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ON A THEOREM OF V. PTÁK CONCERNING BEST APPROXIMATION OF CONTINUOUS FUNCTIONS IN THE METRIC $\int_a^b |x(t)|\,\mathrm{d}t$

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The author derives from his previous results on best approximation in general normed linear spaces some improvements of a recent theorem of V. PTÁK concerning best approximation of continuous functions in the metric $\int_a^b |x(t)|\,\mathrm{d}t$.

Let $T = [a, b]$ be a finite segment of the real axis. We shall denote by $C_{L^1}(T)$ the space of all continuous real-valued functions defined on T, endowed with the natural vector operations and with the norm $\|x\| = \int_T |x(t)|\,\mathrm{d}t$. $C_{L^1}(T)$ is a normed linear space, but it is not a Banach space.

Let E be an arbitrary normed linear space, G a linear subspace of E, and $x \in E$. We call *best approximation of the element* x any element $g_0 \in G$ such that

$$\|x - g_0\| = \inf_{g \in G} \|x - g\| .$$

The main result of the recent paper [5] of V. PTÁK is the following (see [5], theorem 2):

Let G be a finite-dimensional linear subspace of the space $C_{L^1}(T)$. There exists an $x_0 \in C_{L^1}(T)$ with a nonunique best approximation if and only if there exist two disjoint sets U_1 and U_2 open in T and an essentially bounded measurable function $\alpha(t)$ defined on $T - (U_1 \cup U_2)$, with the following properties:

(1) $$|\alpha(t)| \leqq 1 \quad \textit{for every} \quad t \in T - (U_1 \cup U_2)\,,$$

(2) $$\int_{U_1} g(t)\,\mathrm{d}t - \int_{U_2} g(t)\,\mathrm{d}t + \int_{T-(U_1 \cup U_2)} g(t)\,\alpha(t)\,\mathrm{d}t = 0 \quad \textit{for every} \quad g \in G\,,$$

(3) *there exists a nonzero $g_0 \in G$ vanishing on $T - (U_1 \cup U_2)$.*

In the present paper we shall derive from our previous results on best approximation in general normed linear spaces ([7], [8], [9]) some improvements of the above theorem of V. Pták.

1. RECALL OF SOME RESULTS ON BEST APPROXIMATION IN GENERAL NORMED LINEAR SPACES

Let E be an arbitrary normed linear space. Then we have the following two theorems:

Theorem (A). (See [8], p. 184, and [9], theorem 1). *Let G be an arbitrary linear subspace of E. There exists an $x_0 \in E$ with a nonunique best approximation if and only if there exists a linear*[1]) *functional $f \in E^*$ with the following properties*:

(A_1) $\|f\| = 1$,

(A_2) $f(g) = 0$ *for all* $g \in G$,

(A_3) $f(x) = \|x\|$ *for at least two distinct elements $x \in E$ whose difference belongs to G.*

Theorem (B). (See [7], theorem 2.2 and [9], theorem 2). *Let G be an n-dimensional linear subspace of E. There exists an $x_0 \in E$ with a nonunique best approximation if and only if there exist h distinct two by two non-opposite extreme points $f_1, \ldots, f_h$ of the unit sphere $S^* \subset E^*$, where $1 \leqq h \leqq n$, and h positive numbers $\lambda_1, \ldots, \lambda_h$ such that $\sum_{i=1}^{h} \lambda_i = 1$, with the following properties*:

(B_1) $\sum_{i=1}^{h} \lambda_i f_i(g) = 0$ *for all* $g \in G$,

(B_2) $\sum_{i=1}^{h} \lambda_i f_i(x) = \|x\|$ *for at least two distinct elements $x \in E$ whose difference belongs to G.*

The above theorems are stated in [7], [8] and [9] respectively, under the hypothesis that E is a Banach space. However the proofs given there make no use of the completeness of E, hence these theorems are clearly valid (with the same proofs) for an arbitrary normed linear space E.

Concerning some other questions related to theorem (B) and to [7], [8], [9] see also the recent paper [4] of V. Pták.

2. BEST APPROXIMATION IN THE SPACE $C_{L^1}(T)$ BY MEANS OF THE ELEMENTS OF AN ARBITRARY LINEAR SUBSPACE

Since the completion of the space $C_{L^1}(T)$ is nothing else but the space $L^1(T)$, the conjugate spaces of $C_{L^1}(T)$ and $L^1(T)$ are equivalent; consequently, the conjugate space of $C_{L^1}(T)$ is equivalent to the space $M(T)$ of all equivalence classes of essentially bounded measurable functions, endowed with the natural

[1]) I. e. additive and continuous.

vector operations and with the norm $\|\beta\| = \operatorname{vrai}\max_{t \in T} |\beta(t)|$, the equivalence $f \longleftrightarrow \beta$ being given by

$$f(x) = \int_T x(t)\, \beta(t)\, dt \quad \text{for all} \quad x \in C_{L^1}(T)\,.$$

Hence, theorem (A) gives the following:

Proposition 1. *Let G be an arbitrary linear subspace of $C_{L^1}(T)$. There exists an $x_0 \in C_{L^1}(T)$ with a nonunique best approximation if and only if there exists an essentially bounded measurable function $\beta(t)$ with the following properties*:

$$\operatorname{vrai}\max_{t \in T} |\beta(t)| = 1\,, \tag{4}$$

$$\int_T g(t)\, \beta(t)\, dt = 0 \quad \textit{for all} \quad g \in G\,, \tag{5}$$

$$\int_T x(t)\, \beta(t)\, dt = \int_T |x(t)|\, dt \tag{6}$$

for at least two distinct elements $x \in C_{L^1}(T)$ whose difference belongs to G.

Now we shall show that the conditions (4), (5) and (6) are equivalent to (1), (2), (3).

Assume first that we have (4), (5) and (6).

Let $x_0 + g_0$ and $x_0 - g_0$ be two elements of $C_{L^1}(T)$ satisfying (6); clearly, any couple x_1, $x_2 \neq x_1$ satisfying (6) may be written in this form, since for $x_1 - x_2 = 2g_0 \in G$ we have only to take $x_0 = x_1 - g_0$.

Put

$$U_1 = \{t \in T | x_0(t) > 0\}\,, \quad U_2 = \{t \in T | x_0(t) < 0\}\,,$$

and let $\alpha(t)$ be the restriction of $\beta(t)$ to $T - (U_1 \cap U_2)$. Then U_1 and U_2 are disjoint and open in T, and (4) clearly implies (1).

Furthermore, (6) and (4) obviously imply that

$$\beta(t) = 1 \text{ a. e.}^{2)} \text{ on } U_1 \text{ and } \beta(t) = -1 \text{ a. e. on } U_2\,, \tag{7}$$

whence, by (5) we have (2).

Finally, by (6) for $x_0 + g_0$, $x_0 - g_0$ and by (4) we have

$$\int_T |x_0(t)|\, dt \leqq \tfrac{1}{2} \int_T |x_0(t) + g_0(t)|\, dt + \tfrac{1}{2} \int_T |x_0(t) - g_0(t)|\, dt =$$
$$= \int_T x_0(t)\, \beta(t)\, dt \leqq \int_T |x_0(t)|\, dt\,,$$

and thus, the equality, which is possible only if

$$(x_0(t) + g_0(t))(x_0(t) - g_0(t)) \geqq 0 \quad (t \in T)\,,$$

whence we infer that g_0 vanishes on $T - (U_1 \cup U_2)$, i. e. (3).

Conversely, assume that we have (1), (2) and (3).

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[2]) I. e. almost everywhere.

Define

(8) $$x_0(t) = \begin{cases} |g_0(t)| \text{ for } t \in U_1 , \\ - |g_0(t)| \text{ for } t \in U_2 , \\ 0 \quad \text{ for } t \in T - (U_1 \cup U_2) \end{cases}$$

and

(9) $$\beta(t) = \begin{cases} 1 \text{ for } t \in U_1 , \\ -1 \text{ for } t \in U_2 , \\ \alpha(t) \text{ for } t \in T - (U_1 \cup U_2) . \end{cases}$$

Then $x_0(t)$ is continuous, and $\beta(t)$ is measurable. By (1) and (9) we shall have (4), and by (2) and (9) we shall have (5). Finally, (9), (8) and (3) imply

$$\int_T x_0(t)\, \beta(t)\, \mathrm{d}t = \int_T |x_0(t)|\, \mathrm{d}t$$

and

$$\int_T [x_0(t) - g_0(t)]\, \beta(t)\, \mathrm{d}t = \int_T |x_0(t) - g_0(t)|\, \mathrm{d}t ,$$

i. e. (6) with $x_1 = x_0$, $x_2 = x_0 - g_0$.

Thus we have proved the following theorem:

Theorem 1. *Let G be an* arbitrary *linear subspace of the space $C_{L^1}(T)$. There exists an $x_0 \in C_{L^1}(T)$ with a nonunique best approximation if and only if there exist two disjoint sets U_1 and U_2 open in T and a measurable function $\alpha(t)$ defined on $T - (U_1 \cup U_2)$, with the properties* (1), (2) *and* (3).

Remark. This theorem is *implicitely* proved also by V. Pták, in [5] (see also [6]). In fact, though in the formulation of theorem 2 of [5] is stated the hypothesis that G is a *finite-dimensional* subspace of $C_{L^1}(T)$, it is easy to verify that Pták's proof of that theorem, given in [5], [6], makes no use of this hypothesis.

3. BEST APPROXIMATION IN THE SPACE $C_{L^1}(T)$ BY MEANS OF THE ELEMENTS OF A FINITE-DIMENSIONAL LINEAR SUBSPACE

The extreme points of the unit sphere S^* of the conjugate space $[C_{L^1}(T)]^*$ are, by the remark at the beginning of section 2 and by [7], lemma 1.4, the linear functionals f which have the form

$$f(x) = \int_T x(t)\, \beta_M(t)\, \mathrm{d}t \quad \text{for all} \quad x \in C_{L^1}(T) ,$$

where M is a measurable subset of T and where

(10) $$\beta_M(t) \overset{\text{df}}{=} \begin{cases} 1 \quad \text{for a. e. } t \in M , \\ -1 \quad \text{for a. e. } t \in T - M . \end{cases}$$

Hence, theorem (B) gives the following:

Proposition 2. *Let G be an n-dimensional linear subspace of $C_{L^1}(T)$. There exists an $x_0 \in C_{L^1}(T)$ with a nonunique best approximation if and only if there exist h measurable subsets $M_1, \ldots, M_h$ of T, where $1 \leqq h \leqq n$, such that* [3])

$$M_i \not\sim M_j \text{ and } M_i \not\sim T - M_j \text{ for } i \neq j\,,$$

and an essentially bounded measurable functioo $\beta(t)$ of the form

$$(11) \qquad \beta(t) = \lambda_1 \beta_{M_1}(t) + \ldots + \lambda_h \beta_{M_h}(t) \quad (t \in T)\,,$$

where $\lambda_i > 0$, $i = 1, \ldots, h$, $\sum_{i=1}^{h} \lambda_i = 1$, with the properties (4), (5) *and* (6).

Let us remark that the above $\beta(t)$ is a. e. a finitely-valued function which assumes on T (excepting a set of measure zero) at most 2^n distinct values, all between -1 and $+1$.

Now we can prove

Theorem 2. *Let G be an n-dimensional linear subspace of the space $C_{L^1}(T)$. There exists an $x_0 \in C_{L^1}(T)$ with a nonunique best approximation if and only if there exist two disjoint open subsets U_1 and U_2 of T, h measurable subsets $M_1, \ldots$ $\ldots, M_h$ of T, where $1 \leqq h \leqq n$, such that*

$$M_i \not\sim M_j \text{ and } M_i \not\sim T - M_j \text{ for } i \neq j\,,$$

and

$$(12) \qquad M_i \supset U_1\,, \quad T - M_i \supset U_2$$

(excepting a set of measure zero), $i = 1, \ldots, h$, and an essentially bounded measurable function $\alpha(t)$ defined on $T - (U_1 \cup U_2)$, of the form

$$(13) \qquad \alpha(t) = \lambda_1 \alpha_{M_1}(t) + \ldots + \lambda_h \alpha_{M_h}(t) \quad (t \in T - (U_1 \cup U_2))\,,$$

where $\alpha_{M_i}(t)$ denotes the restriction of $\beta_{M_i}(t)$ to $T - (U_1 \cup U_2)$ and where $\lambda_i > 0$, $i = 1, \ldots, h$, $\sum_{i=1}^{h} \lambda_i = 1$, with the properties (1), (2) *and* (3).

Proof. This theorem follows from proposition 2 above, by the method used in the preceding section (for the derivation of theorem 1 from proposition 1). The only necessary additions are the following:

To the necessity part: From (7), (11), $\lambda_i > 0$ $(i = 1, \ldots, h)$ and $\sum_{i=1}^{h} \lambda_i = 1$ it follows that

$$(14) \qquad \beta_{M_i}(t) = \begin{cases} 1 \text{ a. e. on } U_1\,, \\ -1 \text{ a. e. on } U_2\,, \end{cases} \quad i = 1, \ldots, h\,,$$

whence, by (10), we infer (12).

[3]) The symbol $\not\sim$ denotes the non-equivalence of the sets in question (with respect to the Lebesque measure).

To the sufficiency part: From $\sum_{i=1}^{h} \lambda_i = 1$, (12) and (10) it follows, since (12) and (10) imply (14), that the $\beta(t)$ defined by (9) and (13) is nothing else but (11).

Remark 1. The above function $\alpha(t)$ on $T - (U_1 \cup U_2)$ is a. e. by (13), a finitely-valued function, which assumes on $T - (U_1 \cup U_2)$ (excepting a set of measure zero) at most 2^n distinct values, all between -1 and $+1$.

Remark 2. It is easy (we omit the details) to derive from theorem 2 the following result, due essentially to D. JACKSON [3]:

Let G be an n-dimensional linear subspace of $C_{L^1}(T)$. If there exists an $x_0 \in C_{L^1}(T)$ with a nonunique best approximation, then there exists an element $g_0 \in G$ and n distinct inner points t_i of T such that $g_0(t_i) = 0$.

Let us mention that a short direct proof of this theorem has been given by V. Pták [5], [6].

Remark 3. Using the above methods, one can derive from theorem 3.1 of [8] a theorem of characterization of the polynomials of best approximation (in the metric $\int_T |x(t)| \, dt$) of a continuous function $x_0(t)$.

Finally, let us mention that S. Ia. HAVINSON has given, in the papers [1], [2], some theorems concerning the best approximation in the metric $\int_T |x(t)| \, d\mu(t)$ of a function belonging to a certain subclass of $L^1(T, \mu)$, where μ is a nonnegative measure on a completely additive class of subsets of a separable metric space R, containing all the Borel sets, and where T is a μ-measurable set "reduced" with respect to μ, by means of the elements of a linear subspace G consisting of continuous functions. In the present paper we shall not discuss these theorems.

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Резюме

ОБ ОДНОЙ ТЕОРЕМЕ В. ПТАКА

ИВАН ЗИНГЕР (Ivan Singer), Бухарест

Используя общие результаты из [7], [8], [9], автор доказывает некоторые улучшения теоремы В. Птака [5] об аппроксимации непрерывных функций в норме $\int_a^b |x(t)| \, dt$.