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LOCALLY CONNECTED TOPOLOGIES ASSOCIATED WITH A GIVEN COMPLETE METRIZABLE TOPOLOGY

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It is proved that if (P, τ) is a complete metrizable[1]) space of a countable order of disconnectedness, then $(P, m(\tau))$ is a complete metrizable space and $m(\tau) = s(\tau)$.

Let τ be a topology for a set P. Let us denote by $\mathsf{C}(\tau)$ the family of all connected subspaces of (P, τ). The family of all locally connected sets from $\mathsf{C}(\tau)$ will be denoted by $\mathsf{L}\,\mathsf{C}(\tau)$. Finally, the family of all compact connected and locally connected subspaces of (P, τ) will be denoted by $\mathsf{A}(\tau)$. The notation and terminology of [1] will be used throughout.

According to [1], 1.2, there exists a locally connected topology $s(\tau)$ for the set P such that $s(\tau) \leqq \tau$ (that is, $s(\tau)$ is finer than τ) and if τ_0 is a locally connected topology for the set P with $\tau_0 \leqq \tau$, then $\tau_0 \leqq s(\tau)$.

Let us denote by $m(\tau)$ the finest among all the topologies for P which induce the same topology as τ on every $M \in \mathsf{L}\,\mathsf{C}(\tau)$. According to [1], the topology $m(\tau)$ is locally connected.

If U is an open subset of (P, τ) and if $x \in U$, let $S_1(x, U)$ be the union of all $M \in \in \mathsf{L}\,\mathsf{C}(\tau)$, $x \in M \subset U$, and by induction, let $S_{n+1}(x, U)$ be the union of all $M \in \mathsf{L}\,\mathsf{C}(\tau)$ satisfying $M \subset U$ and $S_n(x, U) \cap M \neq \emptyset$. Put

$$S_\infty(x, U) = \bigcup_{n=1}^{\infty} S_n(x, U) .$$

Let us denote by $c(\tau)$ the topology for which the family

$$\{S_\infty(x, U);\ U \text{ is an open neighborhood of } x\}$$

is a local base at x. According to [1] the topology $c(\tau)$ is locally connected and

$$m(\tau) \leqq c(\tau) \leqq s(\tau) .$$

In general $m(\tau) < c(\tau)$. However, if τ is a complete metrizable topology then

[1]) A space P will be called complete metrizable if there exists a metrix φ generating the topology of P such that (P, φ) is a complete metric space.

$m(\tau) = c(\tau)$. In the present note we shall prove that $m(\tau) = s(\tau)$ in the case when τ is a complete metrizable topology of a countable order of disconnectedness. Moreover, in this case $(P, m(\tau))$ is complete metrizable.

The topology $s(\tau)$ may be obtained by iterating the operator η^* defined as follows: Let η be a topology for a set P. The family of all η-components of all η-open sets is an open base for η^*. Let us define $\tau^0 = \tau$ and for every ordinal $\alpha \geqq 1$,

$$\tau^\alpha = \inf \{(\tau^\beta)^*, \ \beta < \alpha\} .$$

It may be shown that $s(\tau) = \inf \tau^\alpha$. The least ordinal α for which $s(\tau) = \tau^\alpha$ is said to be the order of disconnectedness of the topology τ.

Theorem 1. *If τ is a complete metrizable topology for a set P of a countable order of disconnectedness, then $s(\tau)$ is complete metrizable.*

First we shall prove the following

Lemma 1. *If τ is a complete metrizable topology then τ^* is a complete metrizable topology.*

Proof. Let φ be a complete metric for the space (P, τ). Without loss of generality we may assume that $\varphi(x, y) \leqq 1$ for every x and y in P. According to [1], theorem 1.11, the topology τ^* is generated by the metric ϱ defined as follows: Let $x, y \in P$; if there exists no $M \in \mathsf{C}(\tau)$ containing both x and y, then $\varrho(x, y) = 1$; in the opposite case $\varrho(x, y)$ is the greatest lower bound of the set of diameters (with respect to φ) of all $M \in \mathsf{C}(\tau)$ containing both x and y. We shall prove that (P, ϱ) is a complete metric space. Let $\{x_n\}$ be a Cauchy sequence with respect to ϱ. Since $\varphi(x, y) \leqq \varrho(x, y)$, $\{x_n\}$ is a Cauchy sequence with respect to φ. Thus there exists a point x in P such that

$$(*) \qquad \lim_{n \to \infty} \varphi(x_n, x) = 0 .$$

We shall prove that

$$(**) \qquad \lim_{n \to \infty} \varrho(x_n, x) = 0 .$$

Without loss of generality we may assume

$$\varrho(x_n, x_{n+1}) < 2^{-n} \quad (n = 1, 2, \ldots) .$$

Let us choose C_n in $\mathsf{C}(\tau)$ such that the diameter (with respect to φ) of C_n is less than 2^{-n} and $x_n \in C_n$, $x_{n+1} \in C_n$. If is easy to see that the sets

$$K_n = \bigcup\{C_k; \ k = n, n + 1, \ldots\}$$

are connected and the diameter of K_n $(n = 1, 2, \ldots)$ is less than 2^{-n+1}. It follows that the diameter of the τ-closure L_n of K_n is less than 2^{-n+1} and $L_n \in \mathsf{C}(\tau)$. According to (*), the point x belongs to every L_n. Thus by definition of $\varrho(x, y)$ we have

$$\varrho(x_n, x) \leqq 2^{-n+1} ,$$

which establishes (**) and completes the proof of lemma 1.

Proof of Theorem 1. Let α be the order of disconnectedness of the topology τ. By our assumption, $\alpha < \omega_1$ and the topology $\tau^0 = \tau$ is complete metrizable. Let $1 \leqq \alpha_0 \leqq \alpha$ and let us suppose that the topologies τ^β, $\beta < \alpha_0$, are complete metrizable. By definition of τ^{α_0},

$$\tau^{\alpha_0} = \inf \{(\tau^\beta)^*; \beta < \alpha\} .$$

Since $\beta_1 \geqq \beta_2$ implies $(\tau^{\beta_1})^* \leqq (\tau^{\beta_2})^*$ and $\eta_1 \leqq \eta_2$ implies $\eta_1^* \leqq \eta_2^*$, we have that $\beta_1 \geqq \beta_2$ implies $(\tau^{\beta_1})^* \leqq (\tau^{\beta_2})^*$. Thus we may choose ordinals β_n, $n = 1, 2, \ldots$, such that

$$(*) \qquad \tau^{\alpha_0} = \inf \{(\tau^{\beta_n})^*; \; n = 1, 2, \ldots\} .$$

Since the topologies τ^{β_n} are complete metrizable, by lemma 1 we may choose metrics ϱ_n for the set P such that the metric space (P, ϱ_n), $n = 1, 2, \ldots$, is complete, $\varrho_n(x, y) \leqq 1$ and ϱ_n generates the topology τ^{β_n}. For x and y in P put

$$(**) \qquad \varrho(x, y) = \sum_{n=1}^{\infty} 2^{-n} \varrho_n(x, y) .$$

By $(*)$, ϱ is a metric for the space (P, τ^{α_0}). From $(**)$ it follows at once that ϱ is a complete metric. Indeed, let $\{x_n\}$ be a Cauchy sequence with respect to ϱ, *i. e.*

$$\lim_{\substack{n \to \infty \\ m \to \infty}} \varrho(x_n, x_m) = 0 .$$

It follows that

$$\lim_{\substack{n \to \infty \\ m \to \infty}} \varrho_k(x_n, x_m) = 0 \quad (k = 1, 2, \ldots) .$$

The metrics ϱ_k being complete, we may choose $y_k \in P$, $k = 1, 2, \ldots$, such that

$$\lim_{n \to \infty} \varrho_k(x_n, y_k) = 0 .$$

Since $n \geqq m$ implies $\tau^{\beta_n} \leqq \tau^{\beta_m}$, we may conclude at once that $y_1 = y_k$ for every $k = 1, 2, \ldots$ Now it is easy to see that

$$\lim_{n \to \infty} \varrho(x_n, y_1) = 0 .$$

The proof of Theorem 1 is complete.

If (P, τ) is a space and M is a subset of P then the symbol τ/M denotes the relativisation of τ to M and the symbol τ_M denotes the infimum of all topologies η for the set P satisfying $\eta/M \geqq \tau/M$. In [1] the following theorem (3.7) is proved:

Theorem 2. *Let τ be a complete metrizable topology for a set P. Then*

$$c(\tau) = \sup \{\tau_M; M \in \mathsf{L}\,\mathsf{C}(\tau)\} = \sup \{\tau_M; M \in \mathsf{A}(\tau)\} .$$

Theorem 3. *Let τ be a complete metrizable topology (for a set P) of a countable order of disconnectedness. Then* $s(\tau) = \sup \{\tau_M; M \in \mathsf{A}(\tau)\}$. *In consequence,* $s(\tau) = c(\tau) = m(\tau)$.

Proof. Let us denote by τ_0 the topology $\sup \{\tau_M;\ M \in \mathsf{A}(\tau)\}$. It is easy to see that $\tau \geqq \tau_0$ It may be noticed that $\mathsf{A}(\tau) = \mathsf{A}(\tau_0)$. Indeed, if $M \in \mathsf{A}(\tau)$, then by definition of τ_0 we have $\tau_0/M \geqq \tau/M$. Now from the inequality $\tau \geqq \tau_0$ it follows that $\tau_0/M = \tau/M$. Thus $M \in \mathsf{A}(\tau_0)$. Conversely, if $M \in \mathsf{A}(\tau_0)$ then from the fact that the topology τ_0/M is compact and from the inequality $\tau \geqq \tau_0$ it follows at once that $\tau_0/M = \tau/M$. Thus $M \in \mathsf{A}(\tau)$. Since $\tau \geqq s(\tau) \geqq \tau_0$, from the equality $\mathsf{A}(\tau) = \mathsf{A}(\tau_0)$ we have at once

$$\mathsf{A}(\tau) = \mathsf{A}(s(\tau)) = \mathsf{A}(\tau_0)\,.$$

Since the topology $s(\tau)$ is locally connected, we have

$$s(\tau) = \sup \{\tau_M;\ M \in \mathsf{L\,C}(s(\tau))\}\,.$$

By theorem 2 we have

$$\sup \{\tau_M;\ M \in \mathsf{A}(s(\tau))\} = \sup \{\tau_M;\ M \in \mathsf{L\,C}(s(\tau))\}\,.$$

Finally, combining (*), (**) and (***), we obtain $s(\tau) = \tau_0$. The proof of the theorem 3 is complete.

By theorem 1, if the topology is complete metrizable, then the topology $s(\tau)$ is complete metrizable. Now we shall construct a complete metric for $(P, s(\tau))$.

Theorem 4. *Let (P, τ) be a complete metrizable space. Let φ be a complete metric generating the topology τ such that $\varphi(x, y) \leqq 1$ for every x and y in P. Let us define a metric ϱ for the set P as follows:*

If there exists no $A \in \mathsf{A}(\tau)$ containing both x and y, then $\varrho(x, y) = 1$. In the other case let $\varrho(x, y)$ be the greatest lower bound of the set of diameters of all $A \in \mathsf{A}(\tau)$ containing both x and y.

The metric space (P, ϱ) is complete (and by [1], theorem 3.7, ϱ generates the topology $m(\tau) = c(\tau)$) and by theorem 3 on the present note, the metric ϱ generates the topology $s(\tau)$.

Proof. Let us suppose that $\{x_n\}$ is a Cauchy sequence with respect to the metric ϱ. Since $\varphi(x, y) \leqq \varrho(x, y)$, $\{x_n\}$ is a Cauchy sequence with respect to φ. Thus there exists a point x in P such that

$$(*) \qquad \lim_{n\to\infty} \varphi(x, x_n) = 0\,.$$

We shall prove

$$(**) \qquad \lim_{n\to\infty} \varrho(x, x_n) = 0\,.$$

To prove (**), we may assume without loss of the generality that

$$\varrho(x_n, x_{n+1}) < 2^{-n} \quad (n = 1, 2, \ldots)\,.$$

Let us choose $A_n \in \mathsf{A}(\tau)$ for $n = 1, 2, \ldots$, such that the diameter (with respect to φ) of A_n is less than 2^{-n} and that both x_n and x_{n+1} belong to A_n. Put

$$K_n = \bigcup_{k=n}^{\infty} A_k \quad (n = 1, 2, \ldots)\,.$$

Let us denote by C_n the τ-closure of K_n, $n = 1, 2, \ldots$ Evidently the diameter (with respect to φ) of K_n, and hence that of C_n also, is less than $\sum_{k=n}^{\infty} 2^{-k} = 2^{-n+1}$. By (*) the point x belongs to C_n $(n = 1, 2, \ldots)$. Thus to prove (**) it is sufficient to show that $C_n \in A(\tau)$, $n = 1, 2, \ldots$ Evidently the sets C_n are τ-connected. To prove compactness of C_n, it is sufficient to notice that any infinite subset M of C_n either is contained in the union of a finite number of A_n or the point x is an accumulation point of M. It remains to prove that the sets C_n are locally connected. If $y \in C_n$ and $y \neq x$, then $\varrho(x, y) = \varepsilon > 0$, and consequently, the φ-spheres about x of radius less than ε are contained in the union of a finite number of A_k. Thus C_n is locally connected at every point $y \neq x$. To prove that C_n is locally connected at the point x, it is sufficient to notice that the sets C_k are connected, the sets A_k are locally connected and the diameters with respect to φ of C_k converge to zero with $k \to \infty$. Thus the proof is complete.

References

[1] *Z. Frolík:* Locally connected topologies. Czech. Math. J. *11* (*86*), 1961, 398—412.

Резюме

ЛОКАЛЬНО СВЯЗНЫЕ ТОПОЛОГИИ АССОЦИИРОВАННЫЕ С ДАННОЙ ПОЛНО МЕТРИЗУЕМОЙ ТОПОЛОГИЕЙ

ЗДЕНЕК ФРОЛИК (Zdeněk Frolík), Прага

Топология τ на множестве P называется полно метризуемой, если существует метрика ϱ пространства (P, τ) такая, что (P, ϱ) является полным метрическим пространством.

В работе [1] для всякой топологии τ на множестве P определены локально связные топологии $s(\tau)$, $m(\tau)$ и $c(\tau)$ на множестве P, и рассматриваются соотношения между τ, $s(\tau)$, $c(\tau)$ и $m(\tau)$.

Главным результатом настоящей работы является теорема 3, которая утверждает, что $s(\tau) = c(\tau) = m(\tau)$, если только τ полно метризуема и если τ имеет счетный порядок несвязности. В этом случае также конструируется полная метрика для пространства $(P, s(\tau))$.