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ITERATIONS OF LINEAR BOUNDED OPERATORS IN NON SELF-ADJOINT EIGENVALUE PROBLEMS AND KELLOGG'S ITERATION PROCESS

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The convergence of an iteration process which is a generalization of the Kellogg iteration process for linear bounded operators in Banach spaces is proved. A general formula is given for approximate calculations of the dominant eigenvalue and its corresponding eigenvector. Many commonly used iteration formulas are special cases of this general formula. The results are applied to the case of equations of the type $Lx = \lambda Bx$, where L, B are linear in general unbounded operators whose domains $\mathscr{D}(L)$, $\mathscr{D}(B)$ satisfy $\mathscr{D}(L) \subset \subset \mathscr{D}(B)$.

1. Introduction. The purpose of this paper is to prove the convergence of Kellogg's iterations and of similar processes. The method of these proofs is based on an application of the operator calculus for linear operators in Banach spaces, developed by A. E. TAYLOR [9]. The advantage of this method is that, in contrast with [2], [3], [7], [8], [13], [14], [16], [17] et al., one can drop the condition that the investigated operator must be symmetrizable.

The definitions and necessary designations are given in the second chapter, basic auxiliary assertions on iterations of linear bounded operators (lemma 1, 2, theorem 1) in the third chapter and assertions on the convergence of iterations of the Kellogg type (theorems 2–4) are given in the fourth chapter. If the order q of the dominant eigenvalue μ_0 is known, one can use theorem 5 to determine μ_0. In the fifth and sixth chapter the results of the fourth are applied to the case of equations of type $Lx = \lambda Bx$ with a generally unbounded operator L and a bounded operator B. In the sixth chapter both the operators L, B may be unbounded.

The listed iteration processes are rather general, since they contain or generalize most of the iteration processes in specific Banach and Hilbert spaces.

2. Definitions and designations. Let $\mathbf{X}$ be a complex Banach space and $\mathbf{X}'$ its adjoint space of continuous linear forms. We will denote elements of the space $\mathbf{X}$ by small Roman characters, elements of the space $\mathbf{X}'$ by the same characters with the primes. The symbol o will mean the zero – vector in the space $\mathbf{X}$. Let T be a linear bounded operator which maps $\mathbf{X}$ into itself. The set of such operators forms a Banach space which we will denote by $\mathbf{X}_1$ or $(\mathbf{X} \to \mathbf{X})$. We will distinguish norms in the spaces $\mathbf{X}$, $\mathbf{X}'$, $\mathbf{X}_1$ by the corresponding index next to the norm sign, *i.e.* for $x \in \mathbf{X}$, $x' \in \mathbf{X}'$: $\|x\|_{\mathbf{X}}$, $\|x'\|_{\mathbf{X}'}$, and for $T \in \mathbf{X}_1$:

$$\|T\|_{\mathbf{X}_1} = \sup_{\|x\|_{\mathbf{X}} \leqq 1} \|Tx\|_{\mathbf{X}} . \tag{1}$$

We will drop the indices in cases when no misunderstanding can arise. We denote the zero and identity operators by the symbols O and I. Let Π be the open complex plane. We denote the spectrum of the operator T by the symbol $\sigma(T)$.

Let $T \in \mathbf{X}_1$ and let $R(\lambda, T) = (\lambda I - T)^{-1}$ be the resolvent of the operator T in the point $\lambda \in \Pi$. Let Γ be an open set in the complex plane Π. Let the boundary of the set Γ be the disjoint union of a finite number of rectifiable Jordan curves and let $\Gamma \supset \sigma(T)$. Then we have for every polynomial f [9],

$$f(T) = \frac{1}{2\pi i} \int_C f(\lambda)\, R(\lambda, T)\, \mathrm{d}\lambda , \tag{2}$$

where C is the boundary of the set Γ, oriented in the evident way.

We will say that the operator T has the property R_q in the point $\mu_0 \in \sigma(T)$ if μ_0 is an isolated pole of order q of the resolvent $R(\lambda, T)$.

If the operator T has the property R_q in the point μ_0, the resolvent $R(\lambda, T)$ can be developed into a Laurent series [9],

$$R(\lambda, T) = \sum_{k=0}^{\infty} (\lambda - \mu_0)^k\, T_k + \sum_{k=1}^{q} (\lambda - \mu_0)^{-k}\, B_k , \tag{3}$$

where

$$B_1 = \frac{1}{2\pi i} \int_{C_0} R(\lambda, T)\, \mathrm{d}\lambda , \quad B_{k+1} = (T - \mu_0 I)\, B_k , \quad k = 1, \ldots, q - 1 \tag{4}$$

and C_0 is a positively oriented circle with center at μ_0, and such that no points of the spectrum $\sigma(T)$ except μ_0 lie on or inside the circle C_0.

If f is a polynomial and the operator T has the property R_q in the point μ_0, we put

$$H[\mu_0, T, f(\lambda)] = \frac{1}{2\pi i} \int_{C_0} f(\lambda)\, R(\lambda, T)\, \mathrm{d}\lambda = \sum_{k=1}^{q} \frac{f^{(k-1)}(\mu_0)}{(k-1)!}\, B_k \tag{5}$$

where C_0 has the mentioned meaning.

Specifically we define

$$H_m[\mu_0, T] = H\left[\mu_0 , T, \left(\frac{\lambda}{\mu_0}\right)^m\right], \quad m \geqq 0 .$$

Further we will call the point $\mu_0 \in \sigma(T)$ the dominant point of the spectrum of the operator T if

$$|\lambda| < |\mu_0| \tag{6}$$

holds for every point $\lambda \in \sigma(T)$, $\lambda \neq \mu_0$.

3. Iterations of linear bounded operators. Let the following assumptions be satisfied in all the statements of this chapter if nothing else is asserted:

Assumptions. (a) *Operator T is a linear bounded operator mapping the space $\mathbf{X}$ into itself.*

(b) *The value μ_0 is a pole of order q of the resolvent $R(\lambda, T)$.*

(c) *The value μ_0 is the dominant point of the spectrum of the operator T.*

Let us prove several auxiliary statements which are important for further considerations.

Lemma 1. *In the norm of the space $\mathbf{X}_1$ we have*

$$\lim_{m \to \infty} m^{-q+1} H_m[\mu_0, T] = \frac{\mu_0^{-q+1}}{(q-1)} B_q . \tag{7}$$

Proof. According to definition (5) we have

$$H_m[\mu_0, T] = B_1 + \sum_{k=2}^{q} m(m-1)\dots(m-k+2) \frac{\mu_0^{-k+1}}{(k-1)!} B_k ,$$

and since

$$\frac{m(m-1)\dots(m-k+2)}{m^{k-1}} = 1 + O\left(\frac{1}{m}\right)$$

holds for $k \geqq 2$, $k \leqq m+1$, we obtain

$$\left\| m^{-q+1} H_m[\mu_0, T] - \frac{\mu_0^{-q+1}}{(q-1)!} B_q \right\| \leqq O\left(\frac{1}{m}\right).$$

Lemma 2. *There exists a constant $c_1 > 0$ independent of m such that for all positive integral m we have*

$$\|H_m[\mu_0, T] - \mu_0^{-m} T^m\| \leqq c_1 \left|\frac{\mu}{\mu_0}\right|^m , \tag{8}$$

where μ is the radius of the smallest circle with center in the origin which contains the whole spectrum $\sigma(T)$ except μ_0.

Proof. Let us choose a number μ_1 so that $\mu < \mu_1 < |\mu_0|$. Then we have

$$\mu_0^{-m} T^m = H_m[\mu_0, T] + \frac{1}{2\pi i} \int_{C_1} \left(\frac{\lambda}{\mu_0}\right)^m R(\lambda, T) \, d\lambda$$

where C_1 is a positively oriented circle with center in the origin and radius μ_1. On this circle $\|R(\lambda, T)\|$ is bounded function of the argument λ, so that

$$\sup_{\lambda \in C_1} \|R(\lambda, T)\| = c_2 < +\infty .$$

Thus

$$\|\mu_0^{-m} T^m - H_m[\mu_0, T]\| \leqq \left\| \frac{1}{2\pi i} \int_{C_1} \left(\frac{\lambda}{\mu_0}\right)^m R(\lambda, T) \, d\lambda \right\| \leqq c_2 \mu_1 \left|\frac{\mu_1}{\mu_0}\right|^m = c_1 \left|\frac{\mu_1}{\mu_0}\right|^m$$

and from here our assertion follows directly.

The following theorem is a consequence of the preceding lemmas.

Theorem 1. *In the norm of the space* $\mathbf{X}_1$ *we have*

(9)
$$\lim_{m\to\infty} m^{-q+1}\mu_0^{-m}T^m = \frac{\mu_0^{-q+1}}{(q-1)!} B_q .$$

4. Iteration processes and eigenvalue problems. We will prove the convergence of the Kellogg iteration process for the class of all bounded linear operators satisfying the assumptions (a)–(c) of the third chapter. With the help of these results we will prove the convergence of other iteration processes.

Let $\{x'_m\}$, $\{y'_m\}$, $\{z'_m\}$ be sequences of linear forms mapping $\mathbf{X}$ into Π. Let there exist forms $x' \in \mathbf{X}'$, $y' \in \mathbf{X}'$ such that

(10)
$$x'(x) = \lim_{m\to\infty} x'_m(x)\,, \quad y'(x) = \lim_{m\to\infty} y'_m(x) = \lim_{m\to\infty} z'_m(x)$$

for every vector $x \in \mathbf{X}$.

Let $x^{(0)} \in \mathbf{X}$ be a definite fixed vector for which $B_1x^{(0)} \neq o$, so that there is an index s $(1 \leqq s \leqq q)$ such that

(11)
$$B_sx^{(0)} \neq o\,, \quad B_{s+1}x^{(0)} = o$$

where $B_{q+1} = O$.

Further let

(12)
$$x'(B_sx^{(0)}) \neq o\,, \quad y'(B_sx^{(0)}) \neq 0$$

hold and let us put

(13)
$$x_0 = \frac{B_sx^{(0)}}{x'(B_sx^{(0)})}\,.$$

Kellogg's iterations are constructed according to the following formulas

(14)
$$x^{(m)} = Tx^{(m-1)}\,, \quad x_{(m)} = \frac{x^{(m)}}{x'_m(x^{(m)})}\,,$$

(15)
$$\mu_{(m)} = \frac{z'_m(x^{(m+1)})}{y'_m(x^{(m)})}\,.$$

Lemma 3. *Let* (11) *hold for the vector* $x^{(0)}$. *Then we have, in the norm of the space* $\mathbf{X}$

$$\lim_{m\to\infty} m^{-s+1}\mu_0^{-m}T^mx^{(0)} = \frac{\mu_0^{-s+1}}{(s-1)!} B_sx^{(0)}\,,$$

and the estimate

$$\left\| m^{-s+1}\mu_0^{-m}T^mx^{(0)} - \frac{\mu_0^{-s+1}}{(s-1)!} B_sx^{(0)} \right\| \leqq c_3\, \|x^{(0)}\|\, m^{-1}\,,$$

where c_3 *does not depend on* m.

The proof of this lemma is similar to that of theorem 1 and will therefore be omitted.

Lemma 4. *If* $v_m \in \mathbf{X}$, $v \in \mathbf{X}$; $v'_m \in \mathbf{X}'$, $v' \in \mathbf{X}'$ and if $v'_m(x) \to v'(x)$ for every $x \in \mathbf{X}$, then $v'_m(v_m) \to v'(v)$.

Proof. The assertion follows directly from the Banach theorem ([12] p. 204).

Theorem 2. *Let* (10) *hold for the forms* x'_m, y'_m, z'_m, x', y'. *Let* $x^{(0)} \in \mathbf{X}$ *be a vector such that* (11) *and* (12) *hold. Then*

$$\lim_{m\to\infty} x_{(m)} = x_0 \tag{16}$$

holds for the sequence (14) *in the norm of the space* $\mathbf{X}$ *and*

$$\lim_{m\to\infty} \mu_{(m)} = \mu_0 \tag{17}$$

holds for the numerical sequence (15). *The vector* x_0 *is the eigenvector of the operator* T *corresponding to the value* μ_0.

Proof. According to lemma 3 there exist numbers α_m such that

$$\lim_{m\to\infty} \alpha_m x^{(m)} = B_s x^{(0)} .$$

According to lemma 4 we have

$$x_{(m)} = \frac{\alpha_m x^{(m)}}{x'_m(\alpha_m x^{(m)})} \to x_0 ,$$

$$\mu_{(m)} = \frac{z'_m(\alpha_m T x^{(m)})}{y'_m(\alpha_m x^{(m)})} \to \frac{y'(TB_s x^{(0)})}{y'(B_s x^{(0)})} = \frac{y'(Tx_0)}{y'(x_0)} .$$

It follows from (11) that

$$(T - \mu_0 I)\, x_0 = \frac{1}{x'(B_s x^{(0)})}(T - \mu_0 I)\, B_s x^{(0)} = \frac{1}{x'(B_s x^{(0)})} B_{s+1} x^{(0)} = o$$

so that the vector x_0 is the eigenvector of the operator corresponding to the eigenvalue μ_0. Thus the theorem is proved.

The rate with which the sequence $\{\mu_{(m)}\}$ converges to μ_0 depends on the rate with which the sequences of forms $\{y'_m\}$ and $\{z'_m\}$ converge to y' and on the rate with which the sequence $\{x_{(m)}\}$ converges to the vector x_0.

Remark 1. *Let the operator* T *have the property* R_1 *in the point* μ_0. *Then* $H_m[\mu_0, T] = B_1$ *and, according to lemma* 2,

$$\|\mu_0^{-m} T^m - B_1\| \leqq c_1 \left|\frac{\mu}{\mu_0}\right|^m$$

and hence

$$\|\mu_0^{-m} x^{(m)} - B_1 x^{(0)}\| \leqq c_1 \|x^{(0)}\| \left|\frac{\mu}{\mu_0}\right|^m .$$

Let $\delta > 0$. *Suppose that there exists for every* $x \in X$ *a* $c_4(x)$ *such that*

$$|x'_m(x) - x'(x)| + |y'_m(x) - y'(x)| + \\ + |z'_m(x) - y'(x)| \leqq c_4(x)\, m^{-1-\delta} \tag{18}$$

holds for all $x \in \mathbf{X}$. According to the Banach theorem we have

$$|x'_m(\mu_0^{-m}x^{(m)}) - x'(B_1x^{(0)})| + |y'_m(\mu_0^{-m}x^{(m)}) - y'(B_1x^{(0)})| + \\ + |z'_m(\mu_0^{-m}x^{(m)}) - y'(B_1x^{(0)})| \leqq c_5(x^{(0)})\, m^{-1-\delta}$$

and from this we obtain easily the following estimates

$$(19) \qquad \|x_{(m)} - x_0\| \leqq c_6(x^{(0)})\, m^{-1-\delta}\,, \quad |\mu_{(m)} - \mu_0| \leqq c_7(x^{(0)})\, m^{-1-\delta}\,.$$

Remark 2. *Assume that the operator T has a positive dominant eigenvalue; take sequences of functionals $\{\tilde{x}_m\}$, $\{\tilde{y}_m\}$, $\{\tilde{z}_m\}$ such that $\tilde{x}_m(\lambda x) = |\lambda|\, \tilde{x}_m(x)$, $\tilde{y}_m(\lambda x) = |\lambda|\, \tilde{y}_m(x)$, $\tilde{z}_m(\lambda x) = |\lambda|\, z_m(x)$ instead of the sequences of linear forms $\{x'_m\}$, $\{y'_m\}$, $\{z'_m\}$ in formula (15) and such that there is a c_8 independent of m and with*

$$|\tilde{x}_m(x) - \tilde{x}_m(y)| + |\tilde{y}_m(x) - \tilde{y}_m(y)| + |\tilde{z}_m(x) - \tilde{z}_m(y)| \leqq c_8\|x - y\|$$

for arbitrary vectors $x \in \mathbf{X}$, $y \in \mathbf{X}$. Let there exist functionals $\tilde{x}$, $\tilde{y}$ such that

$$\tilde{x}(x) = \lim_{m\to\infty} \tilde{x}_m(x)\,, \quad \tilde{y}(x) = \lim_{m\to\infty} \tilde{y}_m(x) = \lim_{m\to\infty} \tilde{z}_m(x)$$

hold for all vectors $x \in \mathbf{X}$. Then under the assumptions of theorem 2 we obtain

$$\lim_{m\to\infty} \frac{x^{(m)}}{\tilde{x}_m(x^{(m)})} = x_0\,, \quad \lim_{m\to\infty} \frac{\tilde{z}_m(x^{(m+1)})}{\tilde{y}_m(x^{(m)})} = \mu_0\,.$$

Specifically for $\tilde{x}_m(x) = \tilde{y}_m(x) = \tilde{z}_m(x) = \|x\|$, we obtain Kellogg's classical iteration sequence $\{\|x^{(m+1)}\|/\|x^{(m)}\|\}$ and the formula

$$\lim_{m\to\infty} \frac{\|x^{(m+1)}\|}{\|x^{(m)}\|} = \mu_0\,.$$

If we choose the sequences of forms $\{x'_m\}$, $\{y'_m\}$, $\{z'_m\}$ or functionals $\{\tilde{x}_m\}$, $\{\tilde{y}_m\}$, $\{\tilde{z}_m\}$ in some specific way we obtain other well known iteration processes.

Proof. The assertion in remark 2 can be proved as follows. If $\tilde{y}_m(x) \to \tilde{y}(x)$, $\tilde{z}_m(x) \to \tilde{y}(x)$ hold for every vector $x \in \mathbf{X}$, then for $v_m \to v$, $v_m \in \mathbf{X}$, $v \in \mathbf{X}$ we have $\tilde{y}_m(v_m) \to \tilde{y}(v)$, $\tilde{z}_m(v_m) \to \tilde{y}(v)$, since

$$|\tilde{y}_m(v_m) - \tilde{y}(v)| + |\tilde{z}_m(v_m) - \tilde{y}(v)| \leqq \\ \leqq c_9\|v_m - v\| + |\tilde{y}_m(v) - \tilde{y}(v)| + |\tilde{z}_m(v) - \tilde{y}(v)|$$

and

$$\frac{x^{(m)}}{\tilde{x}_m(x^{(m)})} = v_m \to v = \frac{B_s x^{(0)}}{\tilde{x}(B_s x^{(0)})}\,.$$

The rest of the proof is the same as in the case of sequences of forms.

Corollary 1. *Let $\mathbf{X} = \mathbf{C}(0, 1)$ be the Banach space of functions, continuous on the interval $\langle 0, 1\rangle$. Let T be an integral operator defined by the kernel $t(s, s')$ continuous in the square $\langle 0, 1\rangle \times \langle 0, 1\rangle$,*

$$y(s) = \int_0^1 t(s, s')\, x(s')\, ds'\,.$$

Let μ_0 be the dominant eigenvalue of the operator T. Let $x'_m(x) = y'_m(x) = z'_m(x) = x(s_0)$ where $x \in \mathbf{C}(0, 1)$ and the point $s_0 \in \langle 0, 1\rangle$ has the property that $x_0(s_0) \neq 0$ (x_0 is defined by formula (13)). Thus we obtain the iteration process

$$x_{(m)} = \frac{x^{(m)}}{x^{(m)}(s_0)}, \quad \mu_{(m)} = \frac{x^{(m+1)}(s_0)}{x^{(m)}(s_0)} = \mu_0$$

where

$$x^{(m)}(s) = \int_0^1 t(s, s')\, x^{(m-1)}(s')\, ds' .$$

According to theorem 2 we have uniformly in $\langle 0, 1\rangle$

$$\lim_{m\to\infty} x_{(m)}(s) = x_0(s) \quad \text{and} \quad \lim_{m\to\infty} \frac{x^{(m+1)}(s_0)}{x^{(m)}(s_0)} = \mu_0 .$$

The convergence of this method was proved by H. WIELANDT [15].

In Hilbert spaces iteration methods, in which Schwarz constants [3] figure, are used to determine eigenvalues of symmetric compact operators. The iterations given in [2] and [7] are analogous to this method. It will be shown that the convergence of these processes is a consequence of the assertions proved here.

Corollary 2. *Let $\boldsymbol{X}$ be a Hilbert space with the scalar product (x, y). Let the assumptions of theorem 2 be fulfiled, where $\{x'_m\}$ is an arbitrary sequence for which equations (10) holds. Let the sequences $\{y'_m\}$, $\{z'_m\}$ be defined by one of the formulas*

(20) $$y'_m(x) = z'_m(x) = (x, x_{(m)}) ,$$

(21) $$y'_m(x) = z'_m(x) = (x, Tx_{(m)}) .$$

Then according to theorem 2 [2], [7],

$$\lim_{m\to\infty} \frac{(x^{(m+1)}, x^{(m)})}{(x^{(m)}, x^{(m)})} = \mu_0$$

holds for the case (20) and

$$\lim_{m\to\infty} \frac{(x^{(m+1)}, x^{(m+1)})}{(x^{(m)}, x^{(m+1)})} = \mu_0$$

for the case (21).

In [2] and [7] formulas are given for constructing eigenvectors besides those for constructing eigenvalues. These formulas differ from Kellogg's original formula (14). J. KOLOMÝ [7] proves the convergence of the mentioned sequence for the case of a compact symmetrizable linear operator mapping Hilbert space into itself. Both of these iteration processes can be summed up in a general one,

(22) $$y_{(m+1)} = \frac{1}{\mu_{(m)}} Ty_{(m)}, \; y_{(0)} = x^{(0)} ,$$

where $\mu_{(m)}$ is defined by (15) *i.e.*

$$\mu_{(m)} = \frac{z'_m(Ty_{(m)})}{y'_m(y_{(m)})}. \tag{23}$$

Theorem 3. *Let the operator T have the property* R_1 *in the point* μ_0. *Let the inequalities* (18) *hold for the forms* x'_m, y'_m, z'_m, x', y' *which also satisfy the conditions of theorem* 2. *Let* (11) *and* (12) *hold for the vector* $x^{(0)}$. *Let*

$$\mu_{(m)} \neq 0 \tag{24}$$

for $\mu_{(m)}$ *defined by* (23) *for all m. Then in the norm of the space* $\mathbf{X}$ *we have*

$$\lim_{m \to \infty} y_{(m)} = y_0 \tag{25}$$

where y_0 *is the eigenvector of the operator T corresponding to the eigenvalue* μ_0, *and for this* μ_0 (17) *holds.*

Proof. We know that

$$\lim_{m \to \infty} \mu_0^{-m} T^m x^{(0)} = B_1 x^{(0)}.$$

If we take

$$\beta = \prod_{m=0}^{\infty} \frac{\mu_0}{\mu_{(m)}}$$

(the convergence of the product follows from (19)), then we have

$$y_{(m)} = \frac{1}{\mu_{(0)} \cdots \mu_{(m-1)}} T^m x^{(0)} = \frac{1}{\mu_0^m} \prod_{k=0}^{m-1} \frac{\mu_0}{\mu_{(k)}} T^m x^{(0)} \to \beta B_1 x^{(0)} = y_0 .$$

Further

$$\mu_0^{-1} T y_0 = \lim_{m \to \infty} \mu_0^{-1} T y_{(m)} = \lim_{m \to \infty} \frac{1}{\mu_{(m)}} T y_{(m)} = y_0$$

as was to be proved.

Remark. *The condition* (24) *is not restrictive at all, since we can choose the vector* $T^M x^{(0)}$, *where M is such that* $y'_m(x^{(m)}) \neq 0$, $z'_m(x^{(m)}) \neq 0$ *for* $m \geq M$, *as the initial vector of the iterations* (22) *and omit those terms of the sequences* $\{y'_m\}$, $\{z'_m\}$ *for which* $m < M$.

Corollary 3. [7] *Let* $\mathbf{X}$ *be a Hilbert space with the scalar product* (x, y). *Let T be a compact symmetric definite operator mapping the space* $\mathbf{X}$ *into itself. Let* $\{x'_m\}$ *be a sequence of linear forms for which a form* x' *exists such that* (10) *holds and for every* $x \in \mathbf{X}$

$$|x'_m(x) - x'(x)| \leq c_{10}(x)\, m^{-1-\delta}, \quad \delta > 0 .$$

Let y'_m, z'_m *be defined by one of the formulas* (20), (21) *and let* $(x^{(m+1)}, x^{(m)}) \neq 0$ *for* $m = 0, 1, \ldots$. *Then* (25) *holds in the norm of the space* $\mathbf{X}$, *where* $y_{(m)}$ *are defined by formula* (22). *The vector* y_0 *is the eigenvector of the operator corresponding to the value* μ_0 *and* (17) *holds.*

For specific choice of the sequences $\{y'_m\}$, $\{z'_m\}$ the iteration process (22) reduces to the process (14) with

$$x'_m(x) = \frac{1}{y'_0(x^{(0)})}\, y'_m(x)\,.$$

Theorem 4. *Let $z'_m(x) = y'_{m+1}(x)$ for $m = 0, 1, \dots$ and let $y'_m(x) \to y'(x)$ hold for every $x \in \mathbf{X}$. Let $y'_m(x^{(m)}) \neq 0$ hold for $m = 0, 1 \dots$. Let (11) and (12) hold for the vector $x^{(0)}$. Then (25) holds for the sequence (22) where $\mu_{(m)}$ are defined by formula (23) and y_0 is the eigenvector of the operator T corresponding to the value μ_0. For the eigenvalue μ_0 (17) holds.*

Proof. Because of the special choice of the forms $\{y'_m\}$, $\{z'_m\}$ we have

$$y_{(m+1)} = \frac{y'_m(x^{(m)})}{y'_{m+1}(x^{(m+1)})}\,\frac{y'_{m-1}(x^{(m-1)})}{y'_m(x^{(m)})}\cdots\frac{y'_0(x^{(0)})}{y'_1(x^{(1)})}\,T^{m+1}x^{(0)} =$$

$$= \frac{y'_0(x^{(0)})}{y'_{m+1}(x^{(m+1)})}\,T^{m+1}x^{(0)}\,.$$

According to theorem 2

$$y_0 = \lim_{m\to\infty} y_{(m)}$$

and it can be easily proved that y_0 is the eigevector of the operator T corresponding to the value μ_0. (14) and (15) are sufficient for the applicability of Kellogg's iterations if we know that the multiplicity q of the value μ_0 is finite. Thus it is not necessary to know the value q explicitely. But if we do know q we can use this in the calculations.

Lemma 5. *In the norm of the space $\mathbf{X}_1$ we have*

$$\lim_{m\to\infty} \mu_0^{-m} T^m (T - \mu_0 I)^q = O\,. \tag{26}$$

Proof. According to [9], for $m \geqq 0$

$$H\left[\mu_0, T, \left(\frac{\lambda}{\mu_0}\right)^m (\lambda - \mu_0)^q\right] = O$$

so that

$$\mu_0^{-m} T^m (T - \mu_0 I)^q = \frac{1}{2\pi i}\int_{C_1} \left(\frac{\lambda}{\mu_0}\right)^m (\lambda - \mu_0)^q\, R(\lambda, T)\, \mathrm{d}\lambda$$

where C_1 is a circle with radius $\mu_1 < |\mu_0|$ and center in the origin, in which the set $\sigma(T) - \{\mu_0\}$ lies. Consequently we have

$$\|\mu_0^{-m} T^m (T - \mu_0 I)^q\| \leqq c_{11} \left|\frac{\mu_1}{\mu_0}\right|^m$$

and this implies (26).

Theorem 5. *If the conditions of theorem 2 are satisfied then*

$$(27)\quad \lim_{m\to\infty} \mu_0^{-m} \left\{ y'_m(x^{(m+q)}) - \binom{q}{1} \mu_0 y'_m(x^{(m+q-1)}) + \ldots + (-1)^q \mu_0^q y'_m(x^{(m)}) \right\} = 0 .$$

Proof. Lemma 5 states that for any $\varepsilon > 0$ there exists an index M such that for $m > M$

$$\left\| x^{(m+q)} - \binom{q}{1} \mu_0 x^{(m+q-1)} + \ldots + (-1)^q \mu_0^q x^{(m)} \right\| < \varepsilon |\mu_0^m|$$

holds. Using the Banach theorem we obtain

$$\left| y'_m(x^{(m+q)}) - \binom{q}{1} \mu_0 y'_m(x^{(m+q-1)}) + \ldots + (-1)^q \mu_0^q y'_m(x^{(m)}) \right| \leqq$$

$$\leqq \|y'_m\|_{\mathbf{X}'} \left\| x^{(m+q)} - \binom{q}{1} \mu_0 x^{(m+q-1)} + \ldots + (-1)^q \mu_0^q x^{(m)} \right\|_{\mathbf{X}} \leqq c_{12} |\mu_0|^m \varepsilon$$

where $c_{12} = \sup \|y'_m\|_{\mathbf{X}'}$.

Corollary 4. [4], [5] *Let* $\mathbf{X}$ *be an arithmetic* l*-dimensional vector-space with some Banach norm. Let vectors* $x \in \mathbf{X}$ *have coordinates* $x_1, \ldots, x_l$*. Let the operator* T *be determined in a fixed basis by a matrix which we will also denote by the symbol* T*. Let the operator* T *have the dominant eigenvalue* μ_0*. The preceding gives*

$$\lim_{m\to\infty} \mu_0^{-m} \left\{ x_j^{(m+q)} - \binom{q}{1} \mu_0 x_j^{(m+q-1)} + \ldots + (-1)^q \mu_0^q x_j^{(m)} \right\} = 0$$

so that if for instance $|\mu_0| \leqq 1$ *we obtain the algebraic equation*

$$x_j^{(m+q)} - \binom{q}{1} x_j^{(m+q-1)} \lambda + \ldots + (-1)^q x_j^{(m)} \lambda^q = 0$$

for the approximate determination of the eigenvalue μ_0*.*

A similar assertion is correct if $\mathbf{X}$ is a Banach space of functions defined in the subset G of Euclidean space $\mathbf{E}_l$, such that the additive functional defined by the formula

$$x'(x) = x(s_0)$$

is continuous. As to the operator T we assume that it is an integral operator, mapping $\mathbf{X}$ into itself and that it has a dominant eigenvalue μ_0. Then we have

$$\lim_{m\to\infty} \mu_0^{-m} \left\{ x^{(m+q)}(s_0) - \binom{q}{1} \mu_0 x^{(m+q-1)}(s_0) + \ldots + (-1)^q \mu_0^q x^{(m)}(s_0) \right\} = 0$$

where $s_0 \in G$ is chosen so that $x_0(s_0) \neq 0$, and where x_0 is the eigenfunction of the operator T corresponding to the value μ_0.

5. Modified iteration processes. The iterations investigated in the previous paragraph can also be applied to the construction of characteristic values and eigenvectors of equations of the type

(28) $$Lx = \lambda Bx$$

where L and B are linear operators mapping domains $\mathscr{D}(L)$ and $\mathscr{D}(B)$ into $\mathbf{X}$. Just as in the third chapter, we will list the assumptions about the operators L and B separately, so as not to repeat their formulations in most of the statements.

Assumptions. (A) *The operator B is a bounded linear operator mapping $\mathbf{X}$ into itself.*

(B) *There exists a bounded inverse operator L^{-1} mapping $\mathbf{X}$ into $\mathscr{D}(L)$, where $\mathscr{D}(L)$ is the domain of the operator L.*

(C) *The operator $T = L^{-1}B$ satisfies the assumptions* (a)–(c) *of the third chapter. Let us put $\lambda_0 = \mu_0^{-1}$.*

We search for the characteristic values of the equation and corresponding eigenvectors with the help of the modified Kellogg iterations

(29) $$v^{(m)} = Bu^{(m)}, \quad Lu^{(m+1)} = v^{(m)}, \quad u^{(0)} = x^{(0)},$$

(30) $$u_{(m)} = \frac{u^{(m)}}{x'_m(u^{(m)})},$$

(31) $$\lambda_{(m)} = \frac{y'_m(u^{(m)})}{z'_m(u^{(m+1)})},$$

where $\{x'_m\}$, $\{y'_m\}$, $\{z'_m\}$ are sequences of linear forms which, together with the forms x', y', appear in theorem 2.

Theorem 6. *Let the forms x'_m, y'_m, z'_m, x', y' and vector $x^{(0)}$ satisfy the conditions of theorem 2. We then have in the norm of space $\mathbf{X}$*

(32) $$\lim_{m\to\infty} u_{(m)} = u_0,$$

and

(33) $$\lim_{m\to\infty} \lambda_{(m)} = \lambda_0$$

where u_0 is the eigenvector of equation (28) *corresponding to the characteristic value λ_0.*

Proof. We will prove that the iterations (30) and (31) are Kellogg iterations for the operator $L^{-1}B$, which satisfies the conditions of theorem 2. From (29) we obtain

$$u^{(m+1)} = L^{-1}v^{(m)} = L^{-1}Bu^{(m)} \quad \text{i.e.} \quad u^{(m)} = T^m x^{(0)} = x^{(m)}.$$

According to (15),

$$\lambda_{(m)} = \frac{y'_m(u^{(m)})}{z'_m(u^{(m+1)})} = \mu_{(m)}^{-1}$$

and using theorem 2 we obtain directly the formulas (32) and (33).

The iterations

$$(34)\qquad w^{(m)} = Bw_{(m)},\ Lw_{m+1} = w^{(m)},\ w_{(m+1)} = \lambda_{(m)}w_{m+1},\ w_{(0)} = x^{(0)}$$

are analogous to the iteration process (22), where $\lambda_{(m)}$ are given by the formula (31) *i.e.*

$$\lambda_{(m)} = \frac{y'_m(w_{(m)})}{z'_m(w_{m+1})}.$$

As a special case, when $\mathbf{X}$ is a Hilbert space with the scalar product (x, y), then by convenient choice of the forms x'_m, y'_m, z'_m we obtain some known modified iteration processes, among them the Schwarz-Collatz iterations [3].

Theorem 7. *Let the operator $T = L^{-1}B$ have the property R_1 in the point $\mu_0 = \lambda_0^{-1}$. Let the forms x'_m, y'_m, z'_m, x', y' satisfy the conditions of theorem 3. Let (11) and (12) hold for the vector $x^{(0)}$. Let $y'_m(w_{(m)}) \neq 0$, $z'_m(w_{(m)}) \neq 0$ hold for $m = 0, 1, \ldots$. Then the following holds in the norm of space $\mathbf{X}$*

$$(35)\qquad \lim_{m\to\infty} w_{(m)} = w_0$$

where w_0 is the eigenvector of equation (28) corresponding to the characteristic value λ_0.

Proof. The sequence of vectors $\{w_{(m)}\}$ is an iteration sequence of the process (22) for the operator $L^{-1}B$, since

$$w_{(m+1)} = \lambda_{(m)}L^{-1}w^{(m)} = \lambda_{(m)}L^{-1}Bw_{(m)}.$$

The statement is thus a consequence of theorem 3.

If we know the multiplicity q of the value $\mu_0 = \lambda_0^{-1}$, we can use an analogy of theorem 5.

Theorem 8. *Let the forms y'_m, y' and the vector $x^{(0)}$ satisfy the conditions of theorem 2. Then we have*

$$\lim_{m\to\infty} \lambda_0^m \left\{\lambda_0^q y'_m(u^{(m+q)}) - \binom{q}{1}\lambda_0^{q-1} y'_m(u^{(m+q-1)}) + \ldots + (-1)^q y'_m(u^{(m)})\right\} = 0.$$

The proof is evident.

6. Modified iterations in a reduced part of the space. Let the condition (B) of the chapter 5 hold, and in place of (A) and (C) assume

(A′) *For the domains $\mathscr{D}(L)$, $\mathscr{D}(B)$ of linear (in general unbounded) operators L, B there holds $\mathscr{D}(L) \subset \mathscr{D}(B)$, and*

(C′) *The operator $T = BL^{-1}$ satisfies the conditions* (a)–(c) *of chapter* 3.

It follows from (B) that the equation (28) is equivalent to the equation

$$(36)\qquad x = \lambda L^{-1}Bx.$$

From (36) applying the map B we obtain $Bx = \lambda BL^{-1}Bx$, so that if we put $y = Bx$, we obtain the equation $y = \lambda BL^{-1}y$.

We can now use the above proved statements about the convergence of iteration processes to determine the characteristic values of the operator BL^{-1}. We will prove that an eigenvector of equation (28) corresponds to each eigenvector y of the operator $T = BL^{-1}$.

Lemma 6. *Let $y \in \mathbf{X}$ be the eigenvector of the operator BL^{-1} corresponding to the characteristic value λ_0. Then the vector $x = L^{-1}y$ is an eigenvector of the equation* (28) *corresponding to the same value λ_0.*

Proof. According to our assumption we have

$$Lx = y = \lambda_0 BL^{-1}y = \lambda_0 Bx$$

and the condition $y \neq o$, according to (B), gives $x \neq o$.

We will form Kellogg's iterations for the operator BL^{-1} directly for the equation (28), so as to avoid having to construct the operator BL^{-1} and its powers. Thus we obtain the following iteration process

$$Lu^{(m+1)} = v^{(m)}, \quad v^{(m+1)} = Bu^{(m+1)}, \quad v^{(0)} = Bx^{(0)}, \tag{37}$$

$$u_{(m)} = \frac{u^{(m)}}{x'_{m-1}(v^{(m-1)})}, \tag{38}$$

$$\lambda_{(m)} = \frac{y'_m(v^{(m)})}{z'_m(v^{(m+1)})} \tag{39}$$

where $\{x'_m\}$, $\{y'_m\}$, $\{z'_m\}$ are the sequences of linear forms which, together with the forms x', y', appear in theorem 2.

Theorem 9. *Let the forms x'_m, y'_m, z'_m, x', y' satisfy the conditions of theorem* 2 *for the operator $T = BL^{-1}$. Let $B_1 y^{(0)} \neq o$ hold for the vector $y^{(0)} = Bx^{(0)}$, so that there exists such an index s, $1 \leqq s \leqq q$ that*

$$B_s y^{(0)} \neq o, \quad B_{s+1} y^{(0)} = o. \tag{40}$$

Further let

$$x'(B_s y^{(0)}) \neq 0, \quad y'(B_s y^{(0)}) \neq 0. \tag{41}$$

Then

$$\lim_{m \to \infty} u_{(m)} = u_0 \tag{42}$$

holds for the sequence (40) *in the norm of the space* $\mathbf{X}$, *where u_0 is the eigenvector of equation* (28) *corresponding to the characteristic value λ_0 and this value is the limit of the sequence* (39),

$$\lim_{m \to \infty} \lambda_{(m)} = \lambda_0. \tag{43}$$

Proof. The sequence $\{v^{(m)}/x'_m(v^{(m)})\}$ is the Kellogg iteration sequence for the operator BL^{-1}, since

$$v^{(m+1)} = Bu^{(m+1)} = BL^{-1}v^{(m)}.$$

According to theorem 2 there exists a vector v_0 such that

$$\lim_{m\to\infty} \frac{v^{(m)}}{x'_m(v^{(m)})} = v_0$$

holds in the norm of space $\mathbf{X}$ and v_0 is the eigenvector of the operator BL^{-1} corresponding to the characteristic value λ_0. According to the same theorem λ_0 is the limit of the sequence (39) and thus (43) is correct. Further we have

$$u_{(m+1)} = \frac{u^{(m+1)}}{x'_m(v^{(m)})} = L^{-1}\left(\frac{v^{(m)}}{x'_m(v^{(m)})}\right) \to L^{-1}v_0\,,$$

so that (42) holds, where $u_0 = L^{-1}v_0$. According to lemma 6 u_0 is the eigenvector of equation (28) corresponding to the value λ_0.

The following iterations are analoguous to iterations (34)

$$(44)\qquad Lz_{(m)} = z^{(m)},\ z_{m+1} = Bz_{(m)},\ z^{(m+1)} = \lambda_{(m)}z_{m+1},\ z^{(0)} = Bx^{(0)}$$

where $\lambda_{(m)}$ are defined by the formula

$$(45)\qquad \lambda_{(m)} = \frac{y'_m(z^{(m)})}{z'_m(z_{m+1})}\,.$$

Theorem 10. *Let the forms x'_m, y'_m, z'_m, x', y' satisfy the conditions of theorem* 3. *Let the operator $T = BL^{-1}$ have the property* R_1 *in the point $\mu_0 = \lambda_0^{-1}$. Let* (40) *and* (41) *hold for the vector $y^{(0)} = z^{(0)} = Bx^{(0)}$. Let $y'_m(z^{(m)}) \neq 0$, $z'_m(z^{(m)}) \neq 0$ hold for $m = 0, 1, \ldots$. Then*

$$(46)\qquad \lim_{m\to\infty} z_{(m)} = z_0$$

holds in the norm of the space $\mathbf{X}$. The vector z_0 is the eigenvector of equation (28) *corresponding to the characteristic value λ_0 and $\lim_{m\to\infty} \lambda_{(m)} = \lambda_0$.*

Proof. The sequence $\{z^{(m)}\}$ is the iteration sequence (22) for the operator $T = BL^{-1}$, since

$$z^{(m+1)} = \lambda_{(m)}z_{m+1} = \lambda_{(m)}Bz_{(m)} = \lambda_{(m)}BL^{-1}z^{(m)}\,.$$

According to theorem 3 the vector $w = \lim_{m\to\infty} z^{(m)}$ is the eigenvector of the operator BL^{-1} corresponding to the characteristic value λ_0, which is the limit of the sequence (45). According to lemma 6 the vector $z_0 = L^{-1}w$ is an eigenvector of the equation (28), and (46) holds for $z_{(m)} = L^{-1}z^{(m)}$.

Corollary 5. *Let $\mathbf{X}$ be a Hilbert space with the scalar product (x, y). Let $T = BL^{-1}$ be a compact operator mapping $\mathbf{X}$ into itself, symmetrizable by a positively definite operator H, so that*

$$(47)\qquad (HBL^{-1}x, x) > 0 \quad \textit{for}\ x \in \mathbf{X},\ x \neq o\,.$$

Let (40) *and* (41) *hold for the vector $y^{(0)} = Bx^{(0)}$. Let the forms x'_m, y'_m, z'_m, x', y' satisfy the conditions of theorem* 3. *Then* (42) *and* (46) *hold for the sequences* (38) *and* (44). *The limits of sequences* (38) *and* (46) *are the eigenvectors of the equation* (28) *corresponding to the characteristic value* λ_0 *and* (39) *and* (45) *hold.*

Proof. Let us investigate whether the conditions of theorems 2 and 3 are satisfied for the operator $T = BL^{-1}$ and for the inital vector $y^{(0)}$ of the iterations. The operator $T = BL^{-1}$ is symmetrizable and compact, so that according to [6] its spectrum lies on the real axis and every non-zero point of the spectrum of the operator T is a simple pole of the resolvent $R(\lambda, T)$. It follows from (47) that the spectrum lies on the positive semi-axis and hence a dominant eigenvalue of the operator T exists. Thus the operator T satisfies the conditions of theorems 2 and 3. The conditions (40) and (41) are analogous to (11) and (12) in theorems 2 and 3. Since the conditions of the mentioned theorems are satisfied, the assertion of theorem 10 is proved.

It can be seen from the remark to theorem 2 that corollary 5 is a generalization of the results of V. S. Vladimirov [10], [11] relating to successive approximations.

Theorem 11. *Let the conditions of theorem* 9 *be fulfilled for the operator* $T = BL^{-1}$. *Then the following holds for the sequence defined by* (37)

$$\lim_{m\to\infty} \lambda_0^m \left\{\lambda_0^q y_m'(u^{(m+q)}) - \binom{q}{1} \lambda_0^{q-1} y_m'(u^{(m+q-1)}) + \ldots + (-1)^q y_m'(u^{(m)})\right\} = 0 .$$

Proof. According to lemma 5 the vector

$$v(m) = \lambda_0^m \left[\lambda_0^q v^{(m+q)} - \binom{q}{1} \lambda_0^{q-1} v^{(m+q-1)} + \ldots + (-1)^q v^{(m)}\right]$$

converges in the norm of the space $\mathbf{X}$ to the zero-vector. The vector $L^{-1}v(m)$ converges to the zero-vector together with $v(m)$ and thus the sequence $\{y_m'(L^{-1}v(m))\}$ converges to zero.

7. Concluding remarks. Let us recapitulate in short the obtained results. The conditions necessary for the convergence of Kellogg processes can be divided into several groups. The first group of conditions refers to the space in which the interations are investigated, the second one concerns the spectral properties of the investigated operator. In the third group are conditions concerning the iteration formulas and the choice of the initial element. As to the space in which the iterations are constructed, we have shown that most of the results hold in general Banach spaces. We have found it possible to carry over into Banach spaces some of the iteration processes, originally introduced in Hilbert spaces and to prove their convergence. Let us make a simple remark about the second group of conditions. The condition that the value μ_0 be dominant can be replaced by a somewhat weaker condition.

Remark. *Let the conditions* (a) *and* (b) *of the third chapter be fulfilled for the operator T. Let p eigenvalues* $\mu_1, \ldots, \mu_p$ *exist, for which the inequalities* $|\lambda| < |\mu_1| = |\mu_2| = \ldots = |\mu_p|$ *hold for all points* $\lambda \in \sigma(T)$, $\lambda \neq \mu_j$, $j = 1, \ldots, p$. *Let us choose a fixed index j. If a* $\nu_j \in \Pi$ *exists such that* $\mu_j - \nu_j$ *is the dominant point of the* *spectrum of the operator $T - \nu_j I$, then the following equation holds under the assumptions of theorem* 2:

$$\mu_j = \nu_j + \lim_{m \to \infty} \mu^{(j)}_{(m)} \quad where \quad \mu^{(j)}_{(m)} = \frac{z'_m\big((T - \nu_j I)^{m+1} x^{(0)}\big)}{y'_m\big((T - \nu_j I)^m x^{(0)}\big)}.$$

Thus by translating the operator T, all p values on the boundary of the spectral circle can be determined.

As to the choice of the initial element of the iterations it is evident that if $B_1 x^{(0)} = o$, then (11) does not hold for any index s and the sequence $\{x_{(m)}\}$ generally does not converge. The situation is analoguous in the case of modified iterations in a reduced part of the space, *i.e.* if (40) does not hold for any index s.

We have actually given two iteration formulas for the construction of the eigenvector of the operator T – the original Kellogg one (14) and the generalized Schwarz-Collatz formula. We have proved that this iteration process converges for general forms y'_m and z'_m if μ_0 is a simple dominant eigenvalue. On the other hand, we have proved that the iterations (23) converge for specifically chosen forms y'_m, z'_m even if the eigenvalues are higher order poles of the resolvent $R(\lambda, T)$ (theorem 4). We did not give similar assertions for modified iterations explicitely, because it is quite evident how theorem 4 can be extended to this case.

Estimates of errors are not performed explicitely. It is clear, however, that these estimates can be obtained according to lemmas 1, 2, 3 and the corresponding proofs in which errors are estimated.

Some of the iteration processes cannot be used for practical calculations in the form given in this paper, especially if the order q of the dominant value μ_0 is much greater than 1. In such a case it is necessary to use methods that increase the rate of convergence of the iteration sequence.

The proved assertions concerning the convergence of iterative processes can be used to prove the convergence of some numerical methods of solving linear equations of type (28). One such example is the Boltzmann equation in the kinetic theory of neutron transport in the multigroup energetic approximation. There are of course many other similar applications.

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Резюме

ИТЕРАЦИИ ОГРАНИЧЕННЫХ ЛИНЕЙНЫХ ОПЕРАТОРОВ В НЕСАМОСОПРЯЖЕННЫХ ПРОБЛЕМАХ СОБСТВЕННЫХ ЗНАЧЕНИЙ И ИТЕРАЦИОННЫЙ ПРОЦЕСС КЕЛЛОГА

ИВО МАРЕК (Ivo Marek), Прага

В статье доказано, что достаточным условием для сходимости итерационного процесса Келлога для отыскания собственных векторов и собственных значений линейных операторов отображающих банахово пространство в себя, является существование неподвижных точек специального типа для этих отображений.

Метод доказательств основан на использовании операционного исчисления для линейных операторов в пространстве Банаха [9]. Преимущество этого метода заключается в том, что при доказательствах сходимости итерационных процессов можно освободиться от предположения симметризуемости рассматриваемых операторов.

Итерационный процесс приведенный в (14) и (15) является по форме довольно общим, так как он обобщает одновременно ряд частных интерационных процессов [2], [3], [7], [8] [13], [14], [15], [16].

Сходимость итерационных процессов является следствием некоторых спектральных свойств рассматриваемых операторов.

Будем рассматривать оператор T и будем предполагать следующее:

Предположения. (a) Оператор T — линейный ограниченный оператор отображающий банахово пространство $\mathbf{X}$ в себя.

(b) Значение μ_0 является полюсом q-ной степени резольвенты $R(\lambda, T)$.

(c) Неравенство

(6) $$|\mu_0| > |\lambda|$$

справедливо для всех $\lambda \in \sigma(T)$, $\lambda \neq \mu_0$, где $\sigma(T)$ спектр оператора T.

Пусть $\mathbf{X}'$ сопряженное к $\mathbf{X}$ пространство линейных форм и $\mathbf{X}_1 = (\mathbf{X} \to \mathbf{X})$ пространство линейных ограниченных отображений $\mathbf{X}$ в $\mathbf{X}$. Нормы в $\mathbf{X}'$ и в $\mathbf{X}_1$ определены обычным способом.

Основным утверждением при доказательстве сходимости итерационных процессов является следующая теорема:

Теорема 1. *В норме пространства* $\mathbf{X}_1$ *имеет место формула*

(9) $$\lim_{m \to \infty} m^{-q+1} \mu_0^{-m} T^m = \frac{\mu_0^{-q+1}}{(q-1)!} Bq ,$$

где B_q *определяется с помощью ряда Лорана резольвенты* $R(\lambda, T)$ [9] *в окрестности точки* μ_0:

$$R(\lambda, T) = \sum_{k=0}^{\infty} (\lambda - \mu_0)^k T_k + \sum_{k=1}^{q} (\lambda - \mu_0)^k B_k .$$

Пусть имеются последовательности форм $\{x'_m\}$, $\{y'_m\}$, $\{z'_m\}$ из пространства $\mathbf{X}'$ такие, что в $\mathbf{X}'$ существуют x', y' для которых

(10) $$x'(x) = \lim_{m \to \infty} x'_m(x) , \quad y'(x) = \lim_{m \to \infty} y'_m(x) = \lim_{m \to \infty} z'_m(x)$$

для всех векторов $x \in \mathbf{X}$.

Пусть для $x^{(0)} \in \mathbf{X}$, $B_1 x^{(0)} \neq o$ (o нулевой вектор), так что существует такое s, $(1 \leqq s \leqq q)$ что

(11) $$B_s x^{(0)} \neq o , \quad B_{s+1} x^{(0)} = o .$$

Предолагая, что

(12) $$x'(B_s x^{(0)}) \neq o , \quad y'(B_s x^{(0)}) \neq o ,$$

положим

(13) $$x_0 = \frac{B_s x^{(0)}}{x'(B_s x^{(0)})} .$$

Тогда итерационные последовательности Келлога строятся по формулам

(14) $$x^{(m)} = T x^{(m-1)} , \quad x_{(m)} = \frac{x^{(m)}}{x'_m(x^{(m)})} ,$$

(15) $$\mu_{(m)} = \frac{z'_m(x^{(m+1)})}{y'_m(x^{(m)})} .$$

Теорема 2. *Если для форм* x'_m, y'_m, z'_m, x', y' *справедливы формулы* (10) *и для вектора* $x^{(0)}$ *справедливы условия* (11) *и* (12), *то в норме пространства* $\mathbf{X}$

$$(16) \qquad \lim_{m\to\infty} x_{(m)} = x_0$$

и для (15)

$$(17) \qquad \lim_{m\to\infty} \mu_{(m)} = \mu_0 .$$

Вектор x_0 *определенный в* (13) *является собственным вектором оператора* T, *соответствующим собственному значению* μ_0.

Кроме процесса (14), (15) рассматривается итерационная последовательность, члены которой строятся по формуле

$$(22) \qquad y_{(m+1)} = \frac{1}{\mu_{(m)}} T y_{(m)} ,$$

где

$$(23) \qquad \mu_{(m)} = \frac{z'_m(Ty_{(m)})}{y'_m(y_{(m)})} ,$$

то есть $\mu_{(m)}$ определенные формулой (23) те-же самые как в (15).

Теорема 3. *Пусть* μ_0 *простой полюс резольвенты* $R(\lambda, \mathbf{T})$ *и пусть выполнены условия теоремы* 2. *Кроме того пусть для* $x \in \mathbf{X}$

$$(18) \qquad |x'_m(x) - x'(x)| + |y'_m(x) - y'(x)| + |z'_m(x) - y'(x)| \leqq c_4(x) m^{-1-\delta} , \quad \delta > 0 .$$

Потом в норме пространства $\mathbf{X}$ *имеем* (*если конечно* $y'_m(y_{(m)}) \neq 0$, $z'_m(y_{(m)}) \neq 0$)

$$(25) \qquad \lim_{m\to\infty} y_{(m)} = y_0 ,$$

где y_0 *собственный вектор оператора* T *соответствующий собственному значению* μ_0 *и справедливо равенство* (17).

Если известна кратность q полюса μ_0, можно использовать этот факт при вычислениях.

Теорема 5. *Если для* y'_m *и* y' *справедливо равенство* (10) *для всех* $x \in \mathbf{X}$, *то*

$$\lim_{m\to\infty} \mu_0^{-m} \left\{ y'_m(x^{(m+q)}) - \binom{q}{1} \mu_0 y'_m(x^{(m+q-1)}) + \ldots + (-1)^q \mu_0^q y'_m(x^{(m)}) \right\} = 0 .$$

В пятой и шестой частях результаты четвертой части перенесены на случай характеристических значений уравнений типа

$$(28) \qquad Lx = \lambda Bx ,$$

где L в общем неограниченный оператор отображающий область определения $\mathscr{D}(L)$ в $\mathbf{X}$ и $B \in \mathbf{X}_1$.

В части шестой могут быть оба оператора L, B в общем неограниченными. Предполагаем только, что для областей определения этих операторов $\mathscr{D}(L)$, $D(B)$ имеет место $\mathscr{D}(L) \subset D(B)$. В пятой части $T = L^{-1}B$ (теоремы 6–8) и в шестой $T = BL^{-1}$ (теоремы 9–11).