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THE FRATTINI SUBGROUP OF A DIRECT PRODUCT OF GROUPS

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This note is devoted to the investigation of the conditions under which the Frattini subgroup of a direct product of groups is the direct product of Frattini subgroups of the direct factors. For this, necessary and sufficient conditions are found and other simple, but only sufficient, conditions are deduced. These conditions show that the above assertion is true for all soluble groups and for all finitely generated groups and that it is generally valid if and only if every simple group has maximal subgroups.

The Frattini subgroup $\Phi(G)$ of a group G is the intersection of all maximal subgroups of G, if such subgroups exist; otherwise $\Phi(G) = G$. The principal property of $\Phi(G)$ yields a second definition for it which can be expressed as follows: $\Phi(G)$ is a subset of G such that each of its elements can be removed from a generating system of G which contains it without altering the generating property of the system.

This paper is devoted to the following investigation. Suppose we have a decomposition of G in a direct product

(1) $$G = \prod_{\varrho \in P}^{\times} G_\varrho ,$$

where the set of indices P has a quite arbitrary cardinality. The question is, under which conditions the equality

(2) $$\Phi(G) = \prod_{\varrho \in P}^{\times} \Phi(G_\varrho)$$

holds?

The Frattini subgroup was introduced into the group theory first by G. FRATTINI [3] in 1885. The implication (1) $\Rightarrow$ (2) for all finite groups G was proved by G. A. MILLER [7] in 1915. (See also W. GASSCHÜTZ [4]). Recently VL. DLAB [1] and [2] proved it for every abelian and every finitely generated G. It is proved in this paper that (2) holds for every soluble G (and even for a slightly more general class of groups) and for two other classes of groups one of which contains all finitely generated groups. The problem whether, in general, the implication (1) $\Rightarrow$ (2) holds or not is equivalent to the problem whether simple groups without maximal subgroups do not or do exist. This second problem seems to be a very difficult one.

VL. KOŘÍNEK first undertook these general investigations, formulated and proved Theorems 3, 4, 5, 6. His proofs were based upon the lemma contained in Remark 1 and Lemmas 3 and 4. Later VL. DLAB discovered the importance of the relation (14), gave the definition of the property F, formulated and proved Theorems 1 and 2. By these two theorems, not insignificant simplifications were brought into the proofs of subsequent theorems. Theorem 7 is due to Dlab.

First, let us explain the notations used in this paper and give some definitions. The signs ϵ, $\cap$, $\cup$ are used in the ordinary sense. The inclusion $A \subset B$ does not exclude the case $A = B$. If this equality is excluded, we write $A \subsetneq B$. $A \dot{-} B$ denotes the set theoretical difference. A subgroup M of G is called *maximal* in G (*max. subgroup*), if $M \neq G$ and if there is no subgroup K such that $M \subsetneq K \subsetneq G$. Similarly the normal subgroup N of G is called *maximal normal subgroup* (*max. normal subgroup*) of G, if $N \neq G$ and if there is no normal subgroup L such that $N \subsetneq L \subsetneq G$.

A direct decomposition of G will be written in the form (1) or in the form

$$G = G_1 \times G_2 \times \ldots \times G_r$$

if the number of direct factors is finite. For every $g \in G$ (1) gives a unique decomposition of g,

$$g = \prod_{\varrho \in P} g_\varrho\,, \quad g_\varrho \in G_\varrho\,. \tag{3}$$

Here, all but a finite number of the components g_ϱ are equal to the group unity of G_ϱ. Let H be an arbitrary subgroup of G. If g runs through all H, the components g_ϱ of g in G_ϱ form a subgroup of G_ϱ, *the component subgroup of* H in G_ϱ which will be denoted by H'_ϱ. The intersection

$$H''_\sigma = H \cap G_\varrho$$

is a normal subgroup of H and therefore also of H'_ϱ. We call it the *subgroup of free components of* H in G_ϱ. We shall often write the decomposition (1), σ_1 being a fixed index from P, in the form

$$G = G_{\sigma_1} \times \overline{G}_{\sigma_1}\,, \tag{4}$$

where

$$\overline{G}_{\sigma_1} = \prod^{\times}_{\varrho \in P \dot{-} (\sigma_1)} G_\varrho\,. \tag{5}$$

The component subgroups and the subgroups of free components of H in the decomposition (4) will be denoted by H'_{σ_1}, $\overline{H}'_{\sigma_1}$, H''_{σ_1}, $\overline{H}''_{\sigma_1}$. $\{A, B, \ldots\}$ is the subgroup generated by the sets A, B, $\ldots$ of elements of G.

We begin with a general lemma which probably is not new:

Lamma 1. *For all groups G and all their direct decompositions* (1) *the inclusion*

$$\Phi(G) \subset \prod_{\varrho \in P}^{\times} \Phi(G_\varrho)$$

holds.

Proof. Take an index $\sigma \in P$ and suppose G_σ has max. subgroups. If M_σ is one of them,

$$M_\sigma \times \prod_{\varrho \in P \dot{-} (\sigma)}^{\times} G_\varrho \tag{6}$$

is a max. subgroup of G, as it can easily be proved. Let H_λ, $\lambda \in \Lambda_\varrho$, be a system of subgroups of G_ϱ containing all max. subgroups of G_ϱ, if such subgroups exist, and the group G_ϱ. We have

$$\Phi(G_\varrho) = \bigcap_{\lambda \in \Lambda_\varrho} H_\lambda$$

and

$$\Phi(G) \subset \bigcap_{\lambda \in \Lambda_\sigma} (H_\lambda \times \prod_{\varrho \in P \dot{-} (\sigma)}^{\times} G_\varrho) = \bigcap_{\lambda \in \Lambda_\sigma} H_\lambda \times \prod_{\varrho \in P \dot{-} (\sigma)}^{\times} G_\varrho = \Phi(G_\sigma) \times \prod_{\varrho \in P \dot{-} (\sigma)}^{\times} G_\varrho$$

for all $\sigma \in P$. Therefore

$$\Phi(G) \subset \bigcap_{\sigma \in P} [\Phi(G_\sigma) \times \prod_{\varrho \in P \dot{-} (\sigma)}^{\times} G_\varrho] = \prod_{\varrho \in P}^{\times} [\Phi(G_\varrho) \cap G_\varrho] = \prod_{\varrho \in P}^{\times} \Phi(G_\varrho) .$$

Lemma 1 shows that it is sufficient for our problem to examine the inverse inclusion. For this purpose we prove first some lemmas.

Lemma 2. *Let* (1) *be a direct decomposition of G. Each max. subgroup M of G satisfies one of these two conditions*

a) *There is in P an index* σ_1 *such that*

$$M'_{\sigma_1} = M''_{\sigma_1} \tag{7}$$

is a max. subgroup of G_{σ_1} *and*

$$M'_\varrho = M''_\varrho = G_\varrho \quad \text{for} \quad \varrho \in P \dot{-} (\sigma_1) . \tag{8}$$

b) *It is*

$$M'_\varrho = G_\varrho \quad \textit{for all} \quad \varrho \in P \tag{9}$$

and either $M''_\varrho = G_\varrho$, *or* M''_ϱ *is a max. normal subgroup of* G_ϱ. *There is at least one index* $\sigma_1 \in P$ *for which* $M''_{\sigma_1} \neq G_{\sigma_1}$[1]) *and for this index we have the isomorphism*

$$G_{\sigma_1}/M''_{\sigma_1} \cong \overline{G}_{\sigma_1}/\overline{M}''_{\sigma_1} \tag{10}$$

in the direct decomposition (4).

Proof. *Suppose there are two indices* σ_1, σ_2 so that $M'_{\sigma_1} \neq G_{\sigma_1}$, $M'_{\sigma_2} \neq G_{\sigma_2}$.

[1]) In fact there must be at least two such indices, but we shall not need this fact.

Then

$$M \subset M'_{\sigma_1} \times M'_{\sigma_2} \times \prod^{\times}_{\varrho \in P \dot{-} (\sigma_1, \sigma_2)} G_\varrho \subset M'_{\sigma_1} \times \prod^{\times}_{\varrho \in P \dot{-} (\sigma_1)} G_\varrho \subset \prod^{\times}_{\varrho \in P} G_\varrho = G$$

and M is not maximal. Therefore we have to consider only two cases.

1) There is one index σ_1 for which $M'_{\sigma_1} \subset G_{\sigma_1}$. Suppose either $M''_{\sigma_1} \subset M'_{\sigma_1}$ or $M''_\sigma \subset M'_\sigma$ for another index $\sigma \in P \dot{-} (\sigma_1)$. In both cases we have

$$M \subset M'_{\sigma_1} \times \prod^{\times}_{\varrho \in P \dot{-} (\sigma_1)} G_\varrho \subset G$$

and M is not maximal. Therefore (7) and (8) hold. If M'_{σ_1} were not max. subgroup in G_{σ_1}, there would be a subgroup H_{σ_1} of G_{σ_1} with $M'_{\sigma_1} \subset H_{\sigma_1} \subset G_{\sigma_1}$ and we should have

$$M = M'_{\sigma_1} \times \prod^{\times}_{\varrho \in P \dot{-} (\sigma_1)} G_\varrho \subset H_{\sigma_1} \times \prod^{\times}_{\varrho \in P \dot{-} (\sigma_1)} G \subset G\,,$$

again in contradiction to the maximality of M. We have proved a).

2. The other possibility is that (9) holds for all $\varrho \in P$. $M \neq G$ implies the existence of an index $\sigma_1 \in P$ for which $M''_{\sigma_1} \neq G_{\sigma_1}$.[2]) Take the decomposition (4) of G for this σ_1. Any $g \in M$ can be written in the form

$$(11) \qquad g = g_{\sigma_1} \bar{g}_{\sigma_1}\,, \quad g_{\sigma_1} \in G_{\sigma_1}\,, \quad \bar{g}_{\sigma_1} \in \bar{G}_{\sigma_1}\,.$$

$M \neq G$ implies $\bar{M}''_{\sigma_1} \neq \bar{G}_{\sigma_1}$. Take into consideration the classes $g_{\sigma_1} M''_{\sigma_1}$ and $\bar{g}_{\sigma_1} \bar{M}''_{\sigma_1}$. In view of the definitions of M''_{σ_1} and $\bar{M}''_{\sigma_1}$ one finds, if g runs throughout M, that $g_{\sigma_1} M'_{\sigma_1} \longleftrightarrow \bar{g}_{\sigma_1} \bar{M}''_{\sigma_1}$ is a one to one mapping of $G_{\sigma_1}/M''_{\sigma_1}$ onto $\bar{G}_{\sigma_1}/\bar{M}''_{\sigma_1}$. One easily proves that this mapping which will be denoted ψ is an isomorphism of these two quotient groups. Briefly speaking, M is a subdirect union of the groups G_{σ_1} and $\bar{G}_{\sigma_1}$ with kernels M''_{σ_1} and $\bar{M}''_{\sigma_1}$. Thus we have (10).

Suppose M''_{σ_1} is not a max. normal subgroup of G_{σ_1}. Let N_{σ_1} be a normal subgroup in G_{σ_1} such that $M''_{\sigma_1} \subset N_{\sigma_1} \subset G_{\sigma_1}$. In view of (10) there must be a normal subgroup $\bar{N}_{\sigma_1}$ in $\bar{G}_{\sigma_1}$ such that $\bar{M}''_{\sigma_1} \subset \bar{N}_{\sigma_1} \subset \bar{G}_{\sigma_1}$. The isomorphism ψ of $G_{\sigma_1}/M''_{\sigma_1}$ onto $G_{\sigma_1}/\bar{M}''_{\sigma_1}$ generates an isomorphism $\bar{\psi}$ of $G_{\sigma_1}/N_{\sigma_1}$ onto $\bar{G}_{\sigma_1}/\bar{N}_{\sigma_1}$. It is easily seen that all elements (11), where the classes $g_{\sigma_1} N_{\sigma_1}$, $\bar{g}_{\sigma_1} \bar{N}_{\sigma_1}$ correspond one to the other by this isomorphism ψ, constitute a subgroup N of G such that $M \subset N \subset G$, in contradiction to the maximality of M. Hence M''_{σ_1} and $\bar{M}''_{\sigma_1}$ are, in addition, max. normal subgroups in G_{σ_1} and $\bar{G}_{\sigma_1}$. Lemma 2 has been proved.

Remark 1. From the above proof of Lemma 2 we can deduce a more precise assertion concerning the decomposition of G into a direct product of two factors:

$$(12) \qquad G = G_1 \times G_2\,.$$

[2]) In fact there must be in P at least two such indices, otherwise $M = G$.

Let (12) *be a direct decomposition of G. A subgroup M of G is a max. subgroup in G, if and only if it belongs to one of these two types*:

a) $M = H_1 \times G_2$ *or* $M = G_1 \times H_2$, *where H_i is a max. subgroup in G_i*, $i = 1, 2$.

b) *It is* $M_i' = G_i$, M_i'' *is max. normal subgroup in* G_i, $i = 1, 2$ *and M is a subdirect union of G_1 and G_2 with kernels M_1'' and M_2''*. But we shall not need this lemma.

Lemma 3. *Let* (1) *be a direct decomposition of the group G. G can be homomorphically mapped onto a simple group S, if and only if the same thing holds at least for one direct factor G_σ in* (1).

Proof. If a direct factor G_σ of (1) can be homomorphically mapped onto S, G does obviously the same. Suppose G can be homomorphically mapped onto S and let N be the kernel of this homomorphism. Then, N must be a max. normal subgroup in G. $N \neq G$ implies the existence of an index $\sigma \in P$ such that $N \cap G_\sigma \neq G_\sigma$. By the maximality of N we get $\{N, G_\sigma\} = G$ and by the first theorem on isomorphism:

$$S \cong G/N = \{N, G_\sigma\}/N \cong G_\sigma/(N \cap G_\sigma) .$$

Now we are able to prove the following theorem:

Theorem 1. *Let* (1) *be a direct decomposition of the group G. The equality* (2) *for its Frattini subgroup $\Phi(G)$ does not hold, if and only if there exist in* (1) *two direct factors G_{σ_1}, G_{σ_2} with two max. normal subgroups N_{σ_1}, N_{σ_2} such that*

$$G_{\sigma_1}/N_{\sigma_1} \cong G_{\sigma_2}/N_{\sigma_2} \tag{13}$$

and

$$\Phi(G_{\sigma_1}) \not\subset N_{\sigma_1} . \tag{14}$$

Proof. 1. Suppose (2) does not hold. If $\Phi(G_\varrho) \subset \Phi(G)$ for all $\varrho \in P$ we should have (2) by Lemma 1. Thus there is an index $\sigma_1 \in P$ such that $\Phi(G_{\sigma_1}) \not\subset \Phi(G)$. This implies the existence of a max. subgroup M in G such that

$$\Phi(G_{\sigma_1}) \not\subset M .$$

The equality $M_{\sigma_1}'' = G_{\sigma_1}$ and the case a) from Lemma 2 for M and the index σ_1 are impossible, as in both cases we should have $\Phi(G_{\sigma_1}) \subset M_{\sigma_1}'' \subset M$. Therefore M must satisfy the conditions of case b) in Lemma 2; i. e. M_{σ_1}'' is a max. normal subgroup in G_{σ_1}. We put $M_{\sigma_1}'' = N_{\sigma_1}$ and we obtain in this way (14) and the relation (10) for the direct decomposition (4) of G. $G_{\sigma_1}/N_{\sigma_1}$ is a simple group and so is $\overline{G}_{\sigma_1}/\overline{M}_{\sigma_1}''$ in (10). By Lemma 3 and (5), there are an index $\sigma_2 \in P \dot{-} (\sigma_1)$ and a max. normal subgroup N_{σ_2} in G_{σ_2} for which (13) holds.

2. Suppose one can find two indices σ_1 and σ_2 in P and max. normal subgroups N_{σ_i} in G_{σ_i}, $i = 1, 2$, such that (13) a (14) hold. Write

$$K = \prod_{\varrho \in P}^{\times} \Phi(G_\varrho) .$$

It is obviously

$$K''_{\sigma_1} = \Phi(G_{\sigma_1}) . \tag{15}$$

On the other side fix an isomorphism in (13) and form all products $g_{\sigma_1}g_{\sigma_2}$, $g_{\sigma_i} \in G_{\sigma_i}$, $i = 1, 2$ such that the classes $g_{\sigma_1}N_{\sigma_1}$, $g_{\sigma_2}N_{\sigma_2}$ correspond one to the other in the fixed isomorphism of $G_{\sigma_1}/N_{\sigma_1}$ onto $G_{\sigma_2}/N_{\sigma_2}$. The set $M_{\sigma_1\sigma_2}$ of all such products is a subgroup in $G_{\sigma_1} \times G_{\sigma_2}$ and it can be easily shown that this subgroup is max. in $G_{\sigma_1} \times G_{\sigma_2}$. Therefore

$$M = M_{\sigma_1\sigma_2} \times \prod^{\times}_{\varrho \in P \dot{-} (\sigma_1, \sigma_2)} G_\varrho$$

is a max. subgroup of G and we deduce from it $\Phi(G) \subset M$. This gives

$$[\Phi(G)]''_{\sigma_1} \subset M''_{\sigma_1} = N_{\sigma_1} .$$

In view of (14) and (15) we have $[\Phi(G)]''_{\sigma_1} \neq K''_{\sigma_1}$ and therefore $\Phi(G) \neq K$.

The condition (14) suggests the following definition.

Definition. We say the group G has *the property F* (*Frattini property*) if either

$$\Phi(G) \subset N$$

holds for all max. normal subgroups N in G or G has no max. normal subgroups.

Now we easily obtain

Theorem 2. *Let* (1) *be a direct decomposition of the group G and suppose each direct factor G_ϱ of* (1) *has the property F. Then* (2) *holds.*

Proof. Indeed the condition (14) of Theorem 1 cannot be fulfilled.

Lemma 4. *Let N be a max. normal subgroup in G and suppose there exists in G a max. subgroup M such that*

$$N \subset M \subsetneq G . \tag{16}$$

Then we have

$$\Phi(G) \subset N .$$

In other words if $\Phi(G) \not\subset N$, then G/N is a simple group without max. subgroups.

Proof. (16) implies

$$N \subset M' \tag{17}$$

for any subgroup M', conjugate to M. M' is also a max. subgroup in G. By N_0 denote the intersection of all these M'. N_0 is normal in G and (16) and (17) give $N \subset N_0$. In view of the maximality of N this means $N = N_0$ and therefore $\Phi(G) \subset N_0 = N$.

Theorem 3. *Let* (1) *be a fixed direct decomposition of G. Suppose there is no pair of direct factors G_{σ_1}, G_{σ_2} in* (1), *which can be homomorphically mapped onto the same simple group without max. subgroups. Then* (2) *holds.*

Proof. Suppose (2) does not hold. By Theorem 1 there are two indices σ_1, $\sigma_2 \in P$ such that we have (13) and (14). By Lemma 4 and by (14), $G_{\sigma_1}/N_{\sigma_1}$ is a simple group without max. subgroups and so is by (13) $G_{\sigma_2}/N_{\sigma_2}$ too. Thus G_{σ_1} and G_{σ_2} permit a homomorphic mapping onto the same simple group without max. subgroups, in contradiction to the supposition.

Theorem 4. *Suppose that, in every direct decomposition* (12) *of the group G in two direct factors, these factors G_1 and G_2 cannot be homomorphically mapped onto the same simple group without max. subgroups. Then for every direct decomposition* (1) *of G the equality* (2) *holds.*

Proof. Let (1) be a given direct decomposition of G. Suppose there are two indices $\sigma_1, \sigma_2 \in P$ such that G_{σ_1} and G_{σ_2} can be homomorphically mapped onto the same simple group S without max. subgroups. Form the decomposition (4). σ_2 is an index from $P \dot{-} (\sigma_1)$. We see by Lemma 3 and (5) that $\overline{G}_{\sigma_1}$ can be homomorphically mapped on S as well as G_{σ_2}. This is a contradiction to the supposition of the theorem. We conclude that the suppositions of Theorem 3 are fulfilled for (1). Therefore (2) holds.

We can summarize the results, we have got til now, in the theorem:

Theorem 5. *The assertion* A: *„For any direct decomposition* (1) *of an arbitrary group G the equality* (2) *holds,“ is logically equivalent to the assertion* B: *“No simple group without max. subgroups exists”.*

Proof. If B is true, A is valid by Theorem 3. If B is false, there exists a simple group S without max. subgroups. Take two copies S_1 and S_2 of this group and put $G = S_1 \times S_2$. The conditions (13) and (14) with S_i instead of G_{σ_i}, $i = 1, 2$ are easily verified. By Theorem 1 the equality (2) for the decomposition $G = S_1 \times S_2$ does not hold.

Now we can seek some classes of groups for which the equality (2) always holds. In this direction we give here two results.

Theorem 6. *For any soluble group G (thus for any nilpotent or abelian group) and for any of its direct decompositions* (1), *the equality* (2) *always holds.*

Proof. A soluble group can be homomorphically mapped onto a simple group S only if this group S is cyclic of prime order, but then S obviously has one max. subgroup, the unity subgroup. Hence, by Theorem 4 and Lemma 3, the equality (2) always holds.

Remark 2. Using the proof of Theorem 3 we see that for a given direct decomposition of a group G we can only suppose that each of the direct factors G_ϱ in (1) is a soluble group which is a slightly more general supposition. But we can formulate the theorem 6 more generally in an other way: *For any RI^*-group and for any of its direct decompositions* (1) *the equality* (2) *holds.* A RI^*-group G (see [5] p. 29 and [6] p. 368) is a group which possesses an ascending well ordered chain of normal subgroups: $U \subset G_1 \subset G_2 \subset \ldots \subset G_\alpha \subset G_{\alpha+1} \subset \ldots$ in which $\bigcup_\alpha G_\alpha = G$ and all the quotient groups $G_{\alpha+1}/G_\alpha$ are abelian.

Theorem 7. *Let* (1) *be a given direct decomposition of a group G. Suppose that one of these two conditions is fulfilled:*

a) *Each $\Phi(G_\varrho)$, $\varrho \in P$, is finitely generated.*

b) *Each direct factor G_ϱ, $\varrho \in P$, is finitely generated.*

Then the equality (2) *holds.*

Proof. Suppose there is at least one direct factor G_σ, $\sigma \in P$, not having the property F. We take the max. normal subgroup N_σ for which $\Phi(G_\sigma) \not\subset N_\sigma$. In view of the maximality of N_σ we have $\{\Phi(G_\sigma), N_\sigma\} = G_\sigma$. Let Γ be a system of generators for $\Phi(G_\sigma)$ and Δ for N_σ. Then $\Gamma \cup \Delta$ is a system of generators for G_σ. In the case a) take Γ finite, in the case b) there is a finite subset of $\Gamma \cup \Delta$ which is also a system of generators of G_σ. In the latter case, write this system in the form $\Gamma_1 \cup \Delta_1$, $\Gamma_1 \subset \Gamma$, $\Delta_1 \subset \Delta$. Thus we have generating systems $\Gamma \cup \Delta$ or $\Gamma_1 \cup \Delta_1$ for G_σ where the sets Γ and Γ_1 of elements from $\Phi(G_\sigma)$ are finite. The principal property of the Frattini subgroup shows, that we can drop Γ and Γ_1 from these generating systems of G_σ and Δ and Δ_1 must generate the whole group G_σ. But Δ and Δ_1 generate only N_σ. We have proved by contradiction that every direct factor G_ϱ in (1) has the property F and therefore (2) holds by Theorem 2.

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Резюме

ПОДГРУППА ФРАТТИНИ ПРЯМОГО ПРОИЗВЕДЕНИЯ ГРУПП

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Пусть дана группа G, являющаяся прямым произведением (1) групп. В работе разбирается ворпос, когда подгруппа Фраттини этой группы равняется прямому произведению подгрупп Фраттини отдельных прямых сомножителей, т. е. когда справедлива импликация (1) $\Rightarrow$ (2). Г. А. Миллер [7] доказал эту импликацию для конечных групп, а недавно В. Длаб [1] и [2] для абелевых групп и для групп с конечным числом образующих. В настоящей работе этот вопрос исследуется в общем виде, и установлены необходимые и достаточные условия справедливости этой импликации (Теорема 1). Из этой теоремы выводятся более простые условия справедливости этой импликации (Теоремы 2, 3 и 4), являющиеся, однако, только достаточными. Из них следует, что указанная импликация справедлива для всех разрешимых групп (Теорема 6) и для групп G, в прямом разложении (1) которых: а) каждый сомножитель G_ϱ или б) подгруппа Фраттини $\Phi(G_\varrho)$ каждого сомножителя обладают конечной системой образующих (Теорема 7). Наконец показано, что утверждение ,,Для каждого прямого разложения (1) любой группы G справедливо (2)“ равносильно утверждению ,,Не существует простой группы без максимальных подгрупп“ (Теорема 5). О существовании или несуществовании таких простых групп нам ничего неизвестно.