)$ with $\mu(B_i) > 0$, $B_1 \cap B_2 = \emptyset$, $B_1 \cup B_2 = B$. Clearly, $B_i \in \mathfrak{B}$ $(i = 1, 2)$ and, consequently, $\mu(B) \geqq 2\iota \geqq \iota + \delta$, which is a contradiction. We have thus shown that $\sigma > \frac{1}{2}\delta$. Our lemma follows easily.

Lemma 4. *Let $\mu(X) < +\infty$, $\delta > 0$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of μ-integrable functions on X such that $\sup_n \int_Y |f_n| \,\mathrm{d}\mu < \infty$ for every $Y \in \mathbf{S}$ with $\mu(Y) \leqq \delta$ and such that $\sup_n |\int_M f_n \,\mathrm{d}\mu| < \infty$ whenever $M \in \mathbf{S}$. Then $\sup_n \int_X |f_n| \,\mathrm{d}\mu < \infty$.*

Proof. Let us express X in the form $X = A_1 \cup \ldots \cup A_p \cup \hat{X}$, where $A_1, \ldots, A_p$, $\hat{X}$ are disjoint elements of $\mathbf{S}$, $A_1, \ldots, A_p$ are μ-atoms and $\mu(A) \leqq \delta$ for every μ-atom $A \subset \hat{X}$. It follows easily from lemma 3 that $\hat{X}$ can be expressed in the form $\hat{X} = Y_1 \cup \ldots \cup Y_m$, where the Y_i are disjoint elements of $\mathbf{S}$, $\mu(Y_i) \leqq \delta$ $(i = 1, \ldots, m)$. (Cf. also [4], th. 3.9, p. 220.) Consequently,

$$\sup_n \int_{\hat{X}} |f_n| \,\mathrm{d}\mu \leqq \sum_{i=1}^{m} \sup_n \int_{Y_i} |f_n| \,\mathrm{d}\mu < \infty\,.$$

[3]) A set $A \in \mathbf{S}$ is called a μ-atom provided $\mu(A) > 0$ and $\mu(M) = 0$ for every $M \in \mathbf{S}$ with $M \subset A$, $\mu(M) < \mu(A)$.

Noting that

$$\int_{A_k} |f_n| \, d\mu = \left| \int_{A_k} f_n \, d\mu \right| \quad (1 \leqq k \leqq p)$$

we conclude that

$$\sup_n \int_X |f_n| \, d\mu \leqq \sup_n \int_{\hat{X}} |f_n| \, d\mu + \sum_{k=1}^{p} \sup_n \left| \int_{A_k} f_n \, d\mu \right| < \infty .$$

In connection with example 1 it is interesting to observe that the following proposition holds. (Prop. 2 and th. 2 follow also from th. 10.8 in [4], p. 275.)

Proposition 2. *Let* $\mu(X) < \infty$. *Let* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of* μ*-integrable functions on* X, *and suppose that* $\sup_n | \int_M f_n \, d\mu| < \infty$ *whenever* $M \in \mathbf{S}$. *Then* $\sup_n \int_X |f_n| \, d\mu < \infty$.

Proof. Assuming that

$$\limsup_{n \to \infty} \int_X |f_n| \, d\mu = +\infty , \tag{27}$$

we shall construct a sequence $M_1, M_2, \ldots$ of mutually disjoint sets $M_k \in \mathbf{S}$ and an increasing sequence $n_1, n_2, \ldots$ of positive integers such that for every k the following relations hold:

$$\sup_n \int_{N_k} |f_n| \, d\mu = +\infty , \quad \text{where} \quad N_k = X - \bigcup_{j=1}^{k} M_j , \tag{28}$$

$$\left| \int_{M_{k+1}} f_{n_{k+1}} \, d\mu \right| > k + 1 + \left| \int_{M_1 \cup \ldots \cup M_k} f_{n_{k+1}} \, d\mu \right| , \tag{29}$$

$$\max_{1 \leqq i \leqq k} \int_{M_{k+1}} |f_{n_i}| \, d\mu < 2^{-k-1} . \tag{30}$$

From (29), (30) we obtain for $M = \bigcup_{k=1}^{\infty} M_k$ that

$$\left| \int_M f_{n_{k+1}} \, d\mu \right| \geqq - \left| \int_{M_1 \cup \ldots \cup M_k} f_{n_{k+1}} \, d\mu \right| + \left| \int_{M_{k+1}} f_{n_{k+1}} \, d\mu \right| - \sum_{p > k+1} \left| \int_{M_p} f_{n_{k+1}} \, d\mu \right| > k .$$

This contradicts the assumption of our proposition.

Put $M_1 = \emptyset$, $n_1 = 1$. Suppose that to a given k, integers $n_1 < \ldots < n_k$ and disjoint sets $M_1, \ldots, M_k \in \mathbf{S}$ have been assigned fulfilling (28). We shall prove that there exist a $n_{k+1} > n_k$ and a $M_{k+1} \subset N_k$, $M_{k+1} \in \mathbf{S}$, such that (29), (30) are true and such that (28) remains valid with k replaced by $k + 1$. For the purpose of proving this we fix a $\delta > 0$ such that

$$(M \in \mathbf{S}, \ \mu(M) \leqq \delta) \Rightarrow \sum_{i=1}^{k} \int_M |f_{n_i}| \, d\mu < 2^{-k-1} . \tag{31}$$

According to (28) we conclude by lemma 4 that there exists an $Y \in \mathbf{S}$ with

$$(32) \qquad Y \subset N_k\,, \quad \mu(Y) \leqq \delta\,, \quad \sup_n \int_Y |f_n| \, \mathrm{d}\mu = \infty\,.$$

Put

$$(33) \qquad \sup_n \left| \int_{M_1 \cup \ldots \cup M_k} f_n \, \mathrm{d}\mu \right| = c\,.$$

Since $c < \infty$, we have by (32) an $n_{k+1} > n_k$ with $\int_Y |f_{n_{k+1}}| \, \mathrm{d}\mu > 2(k + 1 + c)$. Now fix a $\delta_1 > 0$ such that

$$(34) \qquad (T \in \mathbf{S},\ \mu(T) \leqq \delta_1) \Rightarrow \int_{Y-T} |f_{n_{k+1}}| \, \mathrm{d}\mu > 2(k + 1 + c)\,.$$

By (32) and lemma 4 there exists an $\hat{Y} \in \mathbf{S}$ with

$$(35) \qquad \hat{Y} \subset Y\,, \quad \mu(\hat{Y}) \leqq \delta_1\,, \quad \sup_n \int_{\hat{Y}} |f_n| \, \mathrm{d}\mu = \infty\,.$$

From (35), (34) we conclude that

$$\int_{Y-\hat{Y}} |f_{n_{k+1}}| \, \mathrm{d}\mu > 2(k + 1 + c)\,.$$

Let us now define

$$M_{k+1} = \{x;\ x \in Y - \hat{Y}, f_{n_{k+1}}(x) > 0\}$$

or

$$M_{k+1} = \{x;\ x \in Y - \hat{Y}, f_{n_{k+1}}(x) < 0\}$$

according as

$$\int_{Y-\hat{Y}} f^+_{n_{k+1}} \, \mathrm{d}\mu > \int_{Y-\hat{Y}} f^-_{n_{k+1}} \, \mathrm{d}\mu \quad \text{or} \quad \int_{Y-\hat{Y}} f^+_{n_{k+1}} \, \mathrm{d}\mu \leqq \int_{Y-\hat{Y}} f^-_{n_{k+1}} \, \mathrm{d}\mu\,.$$

We have then clearly

$$\left| \int_{M_{k+1}} f_{n_{k+1}} \, \mathrm{d}\mu \right| > k + 1 + c\,,$$

so that (29) is valid (see (33)). Noting that $M_{k+1} \subset Y$ and $\mu(Y) \leqq \delta$ (compare (32)) we conclude on account of (31) that (30) holds. We have $\hat{Y} \subset Y \subset N_k$, $M_{k+1} \subset Y - \hat{Y}$, so that $\hat{Y} \subset N_k - M_{k+1} = N_{k+1}$. This together with (35) secures that (28) remains valid with k replaced by $k + 1$. The proof is complete.

Theorem 2. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of μ-integrable functions on X and suppose that the sequence $\{\int_M f_n \, \mathrm{d}\mu\}_{n=1}^{\infty}$ is bounded whenever $M \in \mathbf{S}$. Then* $\sup_n \int_X |f_n| \mathrm{d}\mu < +\infty$ *and, consequently,* $\liminf_{n\to\infty} |f_n|$ *is μ-integrable on X.*

Proof. Suppose, if possible, that

$$\limsup_{n\to\infty} \int_X |f_n| \, d\mu = +\infty . \tag{36}$$

Let $\{X_i\}_{i=1}^{\infty}$ be a non-decreasing sequence of subsets of X such that

$$\bigcup_{i=1}^{\infty} X_i = X , \quad X_i \in \mathbf{S} , \quad \mu(X_i) < +\infty \quad (i = 1, 2, \ldots) .$$

We shall define two increasing sequences $\{n_k\}_{k=1}^{\infty}$, $\{m_k\}_{k=1}^{\infty}$ of positive integers as follows. Put $n_1 = 1 = m_1$. If integers $n_1 < \ldots < n_k$, $m_1 < \ldots < m_k$ have already been constructed, we first choose an $n_{k+1} > n_k$ with

$$\int_X |f_{n_{k+1}}| \, d\mu > 3c_k + 2k , \quad \text{where} \quad c_k = \sup_n \int_{X_{m_k}} |f_n| \, d\mu \tag{37}$$

(note that, by proposition 2, $0 \leqq c_k < +\infty$). The functions f_n $(n = 1, 2, \ldots)$ being μ-integrable on X, we have

$$\lim_{i\to\infty} \int_{X_i} |f_n| \, d\mu = \int_X |f_n| \, d\mu \quad \text{for every } n .$$

This makes it possible to determine an $m_{k+1} > m_k$ large enough to secure

$$\sum_{i=1}^{k+1} \int_{X - X_{m_{k+1}}} |f_{n_i}| \, d\mu < 2^{-k-1} , \tag{38}$$

$$\int_{X_{m_{k+1}}} |f_{n_{k+1}}| \, d\mu > 3c_k + 2k \tag{39}$$

(cf. (37)). The sequences $\{n_k\}_{k=1}^{\infty}$, $\{m_k\}_{k=1}^{\infty}$ having been defined, we put $Z_k = X_{m_{k+1}} - X_{m_k}$ $(k = 1, 2, \ldots)$. We have then for every k (cf. (37), (39))

$$\int_{Z_k} |f_{n_{k+1}}| \, d\mu = \int_{X_{m_{k+1}}} |f_{n_{k+1}}| \, d\mu - \int_{X_{m_k}} |f_{n_{k+1}}| \, d\mu > 3c_k + 2k - c_k = 2(c_k + k) .$$

Let us now define $M_k = \{x; x \in Z_k, f_{n_{k+1}}(x) > 0\}$ or $M_k = \{x; x \in Z_k, f_{n_{k+1}}(x) < 0\}$ according as

$$\int_{Z_k} f^+_{n_{k+1}} \, d\mu > \int_{Z_k} f^-_{n_{k+1}} \, d\mu \quad \text{or} \quad \int_{Z_k} f^+_{n_{k+1}} \, d\mu \leqq \int_{Z_k} f^-_{n_{k+1}} \, d\mu .$$

Then, clearly,

$$\left| \int_{M_k} f_{n_{k+1}} \, d\mu \right| > c_k + k \quad (k = 1, 2, \ldots) .$$

Writing $L_k = \bigcup_{i=1}^{k-1} M_i$, we have $L_k \subset \bigcup_{i=1}^{k-1} Z_i \subset X_{m_k}$, so that $c_k \geqq \int_{L_k} |f_{n_{k+1}}| \, d\mu$. Hence

$$\left| \int_{M_k} f_{n_{k+1}} \, d\mu \right| > k + \int_{L_k} |f_{n_{k+1}}| \, d\mu . \tag{40}$$

Since $M_{k+1} \subset X - X_{m_{k+1}}$, we conclude from (38) that

$$1 \leqq i \leqq k + 1 \Rightarrow \int_{M_{k+1}} |f_{n_i}| \, d\mu < 2^{-k-1} . \tag{41}$$

Defining $M = \bigcup_{k=1}^{\infty} M_k$, we obtain on account of (40), (41)

$$\left| \int_M f_{n_{k+1}} \, d\mu \right| \geqq - \int_{L_k} |f_{n_{k+1}}| \, d\mu + \left| \int_{M_k} f_{n_{k+1}} \, d\mu \right| - \sum_{p=k+1}^{\infty} \int_{M_p} |f_{n_{k+1}}| \, d\mu > k - 1 ,$$

so that $\sup_n |\int_M f_n \, d\mu| = +\infty$.

Thus (36) is impossible and the proof is complete.

By means of lemmas 3, 4, the theorems 1, 2 can be generalized as follows:

Theorem 1*. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of μ-integrable functions on X. Suppose that there exist a set $Y \in \mathbf{S}$ and a number $\eta > 0$ such that $\mu(Y) < \infty$, $\mu(A) < \eta$ for every μ-atom $A \subset Y$ and such that the sequence $\{\int_M f_n \, d\mu\}_{n=1}^{\infty}$ is bounded from above whenever $M \in \mathbf{S}$, $\mu(M \cap Y) < \eta$. Then $\liminf_{n \to \infty} f_n^+$ is μ-integrable on X (the sequence $\{\int_X f_n^+ \, d\mu\}_{n=1}^{\infty}$, however, need not be bounded).*

Theorem 2*. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of μ-integrable functions on X, $Y \in \mathbf{S}$, $\eta > 0$. Suppose that $\mu(Y) < \infty$ and $\mu(A) < \eta$ for every μ-atom $A \subset Y$. If*

$$\sup_n \left| \int_M f_n \, d\mu \right| < \infty$$

for every $M \in \mathbf{S}$ with $\mu(M \cap Y) < \eta$, then

$$\sup_n \int_X |f_n| \, d\mu < \infty .$$

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Резюме

ЗАМЕТКА О ПОСЛЕДОВАТЕЛЬНОСТЯХ ИНТЕГРИРУЕМЫХ ФУНКЦИЙ

ЙИРЖИ ЕЛИНЕК (Jiří Jelínek) и ЙОСЕФ КРАЛ (Josef Král), Прага

Пусть $(X, \mathbf{S}, \mu)$ — пространство с вполне σ-конечной мерой и пусть $\{f_n\}_{n=1}^{\infty}$ — — последовательность интегрируемых функций на X.

Теорема. *Предположим, что существует $\eta > 0$ и $Y \in \mathbf{S}$ так, что $\mu(A) \leqq \eta$ для каждого μ-атома $A \subset Y$ и последовательность $\{\int_M f_n \,\mathrm{d}\mu\}_{n=1}^{\infty}$ ограничена сверху для каждого множества $M \in \mathbf{S}$, удовлетворяющего условию $\mu(M \cap Y) \leqq \eta$. Тогда функция $\liminf_{n\to\infty} f_n^+$ интегрируема на X.*

Следует подчеркнуть, что в условиях предшествующей теоремы последовательность $\{\int_X f_n^+ \,\mathrm{d}\mu\}_{n=1}^{\infty}$ может и не быть ограниченной (даже если $X = Y$, $\mu(X) < \eta < +\infty$); следовательно, утверждение предшествующей теоремы не может быть получено на основе известной леммы Фату. В связи с этим интересно отметить, что имеет место следующая

Теорема. *Пусть $Y \in \mathbf{S}$, $\eta > 0$. Предположим, что $\mu(A) \leqq \eta$ для каждого μ-атома $A \subset Y$ и последовательность $\{\int_M f_n \,\mathrm{d}\mu\}_{n=1}^{\infty}$ ограничена для каждого множества $M \in \mathbf{S}$, удовлетворяющего требованию $\mu(M \cap Y) \leqq \eta$. Тогда последовательность $\{\int_X |f_n| \,\mathrm{d}\mu\}_{n=1}^{\infty}$ ограничена и, подавно, функция $\liminf_{n\to\infty} |f_n|$ интегрируема на X.*

Эта последняя теорема вытекает тоже из теоремы 10.8, доказанной другим методом в [4], стр. 275—277.

Доказательства теорем в предлагаемой статье основаны на методе „скользящего горба“.