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INTERNAL CHARACTERIZATIONS OF TOPOLOGICALLY COMPLETE SPACES IN THE SENSE OF E. ČECH

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It is proved that a completely regular space P is topologically complete (in the sense of E. ČECH) if, and only if, there exists a complete diameter on the space P, i.e. a non-negative real-valued function on subsets of P satisfying certain axioms. Further internal characterizations are also given.

The terminology and notation of J. KELLEY, General Topology, will be used throughout. All topological spaces are supposed to be completely regular. For convenience we shall use a few not quite common symbols and terms which are listed below.

If $\mathfrak{A}$ is a family of subsets of a space P, then the symbol $\overline{\mathfrak{A}}^P$, or merely $\overline{\mathfrak{A}}$, will be used to denote the family of closures of all sets from $\mathfrak{A}$; in symbols

$$\overline{\overline{\mathfrak{A}}} = \{\overline{A};\ A \in \mathfrak{A}\}\,.$$

If $\mathfrak{A}$ is a family of sets and M is a set, then the symbol $\mathfrak{A} \cap M$ will be used to denote the family $\{A \cap M;\ A \in \mathfrak{A}\}$. The union and the intersection of a family $\mathfrak{A}$ will be denoted by $\bigcup \mathfrak{A}$ and $\bigcap \mathfrak{A}$, respectively. The family of all subsets of a set P will be denoted by exp P. A centered family is a family $\mathfrak{A}$ of sets having the finite intersection property, *i.e.* such that the intersection of every finite sub-family is non-void.

A space R is an extension of a space P if P is a dense subspace of R; R is a proper extension of P if R is an extension and $R \neq P$. The Čech-Stone compactification of a space P will always be denoted by $\beta(P)$.

1. COMPLETE DIAMETERS

Definition 1. A diameter on a space P is a non-negative mapping d of exp P into the extended real line (*i.e.* the values of d are non-negative real numbers or ∞) satisfying the following three conditions:

(d1) If $M \subset N \subset P$, then $d(M) \leqq d(N)$,

(d2) For every $M \subset P$, $d(M)$ is the greatest lower bound of the set of all $d(U)$ with U open and containing M; in symbols: $d(M) = \inf\{d(U);\ U \text{ open},\ U \supset M\}$, (d3) $d(M) = 0$ for every one-point set $M = (x)$.

Clearly, if d is a non-negative function on open sets satisfying (d1) and if we put, for every $M \subset P$,

$$d(M) = \inf \{d(U);\ U \text{ open},\ U \supset M\},$$

then we obtain a function satisfying both (d1) and (d2). From our point of view, among these three conditions the most important one is (d3).

Note 1. If g is a non-decreasing upper-semicontinuous function defined for all numbers $x \geqq 0$ and ∞ and such that $g(0) = 0$, then for every diameter d on a space P the superposition $g \circ d$ is a diameter on P. If d_1 and d_2 are diameters on a space P, then $d = \min (d_1, d_2)$ is a diameter also. If φ is a pseudometric on a space P, then the function d on $\exp P$ defined as follows is a diameter on P:

$$d(\emptyset) = 0\,, \quad d(M) = \sup \{\varphi(x, y);\ x \in M,\ y \in M\}\,.$$

This diameter will be termed generated by φ. If f is a continuous real-valued function on a space P, then $\varphi(x, y) = |f(x) - f(y)|$ is a pseudometric on P and the diameter generated by φ will be termed generated by f.

Definition 2. Let d be a diameter on a space P. A d-Cauchy family is a centered family of subsets of P for which $\inf \{d(M);\ M \in \mathfrak{M}\} = 0$. A diameter d on a space P is said to be complete, if for every d-Cauchy family $\mathfrak{M}$ we have $\bigcap \overline{\mathfrak{M}} \neq \emptyset$. A diameter d is said to be σ-complete, if for every countable d-Cauchy family $\mathfrak{M}$ we have $\bigcap \overline{\mathfrak{M}} \neq \emptyset$.

Proposition 1. *Let d be a diameter on a space P and let $d(M) = 0$. If d is complete, then the closure of M is compact. If d is σ-complete, then the closure of M is countably compact.*

Proof. The space P being completely regular and hence regular, to prove compactness of $\overline{M}$ it is sufficient to show that for every centered family $\mathfrak{M}$ of subsets of M the intersection of $\overline{\mathfrak{M}}$ is non-void. But every centered subfamily of $\exp M$ is a d-Cauchy family. The first assertion follows by completeness of d. The second assertion can be proved analogously.

Clearly every complete diameter is σ-complete. Of course, not every σ-complete diameter is complete. Indeed, $d \equiv 0$ is a diameter on every space. It is easy to see that the diameter $d \equiv 0$ on a space P is complete if and only if P is a compact space. Analogously, $d \equiv 0$ is σ-complete if and only if P is a countably compact space.

Every diameter d generated by a pseudometric satisfies the following condition

$$\text{(d4)} \quad d(M) = d(\overline{M})\,.$$

If d is a diameter on a normal space P and if we define

$$M \subset P \Rightarrow d_1(M) = d(\overline{M})$$

then d_1 is a diameter on P. Since $d_1(M) \geqq d(M)$, if d is complete then d_1 is complete. On the other hand, if d_1 is complete, then d satisfies the following condition

(C) If $\mathfrak{F}$ is a d-Cauchy family consisting of closed sets, then the intersection of $\mathfrak{F}$ is non-void.

If (P, φ) is a complete metric space and if (R, ψ) is a metric space containing P as a dense subspace and $P \neq R$, then φ is not a restriction of ψ, *i.e.* φ has no extension over any proper metrizable extension of P. Conversely, if (P, φ) is a metric space, and φ has no extension over any proper metrizable extension of P, then (P, φ) is a complete metric space. Now we shall investigate relations between complete diameters, "non-extensible" diameters and diameters satisfying condition (C).

Definition 3. Let R be an extension of a space P and let d be a diameter on P. A diameter D on R will be called an extension of d if d is the restriction of D to $\exp P$, *i.e.* if $D(M) = d(M)$ for every $M \subset P$. A diameter d on a space P will be called non-extensible if there exists no extension of d over any proper extension of P.

Proposition 2. *If a diameter d on a space P satisfies the condition* (C), *in particular, if d is complete, then d is non-extensible.*

Proof. Let D be an extension of a diameter d on a space P onto a proper extension R of P. We have to prove that d does not satisfy condition (C). For every $M \subset R$ let us define

$$D_1(M) = \inf \{d(U \cap P);\ U \text{ open in } R,\ U \supset M\}\,.$$

Evidently D_1 satisfies the conditions (d1) and (d2) of Definition 1. Clearly $D_1(M) \leqq D(M)$ for every $M \subset R$. It follows at once that D_1 satisfies the condition (d3) also. Thus D_1 is an extension of d. Let $\mathfrak{A}$ be the family of all closed neighborhoods of a point $x \in R - P$. By (d2) and (d3) $\mathfrak{A}$ is a D_1-Cauchy family. According to the definition of D_1 the family $\mathfrak{A} \cap P$ is a d-Cauchy family. But

$$\bigcap \mathfrak{A} = (x) \subset R - P\,,$$

and consequently, the intersection of $\mathfrak{A} \cap P$ is empty. Thus d does not satisfy condition (C).

The following example shows that a non-extensible diameter may fail to be complete.

Example 1. Let P be the space of all countable ordinals and let J be the set of all isolated points of P. For every $M \subset P$ put $d(M) = 0$ if $M - J$ is countable and $d(M) = 1$ in the other case. It is easy to see that d is a diameter on P. The diameter d is not complete, because if we put $J_\alpha = \{\xi;\ \xi \in J,\ \xi > \alpha\}$, $\alpha \in P$, then $d(J_\alpha) = 0$ for every α in P and

$$\bigcap \{\bar{J}_\alpha;\ \alpha \in P\} = \emptyset\,.$$

On the other hand the diameter d is non-extensible. Indeed, the only proper extension of P is the space R of all ordinals $\alpha \leqq \omega_1$. Now if D is a diameter on R, then

$$D(\{\xi;\ \alpha < \xi < \omega_1\}) = 0$$

for all sufficiently large $\alpha \in P$. Thus D is not an extension of d.

Proposition 3. *If d is a non-extensible diameter on a space P and if S is an extension of P, then P is a G_δ-subset of S.*

Proof. For every positive integer n let U_n be the union of all open subsets U of S for which $d(U \cap P) < 1/n$. The sets U_n are open and $\bigcap_{n=1}^{\infty} U_n = P$. Indeed, clearly d has an extension over $R = \bigcap_{n=1}^{\infty} U_n$ (consider the diameter D_1 from the proof of Proposition 2).

Proposition 4. *Let P be a G_δ-subset of a space R. If there exists a complete diameter on R, then there exists a complete diameter on P.*

Proof. Let D be a complete diameter on a space R. Let d_1 be the restriction of D to $\exp P$. There exists a sequence $\{U_n\}$ of open subsets of R such that

$$P = \bigcap_{n=1}^{\infty} U_n .$$

For every open subset U of P let

$$d_2(U) = \inf \left\{\frac{1}{n} ; \ \overline{U}^R \subset U_n\right\},$$

and for every $M \subset P$ put

$$d_2(M) = \inf \{d(U); \ U \text{ open in } P, U \supset M\},$$
$$d(M) = \max \left[d_1(M), \ d_2(M)\right].$$

Evidently d is a diameter on the space P. We shall prove that d is complete. Let $\mathfrak{M}$ be a d-Cauchy family. By definition of d, $\mathfrak{M}$ is a d_1-Cauchy family, and consequently $\mathfrak{M}$ is a D-Cauchy family. D being a complete diameter, the intersection of $\overline{\mathfrak{M}}^R$ is non-void. Since $\mathfrak{M} \subset \exp P$, to prove $\bigcap \overline{\mathfrak{M}}^P \neq \emptyset$ it is sufficient to show that

$$\bigcap \overline{\mathfrak{M}}^R \subset P .$$

But this is a consequence of the fact that $\mathfrak{M}$ is always a d_2-Cauchy family. Indeed, for every positive integer n there exists an M_n in $\mathfrak{M}$ with

$$d_2(M_n) \leqq d(M_n) \leqq \frac{1}{n},$$

i.e., by definition of d_2, $\overline{M}_n^R \subset U_n$. Thus

$$\bigcap_{n=1}^{\infty} \overline{M}_n^R \subset \bigcap_{n=1}^{\infty} U_n = P ,$$

which completes the proof of proposition 4.

It as easy to prove the following.

Proposition 5. *If d is a complete diameter on a space P and if F is a closed subset of P, then the restriction of d to* $\exp F$ *is a complete diameter on F.*

As an immediate consequence of the preceding theorems and propositions we have the following theorem:

Theorem 2. *The following conditions on a space P are equivalent:*

(1) *There exists a complete diameter on P.*
(2) *There exists a non-extensible diameter on P.*
(3) *P is G_δ in every extension.*
(4) *P is G_δ in some space on which there exists a complete diameter.*
(5) *P is G_δ in $\beta(P)$.*

Definition 4. Every space satisfying the equivalent conditions (1)–(5) of Theorem 2 is said to be topologically complete in the sense of E. Čech.

Note 3. Topologically complete spaces were introduced by E. Čech in [1]. E. Čech defined these spaces by condition (5). The first internal characterization of topologically complete spaces (*i.e.* without reference to extensions) was given in [2] in terms of complete sequences of open coverings. (In [2] topologically complete spaces are called G_δ-spaces.) In [3] an internal characterization using relations of completeness is given. The equivalence of both characterizations will be proved in section 2. For further properties of topologically complete spaces see [4].

The remainder of this section is devoted to the proofs of some assertions concerning diameters. First we shall prove an analogue of Cantor's theorem concerning complete metric spaces.

Proposition 6. *Let d be a complete diameter on a space P. Let $\mathfrak{M}$ be a centred family of sets such that for every $\varepsilon > 0$ there exists an M in $\mathfrak{M}$ and a finite subfamily $\mathfrak{N}$ of P with*

$$(*) \qquad M \subset \bigcup \mathfrak{N}, \; d(N) < \varepsilon \quad \textit{for every} \quad N \in \mathfrak{N} .$$

Then the intersection of $\mathfrak{M}$ is non-void.

Proof. Let $\mathfrak{F}$ be a maximal centered subfamily of exp P containing $\mathfrak{M}$. It is sufficient to prove that $\mathfrak{F}$ is a d-Cauchy family. Given an $\varepsilon > 0$, we have to find an M in $\mathfrak{F}$ with $d(M) < \varepsilon$. There exists a finite subfamily $\mathfrak{N}$ of exp P with (*). $\mathfrak{F}$ being a maximal centered family, for every $M \subset P$ either M or $(P - M)$ belong to $\mathfrak{F}$. It follows that some $N \in \mathfrak{N}$ belongs to $\mathfrak{F}$. Indeed, in the other case $(P - N) \in \mathfrak{F}$ for every $N \in \mathfrak{N}$, and consequently, since $\bigcap \mathfrak{N}$ belongs to $\mathfrak{F}$,

$$M = [\bigcup \mathfrak{N} \cap \bigcap \{(P - N);\; N \in \mathfrak{N}\}] \in \mathfrak{F} ;$$

this is impossible, since $\mathfrak{F}$ is a centered family and the set M is empty. The proof is complete.

As an immediate consequence of the preceding proposition we deduce at once the following

Theorem 3. *Let d be a complete diameter on a space P. For every $M \subset P$ let $\bar{d}(M)$ be the greatest lower bound of the set of all $\varepsilon > 0$ for which there exists a finite subfamily $\mathfrak{N}$ of* exp P *such that $M \subset \bigcap \mathfrak{N}$ and $d(N) < \varepsilon$ for every $N \in \mathfrak{N}$. Then $\bar{d}$ is a complete diameter on the space P.*

Theorem 4. *A σ-complete diameter on a space P satisfies the condition* (C) *if and only if the following condition is fulfilled:*

(K) *If F is a closed subset of P and $d(F) = 0$, then F is a compact subspace of P.*

Proof. Clearly the condition (K) is necessary. Conversely, suppose (K). Let $\mathfrak{F}$ be a d-Cauchy family consisting of closed sets. Without loss of generality we may assume that $\mathfrak{F}$ is a maximal centered family of closed sets. To prove $\bigcap \mathfrak{F} \neq \emptyset$ it is sufficient to show that $\mathfrak{F}$ contains a compact set. Choose $F_n \in \mathfrak{F}$, $n = 1, 2, \ldots$, such that $d(F_n) < 1/n$

Put $F = \bigcap_{n=1}^{\infty} F_n$. From (d1) we have $d(F) = 0$. According to (K), the subspace F is compact. It remains to prove that $F \in \mathfrak{F}$. But if $M \in \mathfrak{F}$, then from the σ-completeness of d it follows at once that $\bigcap_{n=1}^{\infty} (F_n \cap M) \neq \emptyset$, which completes the proof.

Theorem 5. *A diameter on a space P is complete if and only if the following condition is fulfilled:*

(L) *If $\{M_n\}$ is a sequence of non-void subsets of P such that $M_n \supset M_{n+1}$, $n = 1, 2, \ldots$, and $\lim_{n\to\infty} d(M_n) = 0$, then $K = \bigcap_{n\to\infty}^{\infty} \overline{M}_n$ is a non-void compact subspace of P.*

Proof. Let us suppose that the condition is fulfilled.

Let $\mathfrak{M}$ be a d-Cauchy family. To prove $\bigcap \overline{\mathfrak{M}} \neq \emptyset$ it is sufficient to find a compact subspace K of P such that $\overline{\mathfrak{M}} \cap K$ is a centered family. Choose M_n in $\mathfrak{M}$ with $\lim_{n\to\infty} d(M_n) = 0$ and put $K = \bigcap_{n=1}^{\infty} \overline{M}_n$. According to (L), the subspace K of P is compact. If $\mathfrak{N}$ is a finite subfamily of $\mathfrak{M}$, then $\lim_{n\to\infty} d(M_n \cap \bigcap \mathfrak{N}) = 0$ and hence

$$C = \bigcap_{n=1}^{\infty} \overline{M_n \cap \bigcap \mathfrak{N}}$$

is a non-void (compact) subspace of P. But

$$C \subset K \cap \bigcap \overline{\mathfrak{N}}\,.$$

Thus $\overline{\mathfrak{M}} \cap K$ has the finite intersection property. The proof is complete.

A centered family $\mathfrak{M}$ of subsets of a space P will be called completely regular, if for every M in $\mathfrak{M}$ there exists an N in $\mathfrak{M}$ such that N and $P - M$ are completely separated.

Theorem 6. *A diameter d on a space P is non-extensible if and only if the intersection of every d-Cauchy completely regular family is non-void.*

The proof is quite standard (see the proof of proposition 2) and may be left to the reader.

2. COMPLETE SEQUENCES OF OPEN COVERINGS AND RELATIONS OF COMPLETENESS

For convenience we shall recall definitions of complete sequences of open coverings (see [2]) and relations of completeness (see [3]).

Definition 5. A sequence $\{\mathfrak{A}_n\}$ of open coverings of a space P is said to be complete if the following condition is fulfilled:

(c1) If $\mathfrak{A}$ is a centered family of open subsets of P such that $\mathfrak{A} \cap \mathfrak{A}_n \neq \emptyset$ for every $n = 1, 2, \ldots$, then the intersection of $\overline{\mathfrak{A}}$ is non-void.

It is easy to see that this condition is equivalent to the following condition:

If $\mathfrak{M}$ is a centered family of subsets of P such that $\mathfrak{M} \cap \mathfrak{A}_n \neq \emptyset$ for every $n = 1, 2, \ldots$, then the intersection of $\overline{\mathfrak{M}}$ is non-void.

Indeed, if $\mathfrak{A}$ is the family of all open sets A containing an $M \in \mathfrak{M}$, then $\bigcap \overline{\mathfrak{A}} = \bigcap \overline{\mathfrak{M}}$, because in every space any closed set is the intersection of all its closed neighbourhoods.

If there exists a complete sequence $\{\mathfrak{B}_n\}$ of open coverings it is easy to construct a complete sequence $\{\mathfrak{A}_n\}$ of open coverings such that

(c2) $\mathfrak{A}_1 \supset \mathfrak{A}_2 \supset \ldots$.

(c3) If an open set B is contained in some $A \in \mathfrak{A}_n$, then B belongs to $\mathfrak{A}_n$.

Now let $\alpha = \{\mathfrak{A}_n\}$ be a sequence of open coverings satisfying (c1) and (c2). For every open set U put $d(U) = 1$ if $U \notin \mathfrak{A}_1$; in the other case put

$$d(U) = \inf\left\{\frac{1}{n}\,;\ U \in \mathfrak{A}_n\right\}.$$

Now for every $M \subset P$ put

$$d(M) = \inf\{d(U);\ U \text{ open},\ U \supset M\}.$$

It is easy to see that d is a diameter on the space P. This diameter will be denoted by d_α.

Proposition 6. *A sequence $\alpha = \{\mathfrak{A}_n\}$ that satisfies* (c1) *and* (c2) *is complete if and only if the diameter d_α is complete.*

The proof is quite straightforward and may be left to the reader.

Now let d be a diameter on a space P. For every positive integer n let $\mathfrak{A}_n$ be the family of all open sets A for which $d(A) < 1/n$. It is easy to see that $\alpha = \{\mathfrak{A}_n\}$ is a sequence of open coverings. This sequence will be denoted by α_d. It is easy to prove the following proposition:

Proposition 7. *The sequence α_d is complete if and only if d is a complete diameter.*

Using propositions 6 and 7 it is easy to deduce properties of complete sequences of open coverings from the corresponding properties of complete diameters, and conversely.

Definition 6. A relation of completeness on a space P is a binary relation r defined for open subsets of P such that

(r1) $r(A, B)$ implies $A \supset B$,

(r2) If $r(A, B)$, C and D are open, $C \supset A$ and $B \supset D$, then $r(C, D)$,

(r3) If A is a non-void open set, then the family $\{B; r(A, B)\}$ is a base for open subsets of A.

(r4) If $\mathfrak{M}$ is a centered family of sets such that for every positive integer n there exist $A_1, \ldots, A_{n+1}$ with $r(A_i, A_{i+1})$ for all $i = 1, 2, \ldots, n$ and $A_{n+1} \in \mathfrak{M}$, then the intersection of $\mathfrak{M}$ is non-void.

Now let r be a relation of completeness on a space P. For $M \subset P$ let $d(M)$ be the greatest lower bound of the set of all $1/n$ for which there exist $A_1, \ldots, A_{n+1}$, such that $A_{n+1} \supset M$ and $r(A_i, A_{i+1})$, $i = 1, \ldots, n$. If no such n exists, let $d(M) = 1$. It is easy to see that d is a complete diameter on the space P.

Conversely, let d be a complete diameter on a space P. For every pair of open sets U and V let $r(U, V)$ if and only if $U \supset V$ and

$$2d(V) \leqq \min(1, \mathrm{d}(U)) .$$

It is easy to see that r is a relation of completeness on the space P.

Using the preceding two facts it is easy to deduce properties of relations of completeness from corresponding properties of complete diameters. Combining the above results we obtain the following theorem

Theorem 6. *The following conditions on a space P are equivalent:*

(1) *P is topologically complete.*

(2) *There exists a complete sequence of coverings of P.*

(3) *There exists a relation of completeness of P.*

3. DIAMETERS GENERATED BY PSEUDOMETRICS

Let f be a continous mapping of a space P onto a metric space (R, ψ). If for every x and y in P we put

$$(*) \qquad \varphi(x, y) = \psi(f(x), f(y)) ,$$

we obtain a pseudometric φ on P. Conversely, if φ is a pseudometric on a space P, then there exists a continuous mapping f of P onto a metric space (R, ψ) such that (*) holds.

A mapping of a space P onto a space Q is said to be closed if the images of closed subsets of P are closed subsets of Q. We shall prove the following proposition:

Proposition 8. *Let f be a continuous mapping of a space P onto a space (R, ψ) and let φ be the pseudometric on P defined by (*). If the diameter d on P defined by φ (see Note 1) is σ-complete, then the metric space (R, ψ) is complete, f is a closed mapping and the inverses of points are countably compact. Moreover, if d is complete, then the inverse inverses of points are compact.*

Proof. First let us suppose that the diameter d is complete. Evidently (R, ψ) is a complete metric space. Now let us suppose that F is a closed subset of P and that there exists a point y in $\bar{\Phi} - \Phi$, where $\Phi = f[F]$. Let $\mathfrak{N}$ be a countable base at the point y. Denote by $\mathfrak{M}$ the family of all $f^{-1}[N]$, $N \in \mathfrak{N}$. Evidently $\mathfrak{M}$ is a d-Cauchy family. Since $y \in \bar{\Phi} - \Phi$, $\mathfrak{M} \cap F$ is a d-Cauchy family. According to the σ-completeness of d we have

$$F \cap \bigcap \bar{\mathfrak{M}} \neq \emptyset .$$

But $\bigcap \bar{\mathfrak{N}} = (y)$, and by continuity of f

$$f[\bigcap \bar{\mathfrak{M}}] \subset \bigcap \bar{\mathfrak{N}}$$

which is a contradiction. Thus f is a closed mapping. The countable compacteness of inverses of points is a consequence of proposition 1. If d is complete, then the compacteness of inverse of points also follows from Proposition 1.

Proposition 9. *Let f be a closed and contiunous mapping of a space P onto a metric space (R, ψ) such that the inverses of points are compact. If (R, ψ) is a complete metric space, and if φ is defined by (*), then the diameter d generated by φ is complete.*

Proof. Let $\mathfrak{M}$ be a maximal d-Cauchy family. The family of all $f[M]$, $M \in \mathfrak{M}$, will be denoted by $\mathfrak{N}$. From (*) it follows that $\mathfrak{N}$ is a Cauchy family in (R, ψ). Thus $\bigcap \bar{\mathfrak{N}} = (y)$, when y is a point of R. Consider the compact subspace $K f^{-1}[y]$ of P. To prove that the intersection of $\bar{\mathfrak{M}}$ is non-void, it is sufficient to show that $\bar{\mathfrak{M}} \cap K$ is a centered family. $\mathfrak{M}$ being a maximal centered family, it is sufficient to show that $\bar{M} \cap K \neq \emptyset$ for every M in $\mathfrak{M}$. Since f is a closed mapping, we have $f[\bar{M}] = \overline{f[M]}$. Thus $\bar{M} \cap K \neq \emptyset$, since $f[K] \subset f[\bar{M}]$. The proof is complete.

As a consequence of the preceding two Propositions we have the following theorem:

Theorem 7. *Given a space P, there exists a complete diameter generated by a pseudometric if and only if there exists a continuous and closed mapping of P onto a metrizable topologically complete space such that the inverses of points are compact.*

Let us recall that a Hausdorff space P is said to be paracompact if every open cover of P is refined by and open locally-finite cover. It is easy to see that a Hausdorff space P is paracompact if and only if every finitely additive open cover of P is refined by an open locally-finite cover. From this note there follows at once the following well-known

Proposition 10. *Let f be a continuous and closed mapping of a Hausdorff space P onto a space Q such that the inverses of points are compact. If Q is a paracompact space, then P is a paracompact space.*

Proof. Let $\mathfrak{A}$ be a finitely additive open cover of P. Let $\mathfrak{B}$ be the family of all subsets of Q of the form $B(A) = Q - f[P - A]$ for all A in $\mathfrak{A}$. The mapping f being closed, the sets $B(A)$ are open. Since $\mathfrak{A}$ is additive and the inverses of points are compact, the family $\mathfrak{B}$ covers Q. Since Q is paracompact, a locally-finite open covering $\mathscr{C}$ of Q refines $\mathfrak{B}$. The family of all sets of the form $f^{-1}[C]$, $C \in \mathscr{C}$, is a locally-finite open covering of P refining $\mathfrak{A}$. According to the above note the space P is paracompact.

Since every metrizable space is paracompact, from the preceding Proposition and the Theorem 7 it follows at once that if there exists a complete diameter generated by a pseudometric on a space P, then P is a paracompact space.

Conversely, let P be a topologically complete paracompact space. We shall prove that on P there exists a complete diameter generated by a pseudometric. P being complete, there exists a complete sequence $\{\mathfrak{A}_n\}$ of open covering of P. Since P is paracompact, every covering $\mathfrak{A}_n$ is normal, *i.e.*, there exist sequences $\{\mathfrak{B}_K(n)\}_{k=1}^{\infty}$ of open coverings of P such that $\mathfrak{B}_1(n)$ refines $\mathfrak{A}_n$ and $\mathfrak{B}_{K+1}(n)$ is a star-refinement of $\mathfrak{B}_K(n)$. Let us define by induction: $\mathfrak{B}_1 = \mathfrak{B}_1(1)$, $\mathfrak{B}_{n+1}$ is the family of all sets of the form $B \cap A$, with $B \in \mathfrak{B}_n$ and $A \in \mathfrak{B}_{n+1}(n + 1)$. Evidently $\{\mathfrak{B}_n\}$ is a normal sequence of open covering, *i.e.*, $\mathfrak{B}_{n+1}$ is a star-refinement of $\mathfrak{B}_n$. Since every $\mathfrak{B}_n$ refines $\mathfrak{A}_n$, the sequence $\{\mathfrak{B}_n\}$ is complete. The sequence $\{\mathfrak{B}_n\}$ being normal, according to Frink's lemma there exists a pseudometric φ in P such that for every n there is an $\varepsilon > 0$ such that the covering consisting of all φ-spheres of the diameter less than ε refines $\mathfrak{B}_n$, and for every $\varepsilon > 0$ there is an n such that $\mathfrak{B}_n$ refines the covering consisting of all φ-spheres of diameter less than ε. Since $\{\mathfrak{B}_n\}$ is a complete sequence, it follows at once that the diameter generated by φ is complete; this ends the proof.

Combining the the preceding result with Theorem 7 we obtain:

Theorem 8.[1]) *The following conditions on a space P are equivalent*:

(1) *On P there exists a complete diameter generated by a pseudometric.*

(2) *P is topologically complete and paracompact.*

(3) *There exists a closed and continuous mapping of P onto a complete metrizable space such that the inverses of points are compact.*

4. FURTHER SPECIAL TYPES OF DIAMETERS

It is easy to see that a space P is compact if and only if the constant function $d \equiv 0$ on exp P is a complete diameter. It is easy to prove that a space P is locally compact if and only if there exists a complete diameter on P such that $d(M)$ is either 0 or 1. Indeed, if we put

$$d(M) = \begin{cases} 0 \text{ if } \overline{M} \text{ is a compact,} \\ 1 \text{ in the other case,} \end{cases}$$

[1]) *Added in proof:* For further information see ZD. FROLÍK, On the Topological Product of Paracompact Spaces. Bull. Acad. Pol. Sci. 1960, 747—750.

then P is locally compact if and only if d is a complete diameter on P (if d is a diameter, then d is a complete diameter).

Now we shall investigate diameters generated by continuous real-valued functions. Let d be the diameter on a space P generated by a continuous real-valued function f (See Note 1). Let $\bar{d}$ be the diameter defined in Theorem 5. If $d(M)$ is finite, then $\bar{d}(M) = 0$. Indeed, let $\varepsilon > 0$. If $d(M)$ is finite, then the set $f[M]$ is bounded, and consequently there exists a finite covering $\mathfrak{N}$ of $f[M]$ consisting of intervals with length less than ε. Clearly

$$I \in \mathfrak{N} \Rightarrow d(f^{-1}[I]) < \varepsilon$$

and

$$M \subset \bigcap \{f^{-1}[I];\ I \in \mathfrak{N}\}\,.$$

Thus $\bar{d}(M) < \varepsilon$, which completes the proof. From this fact and Theorem 5 we deduce at once the following

Theorem 9. *A space is compact if and only if there exists a complete diameter generated by a bounded continuous real-valued function. A space is locally compact and σ-compact (the union of a countable number of compact subsets) if and only if there exists a complete diameter generated by a continuous real-valued function.*

If $\{f_n\}$ is a sequence of continuous functions and if we define a pseudometric φ as follows,

$$\varphi(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|f_n(x) - f_n(y)|}{1 + |f_n(x) - f_n(y)|}\,,$$

then the diameter d generated by φ is said to be generated by the sequence $\{f_n\}$. The following theorem may then be proved (see [4]).

Theorem 10. *A space P is the intersection of a countable number of N-sets in the Čech-Stone compactification $\beta(P)$ of P if and only if there exists a complete diameter d generated by a sequence of continuous functions.*

Note. A subset M of a space P is said to be a N-set of P if there exists a continuous real-valued continuous function f on P with

$$M = \{x;\ x \in P,\ f(x) \neq 0\}\,.$$

References

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[4] *Z. Frolík:* Relations of Completeness. Czech. Math. J. *12* (1962), 375–381.

[5] *Z. Frolík:* Locally Topologically Complete Spaces. Czech. Math. J. *12* (1962), 346–355.

Резюме

ВНУТРЕННИЕ ХАРАКТЕРИЗАЦИИ ТОПОЛОГИЧЕСКИ ПОЛНЫХ ПРОСТРАНСТВ В СМЫСЛЕ Э. ЧЕХА

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Вполне регулярное топологическое пространство P называют топологически полным в смысле Э. Чеха, если P является G_δ-множеством в своем чеховском расширении $\beta(P)$. В настоящей работе топологически полные пространства характеризуются при помощи так наз. полного диаметра. Диаметром на пространстве P называется такая неотрицательная вещественная функция d, определенная для всех $M \subset P$, что

(1) $$M \subset N \Rightarrow d(M) \leqq d(N)\,,$$

(2) $$d(M) = \inf \{d(U);\; U \supset M,\; U \text{ открыто}\}\,,$$

(3) $$d((x)) = 0 \quad \text{для всех} \quad x \in P\,.$$

Центрированная система $\mathfrak{A}$ подмножеств пространства P называется системой Коши относительно диаметра d, если

$$\inf \{d(A);\; A \in \mathfrak{A}\} = 0\,,$$

диаметр d на пространстве P назовем полным, если для всякой системы Коши $\mathfrak{A}$

$$\bigcap \{\bar{A};\; A \in \mathfrak{A}\} \neq \emptyset\,.$$

Оказывается, что вполне регулярное пространство P является топологически полным в смысле Э. Чеха тогда и только тогда, если существует некоторый полный диаметр на P. Далее в работе рассматриваются некоторые другие внутренние характеризации топологически полных пространств и некоторые специальные типы диаметров. Напр., если для некоторой псевдометрики φ в пространстве P положим для $M \subset P$

$$d(M) = \sup \{\varphi(x, y)\,;\quad x \in M,\; y \in M\}\,,$$
$$d(\emptyset) = 0\,,$$

то d будет диаметром на P, который мы назовем диаметром, порожденным псевдометрикой φ. Оказывается, что для существования на вполне регулярном пространстве P полного диаметра, порожденного псевдометрикой, необходимо и достаточно, чтобы P было топологически полным паракомпактным пространством.