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BERNARD BOLZANO

(*October 5, 1781 — December 18, 1848*)

October 5th, 1961 marked the 180th anniversary of the birth of Prague-born BERNARD BOLZANO; Czechoslovak scientific institutions paid respect to this day by holding a special meeting. Mathematicians throughout the world know Bolzano primarily as one of the pioneers of that direction of mathematical research in the nineteenth century which aimed at a critical revision of the basic concepts of mathematical analysis. In this respect Bolzano's work had much in common with those papers of his great contemporary, A. CAUCHY, which dealt with the basic theorems of analysis. However, there is a substantial difference between the works of these two mathematicians. Bolzano was not merely a mathematician but also a philosopher and logician. He said himself that mathematics interested him primarily as a branch of philosophy and a means towards the practice of correct thought. His work, inasfar as it concerned mathematical analysis, was therefore devoted almost exclusively to obtaining a better foundation of its most basic parts and cannot be compared with the tremendous breadth of Cauchy's work; this certain degree of one-sidedness is probably in connection with the lack of skill in calculations, with which we often meet in Bolzano's work. On the other hand, in studying the basic concepts of mathematical analysis and their interrelations Bolzano goes much further and deeper than any of his contemporaries. The character of his investigations on the foundations of analysis is seen best perhaps in the two works mentioned below.

The first is „*Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege*", published in 1817. This treatise contains a proof of the theorem that a function[1]) $f(x)$, continuous in a closed interval $[a, b]$, takes all the values between $f(a)$ and $f(b)$ inside this interval (Bolzano's formulation is slightly different). In the introduction Bolzano shows why this "intuitively evident" theorem requires proof and criticizes some incorrect "demonstrations". His concept of continuity (in an interval) is identical with today's concept and is the same as that used by Cauchy in his *Cours d'Analyse* (1821). Bolzano's proof is based on the theorem of the existence of the least upper bound of a bounded non-empty set of numbers. The theorem on the least upper bound is, in its turn, derived from the so-called Cauchy

[1]) In the present article "function" always means a real function of a real variable.

(or Bolzano-Cauchy) condition for the convergence of a sequence of real numbers (in Cauchy's work this condition appeared four years later). Bolzano also tries to prove the sufficiency of this condition; here, of course, there is a gap in his proof since the theory of real numbers did not exist at his time.

This work shows Bolzano's deep comprehension of the basic questions of analysis. The same circle of problems is dealt with on a higher level and to a greater extent in a paper found amongst his manuscripts by M. JAŠEK around 1920 and published in 1930 under the title "*Functionenlehre*" by the Royal Bohemian Society of Sciences. This paper was to have formed a part of a large work intitled "*Grössenlehre*". "*Functionenlehre*", which was probably written in the early thirties, consists of a foreword and two chapters, the first dealing with continuity and the second with derivatives. The depth of conception as well as the systematic character and clarity of the exposition is apparent particularly in the chapter on the continuity of functions of one variable. In contrast to the earlier formulations by Bolzano and Cauchy, here the former introduces not only continuity in an interval but also in a point and even continuity on the right and left. That he was fully aware of the importance of this localization of continuity is clear from the fact that he gives an example of a function which is discontinuous at each point as well as an example of a function which is continuous exactly at one point. Bolzano then proves a number of theorems on continuous functions in the following order (we have written them in modern notation and terminology):

I. *If* $\limsup_{x \to c} |f(x)| = \infty$, *$f$ is not continuous at the point c.*

II. *A function which is continuous in a closed interval* $[a, b]$ *is bounded in it.*

III. *If f is continuous in a closed interval* $[a, b]$, *and if a sequence of numbers* $x_n \in [a, b]$ *exists such that* $\lim_{n \to \infty} f(x_n) = c$, *then f takes the value c in* $[a, b]$.

IV. *Each function, continuous in a closed interval* $[a, b]$, *assumes there greatest and smallest values* (*it is interesting how careful are Bolzano's formulations*: "*greatest in the sense that no other value is greater*").

V. *A function f, continuous in an interval J, has the following property*

$$(A): \textit{If } \alpha \in J, \quad \beta \in J, \quad f(\alpha) \neq f(\beta),$$

then f takes all values between $f(\alpha)$ and $f(\beta)$ in the open interval (α, β). (*This is of course the theorem from „Rein analytischer Beweis*".)

The proofs of the theorems are built up systematically from the theorem on the least upper bound and the so-called Bolzano-Weierstrass theorem: *Each bounded sequence of numbers has an accumulation point.* Bolzano was fully aware of the importance of the compactness of a closed interval: he shows that theorems III and IV are not valid for open intervals. He also emphasizes that property (A) is not sufficient for a function to be continuous in J — but here his arguments are not satisfactory.

The theorems mentioned so far have a common characteristic: they show that besides the properties stated in the definition of continuity all continuous functions have certain other properties, which agree with the naive conception of "continuity". But Bolzano's study of the notion of continuity follows also another, almost opposite direction. He constructs examples of functions which are continuous and nevertheless possess some properties which disagree with the naive conception of continuity and therefore appeared paradoxical in his time. The most important and best known is the so-called *Bolzano function.*

This function F is defined in the interval $[a, b]$ as the limit of a sequence of piece-wise linear functions $y_1, y_2, \dots$, the graphs of which are obtained from a segment by iteration of a certain simple geometric construction. Bolzano shows that the function F, although continuous in $[a, b]$, is not monotonic in any interval and has not a finite derivative in any point of a certain set dense in $[a, b]$. We know today that F has no (neither finite nor infinite) derivative in any point of the interval (a, b), but Bolzano neither proved nor asserted it. Bolzano's proof contains one serious gap: he deduces the continuity of the function F simply from the fact that F is the limit of certain continuous functions y_n. He thus did not arrive at the concept of uniformity or recognize its importance; some serious mistakes thus appeared in his work. This is the more surprising since in other places he emphatically warns against errors of this kind.

Bolzano's deep researches of the general properties of continuous functions, and particularly the construction of functions with "paradoxical" properties, show him as the immediate forerunner of the modern theory of real functions. It is a pity that his "Functionenlehre" was not published earlier, at a time when it could have still influenced and accelerated the development of this branch of mathematics.

To complete Bolzano's theory of functions of one real variable, one gap had to be filled, i. e. the theory of real numbers had to be created. It is well known that several such theories were put forward around 1870 (Cantor, Dedekind, Méray, Weierstrass). As K. Rychlík reports (*Theorie der reellen Zahlen im Bolzano's handschriftlichen Nachlasse,* Czechosl. Math. Journal 7(*82*), pp. 553—567, 1957), Bolzano left behind a manuscript containing an attempt at a systematic theory of real numbers; it culminates in a proof of the sufficiency of the Bolzano-Cauchy criterion for the convergence of a sequence, of the theorem on the least upper bound, and of Dedekind's theorem. Bolzano's attempt is remarkable for his time but not quite successful. The fault lies in the fact that he tries to include in his theory all the "expressions" containing infinitely many arithmetical operations with rational numbers ("unendliche Zahlenausdrücke"); under restriction, for example, to series with rational terms, Bolzano's attempt could probably be brought to a successful conclusion.

Just as Bolzano's interest in philosophy and logics is seen in his mathematical work, so his mathematical turn of mind appears in his large book on logic „*Wissenschaftslehre*" (1837); Bolzano's importance as the forerunner of mathematical logic is universally recognized. In this direction, too, new discoveries can be expected from what Bolzano left behind, as is shown in K. Rychlík's report „*Betrachtungen aus der Logik in Bolzano's handschriftlichem Nachlasse*" (Czechoslov. Math. Journal *8* (*83*), pp. 197—202, 1958).

The large number of manuscripts left by Bolzano has not yet been fully explored; many sketches and drafts will perhaps remain undecipherable for ever. However, a careful study of this wealth of material can be expected to throw new light on the work of this deep and original thinker.

Bolzano was not only a mathematician, logician and philosopher but also an outstanding social thinker. After the French Revolution, in the suffocating atmosphere of the Austrian absolute monarchy when reaction was growing even stronger he became one of the first to advance new ideas in Bohemia, and paid for this by being demoted from his post of professor at the university. However, his activities along these lines did not cease, and at the beginning of the thirties of the 19th century he published the first and only comprehensive socialist *Utopia* to appear in our country.

Bolzano started from a criticism of the contemporary state, and regarded the inequality of classes and property to be its greatest evil. He created the image of the ideal state which was to be a Republic where the function of parliament was to be replaced by a plebiscit. The state was to be the governing body also in the field of the national economy, the owner of the land, of the means of production and of part of the consumers' goods. The development of production and particularly of machine production would then have unsuspected possibilities. Trade and finance was to be in the hands of the state. Children were to be educated and fed at the state's expense, and the state also was to run medical services for its citizens and to provide for those unable to work. The weakness of the progressive sections of Bohemian society is reflected in Bolzano's writings in his disbelief that his Utopia will come true, in his fundamental rejection of revolutionary methods, and in his belief that class conflicts could be solved by the method of persuasion. Because of the backwardness of economic conditions Bolzano did not believe that the needs of the whole population could be fully satisfied, and hence he preached the necessity of ascetic restriction and of the equalisation of consumption. On the whole Bolzano's Utopia belongs in content and form to the sphere of socialist utopias of the 18th century, being probably closest to that of MABLY.

Much of the teaching of the great humanist Bolzano certainly belongs to the past. However, in the present tense international conditions whole parts of his work speak to us with great urgency. At the time when an irresponsible decision could bring destruction to large sections of humanity and seriously threaten civilisation, Bolzano's conviction of the power of human reason, the possibility of agreement between peoples and the sensible and universally beneficial arrangement of the world become very topical. In Bolzano this opinion was not only the result of rational considerations but also of painful personal experiences. The Napoleonic Wars which had devastated Europe for a decade broke out when Bolzano was a child, and certainly influenced to a great extent his teaching. For this reason his *Adresses* contain such vivid descriptions of the horrors of war in which on the one hand thousands of people die by the sword and on the other hand thousands succumb to hunger, frost and plague. However, in the darkest periods Bolzano believed in the progressive development of society, in the victory of human reason and healthy optimism, and said prophetically (1811):

„*Es wird — ich sage es mit aller Zuversicht — es wird eine Zeit erscheinen, wo man den Krieg, diess widersinnige Bestreben, sein Recht durchs Schwert zu beweisen eben so allgemein verabscheuen wird, wie man den Zweikampf jetzt schon verabscheuet!*“

It is symbolic that today, exactly 150 years later, this question has so much come to the fore, and that not all members of human society have yet been convinced that war is an impermissible method of solving conflicts between states. In such a situation, responsibility for the development of the world lies mainly on those who preserve healthy reasoning so stressed by Bolzano. The role of the intelligentsia to whose conscience Bolzano primarily appealed is increasing. Only when there is agreement and all people of good will unite, when there is peace can Bolzano's motto „*Fortschreiten soll man*“ be realised. Only under such conditions can real happiness spread over the earth as described by Bolzano:

„*Unser Geschlecht wird endlich auch immer weiterschreiten in wahrer Glückseligkeit, das ist, das Heer der Leiden, welche uns drücken, wird in der Folge der Zeiten sich immer mehr und mehr verringern, je länger, je wirksamer werden die Mittel sein, die man zu ihrer Abhilfe erfunden haben wird, die Zahl derjenigen aus uns, die sich unglücklich fühlen, wird immer kleiner werden, und immer grösser die Anzahl jener, die eine naturgemässe Befriedigung ihrer menschlichen Bedürfnisse auf Erden finden, die ruhig und vergnügt ihr Dasein zubringen, und alt und lebenssatt dem Tode ohne Murren in die Arme sinken, weil auch sie sagen können: dass sie gelebt, und dieser Erde Glück genossen haben.*“ (1811).

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