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THE PRINCIPLE OF UNIFORM BOUNDEDNESS AND THE CLOSED GRAPH THEOREM

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A general theorem of the closed graph type in locally convex vector spaces is proved which contains as special cases the classical theorems of Hellinger and Toeplitz.

1. Among the classical results of Functional Analysis there are several theorems according to which an operator is continuous provided it satisfies some very weak conditions of an algebraic or nearly algebraic nature. Let us recall, as an example, the well known result of E. HELLINGER and O. TOEPLITZ: *A linear transformation T defined for all elements of a Hilbert space H and satisfying $(Tx, y) = (x, Ty)$ is continuous.* The proof is based on the principle of uniform boundedness. Indeed, the relation $(Tx, y) = (x, Ty)$ implies that the adjoint T^* is defined on the whole of H. To show that the operator T is bounded it is sufficient, by the uniform boundedness principle, to prove that (TU, y) is bounded for each $y \in H$. (We denote by U the unit ball in H.) Now it is sufficient to note that $(TU, y) = (U, T^*y)$ and that the last set is bounded. Similarly, in the case of Hilbert space, the closed graph theorem (which clearly contains the preceding result of Hellinger and Toeplitz) may be obtained by the use of the uniform boundedness principle. If T is a closed linear transformation defined for all elements of a Hilbert space H, the proof of the continuity of T proceeds in the following manner. First, the assumption that T is closed implies that the domain of T^* is dense in H. An application of the uniform boundedness principle shows that T^* is bounded in $D(T)$. It follows that T^* is actually defined on the whole of H and the proof is concluded exactly as in the preceding case by another application of the uniform boundedness principle.

It is not difficult to see that the essential point of the proof is the fact that, in both cases, the adjoint of the given transformation is everywhere defined.

There are further results closely connected with these ideas. In an interesting paper [8] of A. E. TAYLOR an application of the closed graph theorem is given to prove the continuity of linear transformations of a certain type under very weak hypotheses. The earlier work on similar questions by S. IZUMI and G. SUNOUCHI [2] is not accessible to the author.

In our previous work we have been able to point out what we believe to be the natural generalizations of the open mapping and closed graph theorems. Especially it turns out that these theorems are essentially based on the properties of one of the two spaces involved only, namely the space from which the mapping goes in the case of the open mapping theorem and the space into which the transformation acts in the case of the closed graph theorem.

No assumption on the other space (the range of the mapping in the case of the open mapping theorem and the domain of definition in the case of the closed graph theorem) need be made if we impose some natural conditions on the mapping (it should be nearly open in the first case and nearly continuous in the second). See theorems (3.8) and (4.7) of [4].

In the present remark we intend to present the general form of the Hellinger-Toeplitz theorems. We analyse first the classical case and show then the general counterpart of each of the steps in the proofs. In this manner the really essential points are exhibited; actually the elimination of metrical methods leads to a considerable simplification of the proofs.

The earlier work on the general form of the open mapping theorem is contained in [3], [4], [5]. The closed graph theorem is discussed in [6]. All definitions, results and necessary informations concerning these questions may be found in [4].

2. Definitions and notation. We use the term "convex space" instead of "locally convex Hausdorff topological vector space over the real field". Terminology and notation is that of [4]. For the convenience of the reader we give here the definitions of some notions which are not yet stabilized in literature and which might lead to a misunderstandig.

Let X and Y be two convex spaces. We denote by $L(X, Y)$ the space of all continuous linear mappings of X into Y. A set $P \subset L(X, Y)$ is said to be total for X if the following implication holds: If $p(x) = 0$ for each $p \in P$ and some x, then $x = 0$. Let f be a linear mapping of X into Y; we denote by $D(f')$ the set of all $y' \in Y'$ such that $\langle f(x), y' \rangle$ is a continuous function of x. The mapping f' of $D(f')$ into X' defined by the relation $\langle f(x), y' \rangle = \langle x, f'(y') \rangle$ for $x \in X$ is called the adjoint of f.

A mapping g of a topological space T into another topological space V is said to be nearly continuous if the following condition is satisfied: for each $t_0 \in T$ and each neighbourhood H of $g(t_0)$ in V the set $\overline{g^{-1}(H)}$ is a neighbourhood of t_0 in T. We shall frequently use the following fact, which is an immediate consequence of this definition.

Let X and Y be two convex spaces and suppose that X is a t-space (if K is a closed absolutely convex set such that the union of its multiples λK is the whole of X then K is a neighbourhood of zero). Then every linear mapping of X into Y is nearly continuous.

3. Application of the uniform boundedness principle. It is a well-known fact that, in the case of Hilbert space, the closed graph theorem may be obtained as a consequence of the principle of uniform boundedness. We present here a simple generalization of this fact. Indeed, we show that the proof is actually based on the reflexivity of the space. At the same time, the method of proof suggests the generalization to nonmetrizable spaces in quite a natural manner. We begin with a well-known result.

(3,1). *Let X and Y be two normed linear spaces (complete or not). Let f be a linear mapping of X into Y. Suppose that the adjoint f' is defined on the whole of Y'. Then f is continuous.*

Proof. Since continuity and boundedness are equivalent in normed spaces, it is sufficient to prove that the set $f(U)$ is bounded, U being the unit ball of X. According to the principle of uniform boundedness, it is sufficient to show that the set $\langle f(U), y' \rangle$ is bounded for each $y' \in Y'$; this, however, follows immediately from the relation $\langle f(U), y' \rangle = \langle U, f'(y') \rangle$ and the fact that $f'(y')$ has a sense for each $y' \in Y'$.

(3,2). *Let X and Y be two normed linear spaces. Suppose that X is a t-space and that Y is reflexive. Let f be a linear mapping of X into Y the graph of which is closed in $X \times Y$. Then f is continuous.*

Proof. The graph of f being closed, the domain $D(f')$ is dense in Y' in the topology $\sigma(Y', Y)$. This is a classical result (for the sake of completeness a proof of this fact is given in (3,3)).

Let us denote now by T the set of those $q \in D(f')$ for which $|q| \leqq 1$. If x is an arbitrary point of X, we have

$$|\langle x, f'(T) \rangle| = |\langle f(x), T \rangle| \leqq |f(x)| .$$

The space X being a t-space, it follows that $|f'(T)| \leqq \alpha$ for some α so that f' is bounded.

We are going to show now that $D(f')$ is equal to the whole of Y'. The space $D(f')$ is $\sigma(Y', Y)$ dense in Y'; the space Y being reflexive, $D(f')$ is dense in Y' in the norm topology as well. Take an arbitrary $y' \in Y'$. There exists a sequence $q_n \in D(f')$ such that $|q_n - y'| \to 0$. For each $x \in X$, we have

$$\langle x, f'(q_n) \rangle = \langle f(x), q_n \rangle \to \langle f(x), y' \rangle$$

so that the functionals $f'(q_n)$ converge to a limit for each $x \in X$. The space X being a t-space, this limit is bounded, hence an element x' of X'. It follows that $\langle x, x' \rangle = \langle f(x), y' \rangle$ for each $x \in X$ so that $y' \in D(f')$. We have thus shown that f' is defined on the whole of Y'. The continuity of f follows then from (3.1).

We conclude this section with a proof of the well-known fact that $D(f')$ is dense if f is a closed operator.

(3,3) *Let E and Y be two convex spaces and let f be a linear mapping of E into Y the graph G of which is closed in $E \times Y$. Then $D(f')$ is dense in $(Y', \sigma(Y', Y))$.*

Proof. Let $y_0 \in Y$ be such that $\langle y_0, D(f') \rangle = 0$. We are going to show that $[0, y_0] \in G$; this implies $y_0 = 0$ and the proof will be complete. To see that $[0, y_0] \in G$, suppose that this is not true. Since G is closed in $E \times Y$, there exists a point $[x', y']$ in $E' \times Y'$ such that $\langle G, [x', y'] \rangle = 0$ and $\langle [0, y_0], [x', y'] \rangle \neq 0$. It follows that

$\langle x, x' \rangle + \langle f(x), y' \rangle = 0$ for every $x \in E$ and $\langle y_0, y' \rangle \neq 0$. The first relation, however, implies $y' \in D(f')$ while the second contradicts $\langle y_0, D(f') \rangle = 0$. Hence $[0, y_0] \in G$ and the lemma is established.

4. The general case. We proceed to show what we believe to be the natural generalizations of the essential steps in the classical proofs. Let us clear up first the meaning of the fact that the adjoint of some mapping is everywhere defined. In the classical case, we use first the completeness of the adjoint space; this enables us to apply the principle of uniform boundedness to show that the given mapping is bounded and consequently continuous. In the general case the adjoint space need not be of the second category; further, continuity does not follow from boundedness. We shall see that the following lemma is an adequate substitute for (3.1):

(4,1) *Let X and E be two convex spaces and let f be a nearly continuous linear mapping of X into E. If the adjoint f' is defined in the whole of E', then f is continuous.*

Proof. Let U be a neighbourhood of zero in E. According to our assumption the closure of $f^{-1}(U)$ is a neighbourhood of zero in X. If we show that $f^{-1}(U)$ is already closed, the proof will be complete. To see that, take an $x_0 \notin f^{-1}(U)$. Since $f(x_0) \notin U$, there exists a $y' \in E'$ such that $y' \in U^\circ$ and $\langle f(x_0), y' \rangle > 1$. Now $y' \in D(f')$ so that $\langle x_0, f'(y') \rangle > 1$. At the same time

$$\langle f^{-1}(U), f'(y') \rangle = \langle f(f^{-1}(U)), y' \rangle \leqq 1 .$$

The proof is complete.

The second essential step in the classical proof consists in showing that, if X is a t-space, then f' is bounded and $D(f')$ contains every limit point of a bounded part of $D(f')$. If we replace the assumption on X by the assumption that f be nearly continuous, we have the following substitute for boundedness of f': the mapping maps each equicontinuous set in Y' into an equicontinuous set in X'. These sets being weakly compact, we are then able to show that the limit points of the equicontinuous parts of $D(f')$ belong to $D(f')$ as well.

(4,2) *Let X and E be two convex spaces and let f be a nearly continuous linear mapping of X into E. Let $Q = D(f')$ be the domain of f'. Then $Q \cap U^\circ$ is $\sigma(E', E)$ closed for every neighbourhood of zero U in E.*

Proof. Let U be given; according to our assumption the set $W = \overline{f^{-1}(U)}$ is a neighbourhood of zero in X. We have

$$\langle f^{-1}(U), f'(Q \cap U^\circ) \rangle = \langle f(f^{-1}(U)), Q \cap U^\circ \rangle = \langle U, Q \cap U^\circ \rangle$$

whence $f^{-1}(U) \subset f'(Q \cap U^\circ)^\circ$ so that $f'(Q \cap U^\circ) \subset (f^{-1}(U))^\circ = W^\circ$. Take now a y' from the $\sigma(E', E)$ closure of $Q \cap U^\circ$. We intend to show that $y' \in D(f')$. If $x_1, \ldots, x_n$ is an arbitrary finite sequence in X and ε an arbitrary positive number, let us denote by $W(x_1, \ldots, x_n; \varepsilon)$ the set of all $x' \in W^\circ$ such that

$$|\langle x_i, x' \rangle - \langle f(x_i), y' \rangle| \leqq \varepsilon \quad \text{for} \quad i = 1, 2, \ldots, n \, .$$

Clearly these sets are closed subsets of $(W^\circ, \sigma(E', E))$. Let us show now that

$$W(x_1, \ldots, x_n; \varepsilon)$$

is nonvoid. Since y' belongs to the $\sigma(E', E)$ closure of $Q \cap U^\circ$, there is a $q \in Q \cap U^\circ$ such that $|\langle f(x_i), q - y' \rangle| \leqq \varepsilon$ for $i = 1, 2, \ldots, n$. Since $q \in D(f')$, we have $f'(q) \in W^\circ$ and

$$\langle f(x_i), y' \rangle = \langle f(x_i), q \rangle + \Theta_i \varepsilon = \langle x_i, f'(q) \rangle + \Theta_i \varepsilon$$

whence $f'(q) \in W(x_1, \ldots, x_n; \varepsilon)$. The system $W(x_1, \ldots, x_n; \varepsilon)$ is thus seen to possess the finite intersection property. It follows that there exists an $x' \in W^\circ$ such that x' belongs to each $W(x_1, \ldots, x_n; \varepsilon)$. Clearly this means that $\langle f(x), y' \rangle = \langle x, x' \rangle$ for each $x \in X$ or, in other words, that $y' \in D(f')$. The proof is complete.

We are now able to formulate the main result:

(4,3) Theorem. *Let X and Y be two convex spaces. Let $S \subset Y'$ be total for Y. Let $a(s)$ be a mapping of S into X' with the following property: for each $x \in X$ there exists a $y \in Y$ such that $\langle x, a(s) \rangle = \langle y, s \rangle$ for each $s \in S$. The mapping $y = Ax$ is a linear mapping of X into Y. If X is a t-space and Y a B_r-complete space, then A is continuous.*

Proof. It is easy to verify the linearity of A. Since X is a t-space, the mapping A is nearly continuous. It follows from (4.2) that $D(A')$ has a $\sigma(Y', Y)$ closed intersection with every set V° where V runs over all neighbourhoods of zero in Y. The identity $\langle x, a(s) \rangle = \langle Ax, s \rangle$ implies that $S \subset D(A')$ and, consequently, $L \subset D(A')$ where L is the linear subspace of Y' spanned by S. Since S is total for Y, L is dense in $(Y', \sigma(Y', Y))$, so that $D(A')$ is dense as well. The space Y being B_r-complete, it follows that $D(A') = Y'$. The conclusion follows from (4.1). ∎

(4,4) Corollary. *Let X, Y, Z be three convex spaces. Let $P \subset L(Y, Z)$ be total for Y. Let $\alpha(p)$ be a mapping of P into $L(X, Z)$ with the following property: for each $x \in X$ there exists a $y \in Y$ such that $\langle x, \alpha(p) \rangle = \langle y, p \rangle$ for each $p \in P$. The mapping $y = Ax$ is a linear mapping of X into Y. If X is a t-space and Y a B_r-complete space, then A is continuous.*

Proof. This is an immediate consequence of the preceding theorem if we take for S the set of all functionals obtained as a superposition of a $p \in P$ and a $z' \in Z'$. If $s = p \circ z'$, we take $a(s) = \alpha(p) \circ z'$.

(4,5) Corollary. *Let X, Y and Z be three convex spaces. Suppose that the elements of Y are functions defined on a set S with values in Z, with addition and scalar multiplication carried out in the usual way. We require that the values $y(s)$ depend continuously on y. Let $a(s)$ be a function on S to $L(X, Z)$ such that $a(s)\,x$, as a function of s, is a member of Y for each $x \in X$. In this manner, a (clearly* *[illegible] A of X into Y is defined. If X is a t-space and Y a B_r-complete space, then A is continuous.*

Proof. An immediate consequence of (4.4).

Bibliography

[1] *E. Hellinger, O. Toeplitz:* Grundlagen für eine Theorie der unendlichen Matrizen. Math. Ann. *69* (1910), 289—330.

[2] *S. Izumi, G. Sunouchi:* Abstract integrals and linear operators. Proc. Imp. Ac. Tokyo *19* (1943), 169—173.

[3] *V. Pták:* On complete topological linear spaces. Czech. Math. J. *78* (1953), 301—364.

[4] *V. Pták:* Completeness and the open mapping theorem. Bull. Soc. Math. de France *86* (1958), 41—74.

[5] *V. Pták:* On the closed graph theorem. Czech. Math. J. *84* (1959), 523—527.

[6] *A. Robertson, W. Robertson:* On the closed graph theorem. Proc. Glasgow Math. Ass. *3* (1956), 9—12.

[7] *A. Robertson, W. Robertson:* Topological vector Spaces. Cambridge Tracts, 1962.

[8] *A. E. Taylor:* Extensions of a theorem of Hellinger and Toeplitz. Math. Zeitschrift *66* (1956), 53—57.

Резюме

ПРИНЦИП РАВНОМЕРНОЙ ОГРАНИЧЕННОСТИ И ТЕОРЕМА О ЗАМКНУТОМ ГРАФИКЕ

ВЛАСТИМИЛ ПТАК (Vlastimil Pták), Прага

Среди классических результатов функционального анализа имеется несколько теорем, позволяющих доказать непрерывность линейного оператора в случае, когда он удовлетворяет лишь весьма слабым условиям алгебраического или почти алгебраического характера. Напомним, например, классический резултат Хеллингера-Теплица:

Линейное преобразование T, определенное во всех точках пространства Гильберта H и выполняющее равенство $(Tx, y) = (x, Ty)$, является уже непрерывным. Доказательство основано на принципе равномерной ограниченности. Этот же принцип позволяет дать доказательство теоремы о замкнутом графике в пространстве Гильберта.

Результаты автора, касающиеся теорем об открытом отображении и о замкнутом графике в общих топологических линейных пространствах, позволяют получить общую теорему (4,3), содержащую результаты типа Хеллингера-Теплица. Проводится *анализ классического случая, и даются естественные обобщения каждого существенного этапа доказательства.* Оказывается, что исключение метрических методов приводит к упрощению доказательств.