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PRIME IDEALS OF THE CARTESIAN PRODUCT OF TWO SEMIGROUPS

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The purpose of this paper is to give necessary and sufficient conditions for a subset of $S \times T$ to be a prime ideal.

A *semigroup* is a non-empty set on which an associative multiplication is defined. If S and T are semigroups, then by $S \times T$ we mean the semigroup consisting of the Cartesian product $S \times T$ of the sets S and T with coordinatewise multiplication. The semigroup $S \times T$ is called the *Cartesian product of the semigroups* S and T. A non-empty subset I of a semigroup S is called an (two-sided) *ideal* of S if $xy, yx \in I$ for all $x \in I$, $y \in S$; if in addition its complement in S is a semigroup (and hence $I \neq S$), then I is called a *prime ideal* of S. We also call the empty set a prime ideal (cf. Definitions 2, 2a, [1]). If A and B are sets, then $A - B$ will denote the set of all elements of A which are not contained in B. A simple inductive argument generalizes the following theorem to the case of any finite number of semigroups.

Theorem. *Let S and T be semigroups. Then a set L is a prime ideal of $S \times T$ if and only if $L = (I \times T) \cup (S \times J)$ where I and J are prime ideals of S and T, respectively.*

Proof. We first prove sufficiency. Let I and J be prime ideals of S and T, respectively, and let

$$L = (I \times T) \cup (S \times J) .$$

If both I and J are empty, then L is also empty and is thus a prime ideal of $S \times T$. Suppose that at least one of the prime ideals I, J is not empty, so that L is also not empty. Let (x, u) and (y, v) be any elements of L and $S \times T$, respectively. Then by definition of L, either $x \in I$ or $u \in J$. Suppose that $x \in I$. (The case $u \in J$ is treated similarly.) Then $xy, yx \in I$ and thus

$$(x, u)(y, v) = (xy, uv) \in I \times T \quad \text{and} \quad (y, v)(x, u) = (yx, vu) \in I \times T .$$

Consequently,

$$(x, u)(y, v), (y, v)(x, u) \in L ,$$

and hence L is a two-sided ideal of $S \times T$. The sets $S - I$ and $T - J$ are not empty and are semigroups. Hence $(S - I) \times (T - J)$ is a semigroup. But

$$(S - I) \times (T - J) = S \times T - L$$

and hence the complement of L is a semigroup. Thus L is a prime ideal of $S \times T$.

We now prove necessity. Let L be a prime ideal of $S \times T$. If L is empty, take I and J to be empty. Suppose then that L is not empty. Let (x, u) be any element of L. We assert that either $\{x\} \times T$ or $S \times \{u\}$ is contained in L. For if we suppose the contrary, then there exists an element $v \in T$ such that $(x, v) \notin L$ and an element $y \in S$ such that $(y, u) \notin L$. We have

$$(1) \qquad (x, v)(y, u)(x, v)(y, u) = (xyxy, vuvu) = (xy, v)(x, u)(y, vu).$$

The expression on the left of (1) is in $S \times T - L$ since $S \times T - L$ is a semigroup; but the expression on the right is in L since $(x, u) \in L$ and L is a two-sided ideal. These two statements are plainly incompatible. This proves the assertion.

Let $I = \{x \in S \mid \{x\} \times T \subseteq L\}$ and $J = \{u \in T \mid S \times \{u\} \subseteq L\}$. Then

$$L = (I \times T) \cup (S \times J).$$

For if $(x, u) \in L$, then either $\{x\} \times T$ or $S \times \{u\}$ is contained in L, which implies that either $x \in I$ or $u \in J$ and thus in either case

$$(x, u) \in (I \times T) \cup (S \times J).$$

The reverse inclusion is obvious.

We now show that I is a prime ideal of S. (One proves similarly that J is a prime ideal of T.) If I is empty, it is a prime ideal by definition. Suppose that I is not empty. The set $S - I$ is not empty for otherwise $S = I$ and hence $L = S \times T$, which is impossible since L is a prime ideal of $S \times T$. Similarly $T - J$ is not empty. Let x and y be any elements of $S - I$, and let u be an element of $T - J$. Then

$$(x, u), (y, u) \in (S - I) \times (T - J).$$

Since L is a prime ideal of $S \times T$, its complement $(S - I) \times (T - J)$ is a semigroup and thus

$$(x, u)(y, u) = (xy, u^2) \in (S - I) \times (T - J).$$

Hence $xy \in S - I$ and thus $S - I$ is a semigroup.

Let x, y, and u be any elements of I, S, and $T - J$, respectively. Then $(x, u) \in L$ since $x \in I$. It follows that

$$(x, u)(y, u) = (xy, u^2) \in L \quad \text{and} \quad (y, u)(x, u) = (yx, u^2) \in L,$$

since L is a two-sided ideal. Since $T - J$ is a semigroup, we have $u^2 \in T - J$. But then

$$(xy, u^2), (yx, u^2) \in L$$

implies that

$$(xy,u^2),\quad (yx,u^2) \in I \times T \quad \text{and thus} \quad xy, yx \in I\,.$$

Hence I is a two-sided ideal, and therefore I is a prime ideal of $S \times T$.

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Резюме

ПРОСТЫЕ ИДЕАЛЫ ПРЯМОГО ПРОИЗВЕДЕНИЯ ДВУХ ПОЛУГРУПП

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Простым идеалом I полугруппы S называется или пустое подмножество, или двусторонний идеал $I \neq S$, для которого $S - I$ является полугруппой.

В статье доказываестя следующая теорема:

Пусть S, T — полугруппы. Множество $L \subset S \times T$ является простым идеалом тогда и только тогда, если $L = (I \times T) \cup (S \times J)$, где I — простой идеал полугруппы S, и J — простой идеал полугруппы T.