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ON DUAL SEMIGROUPS

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The purpose of this paper is to study the structure of a class of so called dual semigroups introduced below. The main results are contained in Theorem 2,2, Theorem 3,3, Theorem 4,3 and Theorem 6,1.

The notion of a dual ring was introduced by R. Baer [2], I. Kaplansky [6] and further developped by F. F. Bonsall and A. W. Goldie [3] and K. G. Wolfson [13]. The first step in this direction was made by T. Nakayama [8], [9]. M. Hall [5] treated related questions. An extensive treatement of these results and the connections to the theory of normed rings are given in the recent book of M. A. Najmark [7].

The introductory notions of these papers use only the multiplicative properties of the elements of the ring under study. It seems therefore natural to ask how their results can be transferred to the theory of semigroups.

In this paper we show that there is some analogy between dual rings and dual semigroups as they are introduced below. The purpose of the paper is to prove fundamental structure theorems concerning such semigroups. In distinction to the papers [3], [6], [13] we shall study in this paper abstract semigroups without topological assumptions. Hence the requirements in the definitions must be somewhat sharpened. The case of compact semigroups will be treated elsewhere.

Some preliminary lemmas which can be found in [6] and [7] and remain unaltered in the theory of semigroups are shortly reproduced for the convenience of the reader.

The proofs and results of this paper are in a loose connection with a paper of A. H. Clifford [4] and an older paper of the author [11].

The following is a consequence of the results of this paper, particularly Theorems 3,1, 4,3, 5,1 and 6,1. Let S be a dual semigroup satisfying the (strong) maximal and minimal conditions for left ideals and for right ideals. Let N be the radical of S. Then N is the intersection of all the maximal left (or right or two-sided) ideals of S. The difference semigroup S/N is a dual semigroup without nilpotent ideals, and is the 0-disjoint union of simple dual semigroups, each of which is a semigroup of finite square matrices over a group with zero. The theorems mentioned are, however, proved under much weaker conditions.

1. INTRODUCTORY NOTIONS

In the whole of the paper S denotes a semigroup with a zero element. The zero element, which will be denoted by 0, is characterized by the property $x0 = 0x = 0$ for every $x \in S$.

A subset $L \subset S (R \subset S)$ is called a left (right) ideal of S if $SL \subset L (RS \subset R)$. A two-sided ideal is a subset of S which is both a left and right ideal of S. The zero element and the whole semigroup S are trivial two-sided ideals of S. (We shall freely use 0 to denote the zero-ideal (0).) Each left (right, two-sided) ideal of S contains the zero element. A left ideal of S is called a minimal left ideal of S if it is $\neq 0$ and does not contain a proper subset $\neq 0$ which is itself a left ideal of S. A left ideal is called maximal if it is $\neq S$ and is not contained as a proper subset in a left ideal $\neq S$. Analogously minimal and maximal right and two-sided ideals are defined

Definition. *Let A be a non-vacuous subset of S. The left annihilator $\mathfrak{L}(A)$ of A is the set of all $x \in S$ with $xA = 0$. The right annihilator $\mathfrak{R}(A)$ of A is the set of all $x \in S$ with $Ax = 0$.*

The definition implies $\mathfrak{L}(A)\, A = A\mathfrak{R}(A) = 0$. The following lemmas can be easily proved:

Lemma 1,1. a) *For every non-vacuous $A \subset S$ the set $\mathfrak{L}(A)$ is a left ideal of S and the set $\mathfrak{R}(A)$ is a right ideal of S.*

b) *For any non-vacuous $A \subset S$ we have $A \subset \mathfrak{R}[\mathfrak{L}(A)]$ and $A \subset \mathfrak{L}[\mathfrak{R}(A)]$.*

c) *If $\emptyset \neq A_1 \subset A_2$, then $\mathfrak{L}(A_1) \supset \mathfrak{L}(A_2)$ and $\mathfrak{R}(A_1) \supset \mathfrak{R}(A_2)$.*

d) *If M is a right [left] ideal of S, then $\mathfrak{R}(M)$ [$\mathfrak{L}(M)$] is a two-sided ideal of S.*[1])

Lemma 1,2. *Let $\{A_\nu \mid \nu \in \Lambda\}$ be a collection of subsets of S. Then we have*

$$\mathfrak{L}(\bigcup_{\nu\in\Lambda} A_\nu) = \bigcap_{\nu\in\Lambda} \mathfrak{L}(A_\nu)\,, \qquad \mathfrak{R}(\bigcup_{\nu\in\Lambda} A_\nu) = \bigcap_{\nu\in\Lambda} \mathfrak{R}(A_\nu)\,.$$

Let **L**(**R**) be the set of all left (right) ideals of S. Let $L \in$ **L**. The correspondence $L \to \mathfrak{R}(L)$ defines a mapping of the elements $\in$ **L** on the elements $\in$ **R**. In general **L** is mapped onto a subset of **R**.

Suppose that for each right ideal R of S the relation

$$\mathfrak{R}[\mathfrak{L}(R)] = R \tag{1}$$

---

[1]) If, for instance, M is a right ideal, we have $M(S\mathfrak{R}(M)) = (MS)\,\mathfrak{R}(M) \subset M\mathfrak{R}(M) = 0$, hence $S\mathfrak{R}(M) \subset \mathfrak{R}(M)$. Further $M(\mathfrak{R}(M)\, S) = (M\mathfrak{R}(M))\, S = 0 .\, S = 0$, hence $\mathfrak{R}(M)\, .\, S \subset \mathfrak{R}(M)$. This proves that $\mathfrak{R}(M)$ is a two-sided ideal.

holds. Then the image of **L** in the mapping $L \to \mathfrak{R}(L)$ is the whole set **R**. For, suppose that R is any element $\in$ **R**. Then $\mathfrak{L}(R) \in$ **L** and $\mathfrak{R}[\mathfrak{L}(R)] = R$, hence R is the image of a certain element $\in$ **L**. Examples show that the correspondence between **L** and **R** need not be one-to-one, since two different elements $\in$ **L** may be mapped into the same element $\in$ **R**.

Suppose moreover that besides the relation (1) also $\mathfrak{L}[\mathfrak{R}(L)] = L$ holds for every left ideal L of S. Then for $L_1, L_2 \in$ **L** with $\mathfrak{R}(L_1) = \mathfrak{R}(L_2)$ we have $\mathfrak{L}[\mathfrak{R}(L_1)] = \mathfrak{L}[\mathfrak{R}(L_2)]$, i. e. $L_1 = L_2$.

Hence the mapping $L \to \mathfrak{R}(L)$ defines a one-to-one correspondence between the elements of **L** and **R**. Clearly $R \to \mathfrak{L}(R)$ is the inverse mapping to the mapping $L \to \mathfrak{R}(L)$.

We introduce the following definition:

Definition. *A semigroup $S \neq 0$ is called dual if for every left ideal L of S we have*

$$\mathfrak{L}[\mathfrak{R}(L)] = L \,, \tag{2}$$

and for every right ideal R of S we have $\mathfrak{R}[\mathfrak{L}(R)] = R$.

Lemma 1,3. *Let $\{R_\nu \mid \nu \in \Lambda\}$ ($\{L_\nu \mid \nu \in \Lambda\}$) be a collection of right (left) ideals of a dual semigroup S. Then we have*

$$\mathfrak{L}(\bigcap_{\nu\in\Lambda} R_\nu) = \bigcup_{\nu\in\Lambda} \mathfrak{L}(R_\nu) \,, \qquad \mathfrak{R}(\bigcap_{\nu\in\Lambda} L_\nu) = \bigcup_{\nu\in\Lambda} \mathfrak{R}(L_\nu) \,.$$

Proof. Using Lemma 1,2 we have

$$\bigcup_{\nu\in\Lambda} \mathfrak{L}(R_\nu) = \mathfrak{L}\{\mathfrak{R}[\bigcup_{\nu\in\Lambda} \mathfrak{L}(R_\nu)]\} = \mathfrak{L}\{\bigcap_{\nu\in\Lambda} \mathfrak{R}[\mathfrak{L}(R_\nu)]\} = \mathfrak{L}\{\bigcap_{\nu\in\Lambda} R_\nu\} \,.$$

The second statement can be proved analogously.

Corollary 1,3. *Let S be dual. Then for any two left ideals L_1, L_2 with $L_1 \cap L_2 = 0$ we have $\mathfrak{R}(L_1) \cup \mathfrak{R}(L_2) = S$. For any two right ideals R_1, R_2 with $R_1 \cap R_2 = 0$ we have $\mathfrak{L}(R_1) \cup \mathfrak{L}(R_2) = S$.*

Let S be dual, **L** the set of all left ideals and $L_1, L_2 \in$ **L**. Since the intersection $L_1 \cap L_2$ is a non-empty left ideal and so is $L_1 \cup L_2$, we may introduce in **L** the operations $\cup$ and $\cap$ under which **L** becomes a lattice $\mathbf{L}_0$. The lattice of right ideals $\mathbf{R}_0$ is defined analogously. Clearly $\mathbf{L}_0$ and $\mathbf{R}_0$ are complete lattices.

Let L_ν, $\nu \in \Lambda$, be a non-vacuous collection of elements $\in \mathbf{L}_0$. Then the one-to-one mapping $L \to \mathfrak{R}(L)$ has the following properties:

$$\mathfrak{R}(\bigcup_{\nu\in\Lambda} L_\nu) = \bigcap_{\nu\in\Lambda} \mathfrak{R}(L_\nu) \,, \qquad \mathfrak{R}(\bigcap_{\nu\in\Lambda} L_\nu) = \bigcup_{\nu\in\Lambda} \mathfrak{R}(L_\nu) \,.$$

Hence the lattices $\mathbf{L}_0$, $\mathbf{R}_0$ are (completely) antiisomorphic.

The set of all two-sided ideals of S is clearly a complete sublattice of both $\mathbf{L}_0$ and $\mathbf{R}_0$. If M is a two-sided ideal of S, then so is $\mathfrak{R}(M)$ (see Lemma 1,1d). This implies:

Theorem 1,1. *Let S be a dual semigroup. Then the complete lattices $\mathbf{L}_0$ and $\mathbf{R}_0$ are antiisomorphic. Hereby the complete sublattice of two-sided ideals is anti-isomorphic to itself.*

The following two lemmas are immediate consequences of Theorem 1,1:

Lemma 1,4. *Let S be dual. Then*

a) $\mathfrak{L}(S) = \mathfrak{R}(S) = 0$.

b) *For every left ideal $L \neq S$ and right ideal $R \neq S$ we have $\mathfrak{L}(R) \neq 0$ and $\mathfrak{R}(L) \neq 0$.*

c) *For $L \neq 0$ we have $\mathfrak{R}(L) \neq S$ and for $R \neq 0$ we have $\mathfrak{L}(R) \neq S$.*

Remark. The result a) will be often used in the form: In a dual semigroup $xS = 0$ (or $Sx = 0$) implies $x = 0$.

Lemma 1,5. *Let S be dual.*

a) *If L is a minimal left of ideal S, then $\mathfrak{R}(L)$ is a maximal right ideal of S.*

b) *If M is a minimal iwo-sided ideal of S, then $\mathfrak{R}(M)$ and $L(M)$ are maximal two-sided ideals of S.*

We omit the explicit formulations of analogous converse statements.

We shall need also the following lemmas:

Lemma 1,6. *If S is dual, then $x \in xS$ and $x \in Sx$ for every $x \in S$.*

Proof. We prove $x \in xS$; the second statement can be proved analogously. Since $xS = \mathfrak{R}[\mathfrak{L}(xS)]$, it is sufficient to prove that $x \in \mathfrak{R}[\mathfrak{L}(xS)]$, i. e. $\mathfrak{L}(xS) \,.\, x = 0$. If $y \in \mathfrak{L}(xS)$ we have $yxS = 0$, hence (see Lemma 1,4a) $yx = 0$. Therefore $\mathfrak{L}(xS)\, x = 0$, q. e. d.

If A is any subset of S, Lemma 1,6 implies $A \subset SA$, $A \subset AS$. In particular, if L is a left ideal, we have $L \subset SL$ and $SL \subset L$; hence $L = SL$. We state explicitly:

Corollary 1,6a. *If $L(R, M)$ is a left (right, two-sided) ideal of a dual semigroup S, we have $L = SL(R = RS,\ M = SM = MS)$.*

We shall need especially:

Corollary 1,6b. *In a dual semigroup we have always $S^2 = S$.*

2. THE FIRST DECOMPOSITION THEOREM

A left (right, two-sided) ideal $L \neq 0$ of S is called nilpotent if there is an integer $\varrho > 0$ such that $L^\varrho = 0$.

If L is a nilpotent left ideal, then we have also $(LS)^\varrho = L(SL)^{\varrho-1} S \subset L \,.\, L^{\varrho-1}S = L^\varrho S = 0$. The set $L \cup LS$ is clearly a two-sided ideal containing L. Further $(L \cup LS)^{2\varrho} = 0$. For each element of the ideal $(L \cup LS)^{2\varrho}$ contains either at least ϱ factors L or at least ϱ factors LS. In both cases the summand equals to zero.

Hence, a semigroup having a nilpotent left (right) ideal contains also a nilpotent two-sided ideal. Therefore instead of supposing in the following that S has (or does not have) a left or right nilpotent ideal we may suppose that S contains (or does not contain) a nilpotent two-sided ideal.

A dual semigroup cannot be nilpotent. For if S is dual, we have $S^2 = S$. This implies $S^\varrho = S$ for every integer $\varrho > 0$. Hence there cannot exist an integer $\tau > 0$ with $S^\tau = 0$.

Lemma 2,1. *Let I be a two- sided ideal of S and L a nilpotent left ideal of S contained in I. Then I contains a nilpotent two-sided ideal of S.*

Proof. The relation $0 \neq L \subset I$ implies $LS \subset IS \subset I$. The ideal $L \cup LS$ is a two-sided ideal $\neq 0$, it is contained in I, and the same argument as above shows that it is nilpotent.

Definition. *The set-theoretical sum of all nilpotent ideals of S is called the radical N of S. In a semigroup without nilpotent ideals we put $N = 0$.*

A semigroup with $N = 0$ will be called more precisely a semigroup without a proper radical.

The radical N is clearly a two-sided ideal of S. It need not be itself nilpotent. However, in the following we shall often impose this condition to N.

We shall see soon that there is an essential difference between two-sided ideals of S that do not contain nilpotent subideals and those which contain (non-zero) nilpotent ideals. A two -sided ideal I contains a nilpotent subideal of S if and only if $I \cap N \neq 0$. For, if $I \cap N = 0$, it is clear that I cannot contain nilpotent ideals of S. Conversely, if $\mathfrak{m} = I \cap N \neq 0$, then I must have a non-zero intersection with at least one nilpotent ideal of S and this intersection is a two- sided nilpotent ideal.

Hence, instead of saying that a two-sided ideal I of S does not contain a nilpotent subideal of S we shall say that $I \cap N = 0$.

Lemma 2,2. *Let S be dual and I a two-sided ideal of S with the property $N \cap \cap I = 0$. Then:*

a) $\mathfrak{R}(I) = \mathfrak{L}(I)$.

b) $I \cap \mathfrak{R}(I) = I \cap \mathfrak{L}(I) = 0, \quad I \cup \mathfrak{R}(I) = I \cup \mathfrak{L}(I) = S.$

c) *If M is a left (right, two-sided) ideal of I, then M is also a left (right, two-sided) ideal of S.*

d) *If M is a left (right, two-sided) ideal of $\mathfrak{R}(I)$, then M is also a left (right, two-sided) ideal of S.*

Proof. a) According to the definition we have $I \,.\, \mathfrak{R}(I) = 0$. We prove first that $I \cap \mathfrak{R}(I) = 0$. Suppose for an indirect proof that $\mathfrak{v} = I \cap \mathfrak{R}(I) \neq \neq 0$. We then have $0 \neq \mathfrak{v} \subset I$, $0 \neq \mathfrak{v} \subset \mathfrak{R}(I)$ and $\mathfrak{v}^2 \subset I \,.\, \mathfrak{R}(I) = 0$. The ideal

I would contain a nilpotent right ideal $\mathfrak{v}$, hence (according to Lemma 2,1) a nilpotent two-sided ideal, contrary to the supposition. Now, since $\mathfrak{R}(I)\,.\,I \subset \mathfrak{R}(I) \cap I$, we have $\mathfrak{R}(I)\,.\,I = 0$, i. e. $\mathfrak{R}(I) \subset \mathfrak{L}(I)$. An analogous argument shows that $\mathfrak{L}(I) \subset \mathfrak{R}(I)$. Hence $\mathfrak{L}(I) = \mathfrak{R}(I)$, q. e. d.

b) The relation $I \cap \mathfrak{R}(I) = 0$ implies (see Corollary 1,3) $I \cup \mathfrak{R}(I) = S$, q. e. d.

c) Let M be a left ideal of I. According to the supposition we have $M \subset I$ and $IM \subset M$. Further we have $\mathfrak{R}(I)\, M \subset \mathfrak{R}(I)\, I = \mathfrak{L}(I)\,.\,I = 0$. Hence $SM = [I \cup \mathfrak{R}(I)]\, M = IM \cup \{0\} \subset M$, q. e. d.

d) Let M be a left ideal of $\mathfrak{R}(I)$. Then $\mathfrak{R}(I)\, M \subset M$. Further $M \subset \mathfrak{R}(I)$ implies $IM \subset I\mathfrak{R}(I) = 0$, hence $SM = [I \cup \mathfrak{R}(I)]\, M = IM \cup \mathfrak{R}(I)\, M \subset \subset \{0\} \cup M = M$, q. e. d.

Example 1. We show on an example that the statements c) and d) of Lemma 2,2 need not hold if I contains nilpotent ideals of S. Let $S = \{0, a, b, c, d\}$ be a commutative semigroup with the following multiplication table:

	0	a	b	c	d
0	0	0	0	0	0
a	0	0	0	b	a
b	0	0	0	a	b
c	0	b	a	d	c
d	0	a	b	c	d

S contains three ideals: S, $N = (0, a, b)$, 0, and it is easy to verify that it is a dual semigroup. Choose $N = I$. Then $(0, a)$ is an ideal of I, but not an ideal of S.

Theorem 2,1. *Let S be a dual semigroup and I a two-sided ideal of S with the property $I \cap N = 0$. Then I and $\mathfrak{R}(I)$ are dual semigroups.*

Proof. a) We prove first that I is a dual semigroup. Denote by $\mathfrak{L}'(A)$ and $\mathfrak{R}'(A)$ the left and right annihilators of A in I, respectively. It is sufficient to prove that for any right ideal M of I we have $\mathfrak{R}'[\mathfrak{L}'(M)] = M$. (The second case, i. e. the case when M is a left ideal can be proved analogously.)

Since $I \cap \mathfrak{R}(I) = 0$, $\mathfrak{L}(I) = \mathfrak{R}(I)$, $\mathfrak{L}(I)\,.\,I = 0$, we have clearly $\mathfrak{L}(M) = \mathfrak{L}(I) \cup \mathfrak{L}'(M)$. Lemma 1,2 implies $\mathfrak{R}[\mathfrak{L}(M)] = \mathfrak{R}[\mathfrak{L}(I)] \cap \mathfrak{R}[\mathfrak{L}'(M)]$. Since (according to Lemma 2,2c) M is a right ideal of S (and S is dual), we have $M = I \cap \mathfrak{R}[\mathfrak{L}'(M)]$. Further since $\mathfrak{L}'(M) \subset I$ and $0 = \mathfrak{R}(I)\, I \supset \mathfrak{R}(I)\, \mathfrak{L}'(M)$, we can write $\mathfrak{R}[\mathfrak{L}'(M)] = \mathfrak{R}(I) \cup \mathfrak{R}'[\mathfrak{L}'(M)]$. Hence we have

$$M = I \cap \{\mathfrak{R}(I) \cup \mathfrak{R}'[\mathfrak{L}'(M)]\} = [I \cap \mathfrak{R}(I)] \cup \{I \cap \mathfrak{R}'[\mathfrak{L}'(M)]\} = \\ = \{0\} \cup \{I \cap \mathfrak{R}'[\mathfrak{L}'(M)]\}\,.$$

But since $\mathfrak{R}'[\mathfrak{L}'(M)] \subset I$, we have finally $M = \mathfrak{R}'[\mathfrak{L}'(M)]$, q. e. d.

b) The proof that $\mathfrak{R}(I)$ is dual follows in the same way, but instead of

Lemma 2,2c we must use Lemma 2,2d.[1a]) Denote the left and right annihilators of $A \subset \mathfrak{R}(I)$ in $\mathfrak{R}(I)$ by $\mathfrak{L}''(A)$ and $\mathfrak{R}''(A)$, respectively, and choose any right ideal M of $\mathfrak{R}(I)$.

Clearly $\mathfrak{L}(M) = I \cup \mathfrak{L}''(M)$, and according to Lemma 1,2 we have

$$\mathfrak{R}[\mathfrak{L}(M)] = \mathfrak{R}(I) \cap \mathfrak{R}[\mathfrak{L}''(M)]\,, \qquad M = \mathfrak{R}(I) \cap \mathfrak{R}[\mathfrak{L}''(M)]\,.$$

Now since $\mathfrak{L}''(M) \subset \mathfrak{R}(I)$ and $\mathfrak{L}''(M) \,.\, I = 0$, we have $\mathfrak{R}[\mathfrak{L}''(M)] = I \cup \cup \mathfrak{R}''[\mathfrak{L}''(M)]$. Therefore

$$M = \mathfrak{R}(I) \cap \{I \cup \mathfrak{R}''[\mathfrak{L}''(M)]\} = \{\mathfrak{R}(I) \cap I\} \cup \{\mathfrak{R}(I) \cap \mathfrak{R}''[\mathfrak{L}''(M)]\} = \\ = \{0\} \cup \{\mathfrak{R}(I) \cap \mathfrak{R}''[\mathfrak{L}''(M)]\} = \mathfrak{R}''[\mathfrak{L}''(M)]\,.$$

This proves Theorem 2,1.

Suppose now that S has a proper radical N and at least one two-sided ideal $I \neq 0$ with $I \cap N = 0$. Denote by S' the class-sum of all two-sided ideals of S each of which has a zero intersection with N. The set S' is a uniquely determined two-sided ideal of S and (according to Theorem 2,1) it is a dual semigroup. Also $\mathfrak{R}(S') = S''$ is a dual semigroup having the property that each two-sided ideal $I \subset \mathfrak{R}(S')$ has a non-zero intersection with N. Therefore we can state:

Theorem 2,2. *Any dual semigroup S with the radical N admits a unique decomposition into a sum of two two-sided ideals $S = S' \cup S''$, where $S'S'' = S''S' = S' \cap S'' = 0$, the summands having the following properties:*

1. *If $N = 0$, $S' = S$, $S'' = 0$.*
2. *If $N \neq 0$, S' is either zero or it is a dual semigroup without nilpotent ideals, S'' is a dual semigroup with the radical N in which each two-sided ideal has a non-zero intersection with N.*

Example 2. We give a simple example to Theorem 2,2. Let $S = \{0, a, b, c\}$ be a semigroup with the multiplication table

	0	a	b	c
0	0	0	0	0
a	0	0	0	a
b	0	0	b	0
c	0	a	0	c

The lattice of ideals is given by the graph in Fig. 1.

It is easy to verify that S is a dual semigroup. The radical of S is $N = (0, a)$. The largest ideal which has a zero intersection with N is $S' = (0, b)$. The

[1a]) Note that the two-sided ideal $\mathfrak{R}(I)$ may contain nilpotent ideals of S. But $\mathfrak{R}(I)$ is not an arbitrary two-sided ideal of S. It is characterized by the property that it is a "complement" of a two-sided ideal without proper nilpotent ideals. (Compare with Example 1.)

"complement" $\mathfrak{R}(S') = S'' = (0, a, c)$ contains N. The decomposition in the sense of Theorem 2,2 is $S = (0, b) \cup (0, a, c)$. (Note that S contains two minimal ideals: $(0, a)$ and $(0, b)$. The first is nilpotent, the second one non-nilpotent.)

Example 3. We show that the existence of a decomposition of the type given in Theorem 2,2 does not imply that S is dual. Consider the semigroup constructed in the following manner: Let G_1, G_2 be two groups and N a semi-

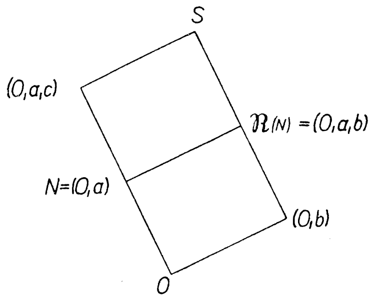

Fig. 1.

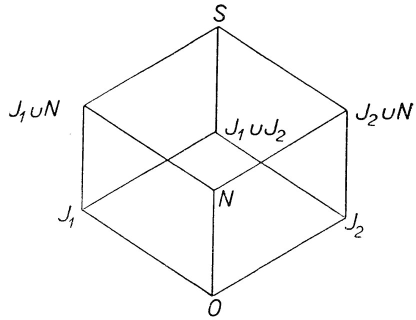

Fig. 2.

group consisting of two elements $N = \{0, a\}$ with the multiplication $0^2 = 0a = a0 = a^2 = 0$. Define further $G_1G_2 = G_2G_1 = G_1N = NG_1 = NG_2 = G_2N = 0$. Then the semigroup $S = G_1 \cup G_2 \cup N$ admits a decomposition $S = J_1 \cup J_2 \cup N$, where $J_1 = \{0\} \cup G_1$, $J_2 = \{0\} \cup G_2$ and N are minimal two- sided ideals of S. Denoting $S' = J_1 \cup J_2$, $S'' = N$ we have a decomposition of the kind given in Theorem 2.2. The lattice of ideals is given by the graph in Fig. 2. Since $\mathfrak{R}(J_1 \cup N) = \mathfrak{R}(J_1) = J_2 \cup N$, our semigroup cannot be dual.

Example 4. To have some material for later purposes we give a more complicated example in which $S' = 0$.

Let S be the multiplicative semigroup of those residue-classes (mod 72) which are represented by the numbers 0, 9, 64, 18, 48, 36. The corresponding multiplication table has the following form:

	0	9	64	18	48	36
0	0	0	0	0	0	0
9	0	9	0	18	0	36
64	0	0	64	0	48	0
18	0	18	0	36	0	0
48	0	0	48	0	0	0
36	0	36	0	0	0	0.

The lattice of ideals has the graph in Fig. 3 (which enables an easy verification that S is a dual semigroup). In this semigroup every two-sided ideal $\neq 0$ has a non-zero intersection with N. There are two minimal two-sided ideals, they are both nilpotent.

3. DUAL SEMIGROUPS WITHOUT NILPOTENT IDEALS

In this section we shall study dual semigroups with $N = 0$. We recall first that a semigroup is said to be simple if it contains no proper two-sided ideal, except the zero ideal (0).

In what follows we shall use the following theorem (see Clifford [4], Theorem 1,1, p. 835): Let S be a semigroup without nilpotent ideals. Then any minimal two-sided ideal of S is a simple semigroup.

Clearly, the intersection of two distinct minimal two-sided ideals of S is the zero element 0. If M_α, M_β are two distinct minimal two-sided ideals of S, we have $M_\alpha M_\beta \subset M_\alpha \cap M_\beta = 0$, hence $M_\alpha M_\beta = 0$.

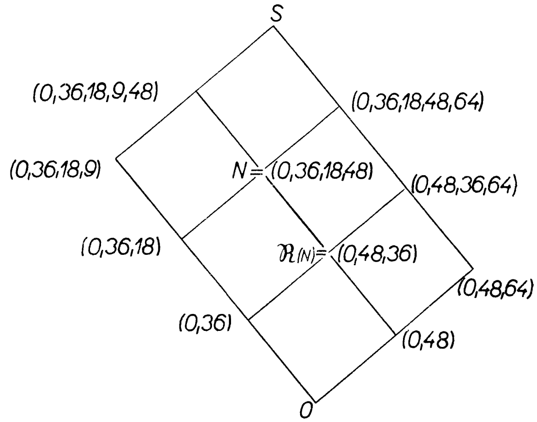

Fig. 3.

Let M_ν, $\nu \in \Lambda$, be the set of all minimal two-sided ideals of a dual semigroup S without nilpotent ideals. According to Theorem 2,1 each M_ν is a dual semigroup. Denote $T = \bigcup_{\nu \in \Lambda} M_\nu$. T is a two-sided ideal. According to Lemma 2,2 we have $S = T \cup \mathfrak{R}(T)$, where T and $\mathfrak{R}(T)$ are dual semigroups with $T \cap \mathfrak{R}(T) = 0$.

Suppose now that every two-sided ideal of S contains at least one minimal two-sided ideal of S. Then $\mathfrak{R}(T) = 0$. For, since $\mathfrak{R}(T)$ is a two-sided ideal of S, $\mathfrak{R}(T) \neq 0$ would imply the existence of a minimal two-sided ideal of S contained in $\mathfrak{R}(T)$. This is a contradiction to $T \cap \mathfrak{R}(T) = 0$.

We have proved:

Theorem 3,1. *Let S be a dual semigroup without nilpotent ideals. Suppose that every two-sided ideal of S contains at least one minimal two-sided ideal of S. Then we have $S = \bigcup_{\nu \in \Lambda} M_\nu$, where $M_\alpha M_\beta = M_\alpha \cap M_\beta = 0$ for $\alpha \neq \beta \in \Lambda$, and M_ν are simple dual semigroups.*

The converse statement is given by

Theorem 3,2. *Let M_ν, $\nu \in \Lambda$, be a collection of simple dual semigroups with $M_\alpha \cap M_\beta = \emptyset$ for $\alpha \neq \beta \in \Lambda$. Identify the zero elements of all M_ν, $\nu \in \Lambda$. Consider the set $S = \bigcup_{\nu \in \Lambda} M_\nu$ and define in S a multiplication $*$ by the following relations: For $a, b \in S$ we have*

$$a * b = \begin{cases} ab & \text{if } a, b \text{ belong to the same } M_\alpha\,, \\ 0 & \text{if } a \in M_\alpha,\ b \in M_\beta,\ \alpha \neq \beta \in \Lambda\,. \end{cases}$$

Then S is a dual semigroup without nilpotent ideals in which every two-sided ideal ($\neq 0$) of S contains at least one minimal two-sided ideal of S.

Proof. We may suppose card $\Lambda > 1$ Let L be a left ideal of S. We show that $\mathfrak{L}[\mathfrak{R}(L)] = L$. Denote $L \cap M_\nu = L_\nu$. Then $L = \bigcup_{\nu \in \Lambda} L_\nu$ and $\mathfrak{R}(L) = \bigcap_{\nu \in \Lambda} \mathfrak{R}(L_\nu)$. Denote further $\overline{M}_\nu = \bigcup_{\alpha \in \Lambda,\, \alpha \neq \nu} M_\alpha$ and let $\mathfrak{R}^\nu(L_\nu)$ be the right annihilator of L_ν in M_ν.[2]) Then $\mathfrak{R}(L_\nu) = \overline{M}_\nu \cup \mathfrak{R}^\nu(L_\nu)$ and $\mathfrak{R}(L) = \bigcap_{\nu \in \Lambda} [\overline{M}_\nu \cup \mathfrak{R}^\nu(L_\nu)]$. Now it follows from a merely set-theoretical conclusion that the last intersection equals to the sum $\bigcup_{\nu \in \Lambda} \mathfrak{R}^\nu(L_\nu)$. Hence $\mathfrak{R}(L) = \bigcup_{\nu \in \Lambda} \mathfrak{R}^\nu(L_\nu)$.

Now we have $\mathfrak{L}[\mathfrak{R}(L)] = \mathfrak{L}[\bigcup_{\nu \in \Lambda} \mathfrak{R}^\nu(L_\nu)] = \bigcap_{\nu \in \Lambda} \mathfrak{L}[\mathfrak{R}^\nu(L_\nu)]$. Denoting by $\mathfrak{L}^\nu[\mathfrak{R}^\nu(L_\nu)]$ the left annihilator of $\mathfrak{R}^\nu(L_\nu)$ in M_ν we have $\bigcap_{\nu \in \Lambda} \mathfrak{L}[\mathfrak{R}^\nu(L_\nu)] = \bigcap_{\nu \in \Lambda} \{\overline{M}_\nu \cup \mathfrak{L}^\nu[\mathfrak{R}^\nu(L_\nu)]\}$. Since M_ν is dual, we have $\mathfrak{L}^\nu[\mathfrak{R}^\nu(L_\nu)] = L_\nu$. Therefore $\mathfrak{L}[\mathfrak{R}(L)] = \bigcap_{\nu \in \Lambda} [\overline{M}_\nu \cup L_\nu]$. It follows again from a merely set-theoretical conclusion that the last expression equals to $\bigcup_{\nu \in \Lambda} L_\nu$. Hence $\mathfrak{L}[\mathfrak{R}(L)] = \bigcup_{\nu \in \Lambda} L_\nu = L$, q. e. d.

By an analogous argument we prove that for any right ideal R of S we have $\mathfrak{R}[\mathfrak{L}(R)] = R$. This proves Theorem 3,2.

Combining Theorems 3,1 and 3,2 we get:

Theorem 3,3. *Let S be a semigroup without nilpotent ideals in which every two-sided ideal of S contains at least one minimal two-sided ideal of S. Then S is dual if and only if S is the union of its minimal two-sided ideals and each of these minimal ideals is a dual semigroup.*

An other criterion for the duality, which we shall use later, is given in Theorem 3,4.

In the following lemma we do not suppose the duality of S.

Lemma 3,1. *Let S be a semigroup with zero, without nilpotent ideals, containing at least two maximal two-sided ideals of S. Let $\{M_\alpha^* \mid \alpha \in \Lambda\}$ be the totality*

[2]) If for some $\nu \in \Lambda$ we have $L_\nu = 0$, then $\mathfrak{R}^\nu(L_\nu) = M_\nu$.

of all different maximal two-sided ideals of S. *Suppose that* $\bigcap_{\alpha\in\Lambda} M^*_\alpha = 0$ *and denote* $P_\alpha = S - M^*_\alpha$. *We then have*:

a) $S = \{0\} \cup [\bigcup_{\alpha\in\Lambda} P_\alpha]$, *where* $P_\alpha \cap P_\beta = \emptyset$ *and* $P_\alpha P_\beta = 0$ *for* $\alpha \neq \beta \in \Lambda$.

b) *For any* $\alpha \in \Lambda$ $\mathfrak{L}(M^*_\alpha) = \mathfrak{R}(M^*_\alpha) = \{0\} \cup P_\alpha$ *holds.*

c) *For any* $\alpha \in \Lambda$ *we have* $S = M^*_\alpha \cup \mathfrak{L}(M^*_\alpha)$, *where* $M^*_\alpha \cap \mathfrak{L}(M^*_\alpha) = 0$ *and* $\mathfrak{L}(M^*_\alpha)$ *is a minimal two-sided ideal of* S.

d) $S = \bigcup_{\alpha\in\Lambda} \mathfrak{L}(M^*_\alpha)$.

e) *Every two-sided ideal of* S *contains at least one minimal two-sided ideal of* S *and* $\{\mathfrak{L}(M^*_\alpha) \mid \alpha \in \Lambda\}$ *is exactly the set of all minimal two-sided ideals of* S.

Proof. a) We prove first that for $\alpha \neq \beta$ we have $P_\alpha \cap P_\beta = \emptyset$. It follows from a purely set-theoretical conclusion that $P_\alpha \cap P_\beta = (S - M^*_\alpha) \cap (S - M^*_\beta) = S - (M^*_\alpha \cup M^*_\beta)$. Since M^*_α, M^*_β are two distinct maximal two-sided ideals of S, we have $M^*_\alpha \cup M^*_\beta = S$, hence $P_\alpha \cap P_\beta = \emptyset$.

We have further $0 = \bigcap_{\alpha\in\Lambda} M^*_\alpha = \bigcap_{\alpha\in\Lambda} (S - P_\alpha) = S - \bigcup_{\alpha\in\Lambda} P_\alpha$, hence S can be written as a union of disjoint sets: $S = \{0\} \cup [\bigcup_{\alpha\in\Lambda} P_\alpha]$.

We show finally that for $\beta \neq \gamma$ we have $P_\beta P_\gamma = 0$. Suppose for an indirect proof that there exists a couple of elements $u_\beta \in P_\beta$, $u_\gamma \in P_\gamma$ such that $u_\beta u_\gamma = u_\delta \neq 0$ and $u_\delta \in P_\delta$. We consider separately the two possibilities: $u_\delta \notin P_\beta$ and $u_\delta \in P_\beta$. Suppose first $u_\delta \notin P_\beta$, i. e. $P_\beta \neq P_\delta$. Since then $P_\beta \subset S - P_\delta = M^*_\delta$ and M^*_δ is a two-sided ideal, we have $P_\beta P_\gamma \subset M^*_\delta P_\gamma \subset M^*_\delta$, hence $u_\delta \in M^*_\delta = S - P_\delta$, which is a contradiction to $u_\delta \in P_\delta$. Suppose next $u_\delta \in P_\beta$. Since $P_\beta \neq P_\gamma$, we have $P_\gamma \subset S - P_\beta = M^*_\beta$ and $P_\beta P_\gamma \subset P_\beta M^*_\beta \subset M^*_\beta$, hence $u_\delta \in M^*_\beta = S - P_\beta$, which contradicts to the supposition $u_\delta \in P_\beta$.

b) Since $P_\alpha M^*_\alpha = P_\alpha \{0 \cup [\bigcup_{\substack{\nu\in\Lambda\\ \nu\neq\alpha}} P_\nu]\} = 0$, we have $\{0\} \cup P_\alpha \subseteqq \mathfrak{L}(M^*_\alpha)$. On the other hand the two-sided ideal $\mathfrak{L}(M^*_\alpha)$ has a zero intersection with M^*_α. For, $\mathfrak{L}(M^*_\alpha) \cap M^*_\alpha = \mathfrak{l} \neq 0$ and $\mathfrak{L}(M^*_\alpha) M^*_\alpha = 0$ would imply $\mathfrak{l}^2 = 0$, i. e. there would exist a nilpotent left ideal $\mathfrak{l} \neq 0$ of S, contrary to the supposition. Therefore $\mathfrak{L}(M^*_\alpha) = \{0\} \cup P_\alpha$. An analogous argument shows that $\mathfrak{R}(M^*_\alpha) = \{0\} \cup P_\alpha$.

c) It is sufficient to prove that $\mathfrak{L}(M^*_\alpha)$ is a minimal two-sided ideal of S. If there were an ideal I of S with $0 \subsetneqq I \subsetneqq \mathfrak{L}(M^*_\alpha)$, the set $M^*_\alpha \cup I$ would be a two-sided ideal of S, which is apparently larger than M^*_α, hence equals to S. But this is impossible since $M^*_\alpha \cup I$ does not contain the elements $\in \mathfrak{L}(M^*_\alpha) - I \neq \emptyset$.

d) This statement is an immediate consequence of $S = \{0\} \cup [\bigcup_{\alpha\in\Lambda} P_\alpha]$, $P_\alpha \cap P_\beta = \emptyset$ for $\alpha \neq \beta$ and $\mathfrak{L}(M^*_\alpha) = \{0\} \cup P_\alpha$.

e) If I is a two-sided ideal of S, then $S = \bigcup_{\alpha \in \Lambda} \mathfrak{L}(M_\alpha^*)$ implies the existence of at least one $\gamma \in \Lambda$ such that $I \cap \mathfrak{L}(M_\gamma^*) \neq \{0\}$. Since $\mathfrak{L}(M_\gamma^*)$ is minimal, we have $\mathfrak{L}(M_\gamma^*) \subset I$. If, moreover, I is minimal we have $I = \mathfrak{L}(M_\gamma^*)$.

This completes the proof of our Lemma.

Remark. Lemma 3,1 trivially holds also in the case if there is a unique maximal ideal M_α^*. We then have $M_\alpha^* = 0$, $\mathfrak{L}(M_\alpha^*) = S$, and $P_\alpha = S - \{0\}$. In this case S is a simple semigroup.

Theorem 3,4. *Let S be a semigroup with zero, without nilpotent ideals, in which every two-sided ideal of S and different from S is contained in a maximal two-sided ideal of S. Suppose that there exist at least two maximal two-sided ideals of S. Let $\{M_\alpha^* \mid \alpha \in \Lambda\}$ be the set of all maximal ideals of S. Then S is dual if and only if*:

a) $\bigcap_{\alpha \in \Lambda} M_\alpha^* = 0$. b) *Every semigroup M_α^*, $\alpha \in \Lambda$, is a dual semigroup.*

Proof. 1 Suppose that S is dual. Condition b) is satisfied according to Theorem 2,1. The duality implies that every two-sided ideal I of S contains a minimal two-sided ideal of S. By Theorem 3,1 we have $S = \bigcup_{\alpha \in \Lambda} \mathfrak{m}_\alpha$, where $\{\mathfrak{m}_\alpha \mid \alpha \in \Lambda\}$ is the set of all minimal two-sided ideals of S. Now since S is dual, we have $0 = \mathfrak{R}(S) = \mathfrak{R}(\bigcup_{\alpha \in \Lambda} \mathfrak{m}_\alpha) = \bigcap_{\alpha \in \Lambda} \mathfrak{R}(\mathfrak{m}_\alpha)$. The set $\{\mathfrak{R}(\mathfrak{m}_\alpha) \mid \alpha \in \Lambda\}$ is exactly the set of all maximal two-sided ideals of S. Hence the first part of our Theorem is proved.

2. Suppose that the conditions a) and b) are satisfied. We show that S is dual. According to Lemma 3,1 we can write $S = M_\alpha^* \cup \mathfrak{L}(M_\alpha^*)$. The two-sided ideal $\mathfrak{L}(M_\alpha^*)$ is contained in a maximal two-sided ideal M_β^* of S and M_β^* is a dual semigroup. $\mathfrak{L}(M_\alpha^*)$, being a two-sided ideal of S, is the more a two-sided ideal of M_β^*. According to Theorem 2,1 $\mathfrak{L}(M_\alpha^*)$ is therefore a dual semigroup. Now condition a) implies (see Lemma 3,1d) that $S = \bigcup_{\alpha \in \Lambda} \mathfrak{L}(M_\alpha^*)$. Further (see Lemma 3,1e) every two-sided ideal of S contains a minimal two-sided ideal of S and S is the union of its minimal two-sided ideals each of which is a dual semigroup. Theorem 3,3 implies that S is a dual semigroup. This proves our Theorem.

Supplement to Theorem 3,4. *Suppose that the suppositions of Theorem 3,4 are satisfied with the exception that there exists a unique maximal two-sided ideal M of S. Then, if S is dual, $M = 0$ and S is a simple dual semigroup.*

Proof. Analogously as in the proof of Theorem 3,4 it follows that every two-sided ideal of S contains a minimal two-sided ideal of S. Since there is a unique maximal two-sided ideal M in S, we conclude that there exists a unique minimal two-sided ideal $\mathfrak{m}$ of S. Theorem 3,1 implies $S = \mathfrak{m}$. Hence S is a simple dual semigroup and $M = 0$.

We can prove similarly:

Theorem 3,4a. *Suppose that the suppositions of Theorem* 3,4 *are satisfied. Then S is dual if and only if*:

a) $\bigcap_{\alpha \in \Lambda} M_\alpha^* = 0$. b) *There is a pair of two-sided ideals M_1, M_2 which are themselves dual semigroups, and we have $S = M_1 \cup M_2$ with $M_1 M_2 = 0$.*

Theorem 3,4b. *Let S be a semigroup with zero element, without nilpotent ideals, in which every two-sided ideal of S is contained in a maximal two-sided ideal of S. Let $\{M_\alpha^* \mid \alpha \in \Lambda\}$ be the set of all maximal ideals of S. Then S is dual if and only if:*

a) $\bigcap_{\alpha \in \Lambda} M_\alpha^* = 0$. b) *Each of the semigroups $\mathfrak{L}(M_\alpha^*)$ is dual.*

4. SIMPLE DUAL SEMIGROUPS

Let S be a simple semigroup with zero. Since $0 \subset S^2 \subset S$ there are only two possibilities: either $S^2 = 0$ or $S^2 = S$. The case $S^2 = 0$ is not interesting since S has then exactly two elements: the zero element and a further unique non-zero nilpotent element.

We recall: Let $S \neq 0$ be a simple semigroup having a zero element with $S^2 \neq 0$. If S contains at least one minimal left ideal, then S is the class sum of all minimal left ideals: $S = \bigcup_\nu L_\nu$, where $L_\alpha \cap L_\beta = 0$ for $\alpha \neq \beta$. If S has at least one minimal left ideal and at least one minimal right ideal, then S contains idempotents $\neq 0$ and each minimal left (right) ideal is generated by an idempotent, i. e. it is of the form Se_α ($e_\beta S$) with a suitably chosen idempotent $e_\alpha \neq 0 (e_\beta \neq 0)$.

Now we shall study simple d u a l semigroups. Acording to Corollary 1,6b we then have necessarily $S^2 \neq 0$ (and $S^2 = S$).

Lemma 4,1. *Let S be a simple dual semigroup having at least one minimal left ideal. Then S has also a minimal right ideal (and hence contains an idempotent $\neq 0$).*

Proof. According to the supposition we can write $S = \bigcup_{\alpha \in \Lambda} L_\alpha$, where L_α runs through all minimal left ideals of S.

a) Suppose card $\Lambda > 1$ and consider the ideal $L'_\beta = \bigcup_{\alpha \in \Lambda, \alpha \neq \beta} L_\alpha$. This is clearly a maximal left ideal of S. According to Lemma 1,5 $\mathfrak{R}(L'_\beta)$ is a minimal right ideal of S. The existence of a minimal right ideal is proved.

b) Suppose next that $S = L_\alpha$ (i. e. S does not contain a left ideal $\neq S$ and $\neq 0$). Then S cannot contain a right ideal $\neq 0$ and $\neq S$. (For, if $\mathfrak{r}$ were a right ideal, $0 \subsetneqq \mathfrak{r} \subsetneqq S$, $\mathfrak{L}(\mathfrak{r})$ would be a left ideal, $0 \subsetneqq \mathfrak{L}(\mathfrak{r}) \subsetneqq S$.) Hence S is at the same time a minimal right ideal of S. The existence of the minimal right ideal is proved.

Remark 1. Let be in this last case $x \in S$ and $x \neq 0$. Then since $x \in Sx$, we have $Sx = S$ and analogously $xS = S$. Therefore the set of all non-zero elements $\in S$ forms a group. The unit element of the group $S - \{0\}$ is the (unique) idempotent $\neq 0$ of S.

Remark 2. Let G be a group. Adjoin to G an idempotent 0 and define $0 \,.\, G = G \,.\, 0 = 0$. Then we term the semigroup $G^{(0)} = G \cup \{0\}$ thus obtained "a group with zero adjoined" (or briefly a "group with zero"). Such a semigroup is simple having only two ideals, namely 0 and $G^{(0)}$.

A group with zero is dual and contains a unique idempotent $\neq 0$. Conversely, it is known that a simple semigroup with zero containing a unique idempotent $\neq 0$ is a group with zero. Hence:

Lemma 4,2. *A simple semigroup with zero and with a unique idempotent* $\neq 0$ *is dual.*

Theorem 4,1. *Let S be a simple dual semigroup having at least one minimal left ideal. Then for any two idempotents $e_\alpha \neq e_\beta$ we have $e_\alpha e_\beta = 0$.*

Proof. With respect to Lemma 4,1 S contains also at least one minimal right ideal and hence at least one idempotent $\neq 0$.

Let $e_\alpha \neq 0$ be any idempotent $\in S$. The left ideal Se_α is a minimal left ideal of S. Therefore $\mathfrak{R}(Se_\alpha)$ is a maximal right ideal of S. Write $S = \bigcup\limits_{\nu \in \Lambda} R_\nu$, where R_ν runs through all different minimal right ideals of S. Then it is clear that every maximal right of S is of the form $\bigcup\limits_{\nu \in \Lambda}{}' R_\nu$, where the dote denotes that precisely one of the summands is excluded. We may therefore write $\mathfrak{R}(Se_\alpha) = \bigcup\limits_{\substack{\nu \in \Lambda \\ \nu \neq \nu_0}} R_\nu$. Now since

$$Se_\alpha \,.\, [\bigcup_{\substack{\nu \in \Lambda \\ \nu \neq \nu_0}} R_\nu] = 0 \,, \tag{3}$$

it follows that R_{ν_0} must be the minimal right ideal of S containing e_α, hence $e_\alpha S$. [For otherwise the product to the left would contain $Se_\alpha \,.\, e_\alpha S$ which is certainly $\neq 0$, containing itself e_α.]

We prove now that Se_α (and hence every minimal left ideal of S) contains precisely one non-zero idempotent. Denote $Se_\alpha = L_{\mu_0}$. We then have

$$L_{\mu_0} = L_{\mu_0} \cap S = [L_{\mu_0} \cap R_{\nu_0}] \cup [L_{\mu_0} \cap (\bigcup_{\substack{\nu \in \Lambda \\ \nu \neq \nu_0}} R_\nu)] \,.$$

The second summand cannot contain an idempotent $\neq 0$ since otherwise the relation (3) would not be possible. Now the minimal ideals L_{μ_0} and R_{ν_0} have the following properties: $L_{\mu_0}R_{\nu_0} = Se_\alpha e_\alpha S = S$, $R_{\nu_0}L_{\mu_0} = e_\alpha S \, . \, Se_\alpha \neq 0$. It is well known that under these conditions $R_{\nu_0} \, . \, L_{\mu_0}$ equals then to $R_{\nu_0} \cap L_{\mu_0}$ and it is a group with zero. (See f. i. Clifford [4], Lemma 3,2—3,5, p. 837.) Hence the first summand (and therefore Se_α) contains a unique non-zero idempotent (which is, of course, e_α).

By an analogous argument we can prove that every minimal right ideal of S contains a unique idempotent $\neq 0$.

Hence the correspondence between the non-zero idempotents and minimal right ideals R_ν, $\nu \in \Lambda$, is a one-to-one. We can write $R_\nu = e_\nu S$, where e_ν is the unique non-zero idempotent $\in R_\nu$.

Now we easily achieve the proof of our Theorem. If S contains a unique non-zero idempotent, Theorem 4,1 trivially holds. We may suppose therefore that there are at least two non-zero idempotents. The relation (3) can be written in the form $(Se_\alpha) \, . \, [\bigcup_{\substack{\beta \in \Lambda \\ \beta \neq \alpha}} e_\beta S] = 0$, i. e. $\bigcup_{\substack{\beta \in \Lambda \\ \beta \neq \alpha}} Se_\alpha e_\beta S = 0$. The summand $Se_\alpha e_\beta S$ contains the element $e_\alpha(e_\alpha e_\beta) \, e_\beta = e_\alpha e_\beta$. Hence $e_\alpha e_\beta = 0$ for every $\beta \neq \alpha$. This proves Theorem 4,1.

Theorem 4,2. *Let S be a simple semigroup with zero containing at least one left and one right minimal ideal. Suppose that for any two idempotents $e_\alpha \neq e_\beta \in S$ we have $e_\alpha e_\beta = 0$. Then S is a dual semigroup.*

Proof. With respect to Lemma 4,2 we may suppose that there exist at least two non-zero idempotents.

We prove first that under our suppositions every minimal left (right) ideal of S contains precisely one non-zero idempotent, and hence there is again a one-to-one correspondence between the set of different minimal left (right) ideals and the set of all non-zero idempotents.

Let Se_α, $e_\alpha \neq 0$, be a minimal left ideal of S. We have $e_\alpha \in Se_\alpha$ and $Se_\alpha e_\alpha S = Se_\alpha S \neq 0$. If there were in Se_α an other idempotent $e_\beta \neq 0$, $e_\beta \neq e_\alpha$, there would hold $Se_\alpha = Se_\beta$ and $Se_\alpha e_\alpha S = Se_\beta e_\alpha S = S \, . \, 0 \, . \, S = 0$. This constitutes an apparent contradiction.

Let $\{e_\alpha \mid \alpha \in \Lambda\}$ be the set of all idempotents $\in S$. Let L be any left ideal of S. We have to show that $\mathfrak{L}[\mathfrak{R}(L)] = L$. The left ideal L can be written in the form $L = \bigcup_{\alpha \in H} Se_\alpha$, where $H \subset \Lambda$.

Suppose first that $\Lambda - H \neq \emptyset$. Then $\mathfrak{R}(L) = \bigcup_{\beta \in \Lambda - H} e_\beta S$. If namely some β were in H the product $L \, . \, \mathfrak{R}(L)$ would contain $Se_\beta e_\beta S$ and since this set contains e_β, the product would not be 0. The left annihilator $\mathfrak{L}[\mathfrak{R}(L)] = \mathfrak{L}[\bigcup_{\beta \in \Lambda - H} e_\beta S]$ is the sum of those minimal left ideals Se_γ for which $Se_\gamma \, (\bigcup_{\alpha \in \Lambda - H} e_\alpha S) = 0$ holds. Clearly, this is exactly the set $\bigcup_{\gamma \in H} Se_\gamma$. Hence $\mathfrak{L}[\mathfrak{R}(L)] = L$.

Suppose next $\Lambda = H$, i. e. $L = S$. Then $\mathfrak{R}(L)$ must be 0. (For if $\mathfrak{R}(L)$ contained at least one summand of the form $e_\gamma S$, the product $L \,.\, \mathfrak{R}(L) = S\mathfrak{R}(L)$ would not be zero, since it contains $e_\gamma \neq 0$.) The relation $\mathfrak{R}(S) = 0$ implies $\mathfrak{L}[\mathfrak{R}(S)] = \mathfrak{L}(0) = S$.

An analogous argument shows that for every right ideal R we have $\mathfrak{R}[\mathfrak{L}(R)] = R$. This proves Theorem 4,2.

Example 5. An example of a dual simple semigroup (which is not a group with zero) is given by the set $S = \{0, a_1, a_2, a_3, a_4\}$ with the following multiplication table:

	0	a_1	a_2	a_3	a_4
0	0	0	0	0	0
a_1	0	a_1	0	a_3	0
a_2	0	0	a_2	0	a_4
a_3	0	0	a_3	0	a_1
a_4	0	a_4	0	a_2	0

A simple semigroup containing at least one minimal left and at least one minimal right ideal is called a completely simple semigroup. Theorem 4,1 and Theorem 4,2 give a necessary and sufficient condition for a completely simple semigroup to be dual.

Our next goal is to describe the construction of completely simple dual semigroups.

We recall some known results first proved by D. Rees [10].

Let $I = \{i, j, k, \ldots\}$ and $\Lambda = \{\varkappa, \lambda, \mu, \ldots\}$ be any two sets of indices and $G^{(0)} = G \cup \{0\}$ a group with zero. By a „$I \times \Lambda$ matrix $\mathbf{M} = \{a_{i\varkappa}\}$ over $G^{(0)}$" we mean a set of elements which we obtain by assigning to each pair $i, \varkappa$ of indices ($i \in I$, $\varkappa \in \Lambda$) a unique element $a_{i\varkappa} \in G^{(0)}$. The set of all elements $a_{i\varkappa}$ of $\mathbf{M}$ with the same index $i \in I$ will be called the i-th row of $\mathbf{M}$. Analogously the $\varkappa$-th column is defined. $\mathbf{M}$ is called regular if no row or column of $\mathbf{M}$ consists of zeros. $\mathbf{M}$ is called monomial if in each row and column exactly one entry is $\neq 0$. For a monomial matrix we have necessarily card I = card Λ. An $I \times \Lambda$ matrix having the element $a \in G^{(0)}$ with $a \neq 0$ in the $(i, \varkappa)$th position and 0 elsewhere will be denoted by $(a)_{i\varkappa}$. The $I \times \Lambda$ matrix for which $a_{i\varkappa} = 0$ for all pairs $(i, \varkappa)$ will be denoted by (0).

Let be $\mathbf{A}$ a $I \times \Lambda$ matrix and $\mathbf{B}$ a $\Lambda \times J$ matrix and suppose that for any $i \in I$, $l \in J$ there is at most one pair $a_{i\lambda}, b_{\lambda l}$ ($a_{i\lambda}$ in the i-th row of $\mathbf{A}$ and $b_{\lambda l}$ in the l-th column of $\mathbf{B}$) such that $a_{i\lambda} b_{\lambda l} \neq 0$. Assuming that 0 has also the properties of an additive zero we define the ordinary matrix multiplication $\mathbf{AB}$ as an $I \times J$ matrix $\mathbf{C}$ with elements $c_{il} = \sum\limits_{\lambda \in \Lambda} a_{i\lambda} b_{\lambda l}$. It is easy to show that this multiplication of matrices is associative, whenever it is defined.

Especially, if $(a)_{i\varkappa}$ is a $I \times \Lambda$ matrix, $(b)_{\lambda j}$ a $\Lambda \times I$ matrix, then their ordinary matrix product is the $I \times I$ matrix, which can be calculated by the rule:

$$(a)_{i\varkappa}(b)_{\lambda j} = \left\langle \begin{array}{ll} (ab)_{ij} & \text{if } \varkappa = \lambda\,, \\ (0) & \text{if } \varkappa \neq \lambda\,. \end{array} \right.$$

Let now be $\mathbf{P} = \{p_{\varkappa i}\}$ a fixed regular $\Lambda \times I$ matrix over $G^{(0)}$. Construct the set S of all $I \times \Lambda$ matrices of the type $(x)_{i\varkappa}$, x running through all elements of the group G, together with the zero matrix (0). Introduce in S a multiplication by the definition $(x)_{i\varkappa} * (y)_{j\lambda} = (x)_{i\varkappa}\, \mathbf{P}(y)_{j\lambda}$, where the product on the right is an ordinary matrix product. Thus

$$(x)_{i\varkappa} * (y)_{j\lambda} = (xp_{\varkappa j}y)_{i\lambda}\,. \tag{4}$$

Then S is a completely simple semigroup with zero. Conversely, every completely simple semigroup with zero can be constructed in this manner by choosing suitably I, Λ, the group G and a regular matrix $\mathbf{P}$ over $G^{(0)}$. Card I and card Λ are the cardinal numbers of the set of different minimal right and left ideals of S, respectively, G is isomorphic to the maximal groups in S and $\mathbf{P}$ is a $\Lambda \times I$ matrix which can be explicitly constructed (but not in a unique way).

To emphasize the means needed to the construction of S we shall write occasionaly $S = S[I \times \Lambda, G, \mathbf{P}]$. Rees proved also that $S[I \times \Lambda, G, \mathbf{P}]$ is isomorphic to $S[I \times \Lambda, G, \mathbf{P}_1]$ if and only if there is an automorphism ϑ of G and two monomial matrices $\mathbf{A}$, $\mathbf{B}$ over $G^{(0)}$ such that $\mathbf{P}_1 = \mathbf{A}\mathbf{P}^{\vartheta}\mathbf{B}$.[3]) Especially, if $\mathbf{P}_2$ is a matrix which arises from $\mathbf{P}$ by row (or column) interchanges, then $S[I \times \Lambda, G, \mathbf{P}_2]$ is isomorphic to $S[I \times \Lambda, G, \mathbf{P}_1]$.

From (4) we conclude easily that each idempotent $\in S[I \times \Lambda, G, \mathbf{P}]$ can be obtained in the following manner: Choose in $\mathbf{P}$ a $(\varkappa, i)$ position for which $p_{\varkappa i} \neq 0$ and construct $(p_{\varkappa i}^{-1})_{i\varkappa}$, then this matrix is an idempotent $\in S$.

Let now S be a completely simple dual semigroup. Write S in the form $S = S[I \times \Lambda, G, \mathbf{P}]$. We wish to know what follows from the fact that S is dual. Let be

$$e = (p_{\varkappa i}^{-1})_{i\varkappa} \neq 0\,, \quad f = (p_{\lambda j}^{-1})_{j\lambda} \neq 0\,,$$

two different idempotents $\in S$. According to the definition we have

$$ef = (p_{\varkappa i}^{-1})_{i\varkappa}\, \mathbf{P}(p_{\lambda j}^{-1})_{j\lambda} = (p_{\varkappa i}^{-1}p_{\varkappa j}p_{\lambda j}^{-1})_{i\lambda}\,, \ fe = (p_{\lambda j}^{-1})_{j\lambda}\, \mathbf{P}(p_{\varkappa i}^{-1})_{i\varkappa} = (p_{\lambda j}^{-1}p_{\lambda i}p_{\varkappa i}^{-1})_{j\varkappa}\,.$$

Since $p_{\varkappa i} \neq 0$, $p_{\lambda j} \neq 0$, the relation $ef = fe = 0$ (which is necessary and sufficient for S to be dual) implies $p_{\varkappa j} = 0$ and $p_{\lambda i} = 0$. This means: If in $\mathbf{P}$ we have in the $(\varkappa, i)$th position and in the (λ, j)th position non-zero elements, then there are necessarily zeros in the $(\varkappa, j)$th position and in the (λ, i)th position.

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[3]) $\mathbf{P}^{\vartheta}$ is the $\Lambda \times I$ matrix the $(\varkappa, i)$th entry of which is $p_{\varkappa i}{}^{\vartheta}$.

Let be $p_{\varkappa i} \neq 0$. Since $\mathbf{P}$ is regular there exists to every $j \in I$ an index $\lambda \in \Lambda$ such that $p_{\lambda j} \neq 0$. Therefore we have $p_{\varkappa j} = 0$ for every $j \in I$ with the exception of $j = i$, i. e. $\mathbf{P}$ contains in the $\varkappa$-th row a unique element $\neq 0$. Analogously there is a unique non-zero element in the i-th column. $\mathbf{P}$ is therefore a monomial matrix, hence card Λ = card I. Since to a given $\varkappa \in \Lambda$ there is exactly one $i \in \Lambda$ such that $p_{\varkappa i} \neq 0$, the sets I and Λ are in a one-to-one correspondence. We may therefore assume that $I = \Lambda$. By a suitable numeration we may arrange that p_{ii} will be the non-zero element in the i-th row or column of the rearranged matrix $\mathbf{P}$. The new matrix $\mathbf{P}_1$ thus obtained is of the form $\mathbf{P}_1 = (q_{\varkappa i})$, where $q_{\varkappa i} = 0$ for $i \neq \varkappa$, and $0 \neq q_{ii} \in G$ for every $i \in I$. Define $\mathbf{P}_1^* = (q_{\varkappa i}^*)$, where $q_{\varkappa i}^* = 0$ for $i \neq \varkappa$, and $q_{ii}^* = q_{ii}^{-1}$. Then $\mathbf{PP}_1^*$ is the $I \times I$ "unit matrix" $\mathbf{E}$, i. e. a $I \times I$ matrix in which exactly the main diagonal elements are $\neq 0$ and equal to the unit element of the group G.

The interchanges which lead from the matrix $\mathbf{P}$ to $\mathbf{E}$ turn out to the multiplications of $\mathbf{P}$ by monomial matrices. Since a product of monomial matrices is again a monomial matrix we can state that our original semigroup S is isomorphic to the semigroup $S[I \times I, G, \mathbf{E}]$.

Now with $\mathbf{E}$ instead of $\mathbf{P}$ in (4) the *-multiplication reduces to the ordinary matrix multiplication. Hence a completely simple dual semigroup is isomorphic to the set of all $I \times I$ matrices $(x)_{ij}$ (where x runs through all elements $\in G$) together with the zero matrix, where the multiplication is the ordinary matrix multiplication.

Conversely, consider the semigroup S_1 of all $I \times I$ matrices of the type $(x)_{ij}$ ($x \in G$, G a group) together with the zero matrix (0), where the multiplication is defined by

$$(x)_{ij}(y)_{rs} = \begin{cases} (xy)_{is} & \text{if } j = r\,, \\ (0) & \text{if } j \neq r\,. \end{cases}$$

This is a completely simple semigroup with zero. The non-zero idempotents $\in S_1$ are clearly the matrices $(e)_{ii}$, $i \in I$, where e is the unit element of the group G. Since for $i \neq j$ we have $(e)_{ii} \cdot (e)_{jj} = (0)$, S_1 is a dual semigroup.

We have proved:

Theorem 4,3. *Let I be any set of indices and $G^{(0)} = G \cup \{0\}$ a group with zero adjoined. Construct the set S of all $I \times I$ matrices over $G^{(0)}$ in which at most one element is $\neq 0$. Then the set S under the usual matrix multiplication (as defined above) is a completely simple dual semigroup. Conversely, every completely simple dual semigroup is isomorphic to a semigroup constructed in this manner with a suitably chosen set of indices I and a suitably chosen group G.*[4])

Remark. It follows from a result of Clifford (see [14], p. 340) that the semigroups described in Theorem 4,3 are just Brandt groupoids with a zero

[4]) This theorem bears a resemblence to Theorems 5 and 8 of Kaplansky's paper [6].

adjoined (as defined in [14]). They can also be characterized as completely simple inverse semigroups, as shown by W. D. Munn [15] (see particularly section 4,2).[4a])

5. THE RADICAL AND MAXIMAL IDEALS

In this section we show (among other results) that in a dual semigroup S satisfying some very general conditions the radical can be characterized as the intersection of all maximal two-sided ideals of S.

Lemma 5,1. *Every minimal left ideal of a dual semigroup S is contained in a minimal two-sided ideal of S.*

Proof. It is well known that if L is a minimal left ideal of S and $c \in S$, then either Lc is also a minimal left ideal of S or $Lc = 0$. The second possibility holds if and only if $c \in \mathfrak{R}(L)$. Denote $Z = S - \mathfrak{R}(L) \neq 0$ and $Z = \{z_\nu \mid \nu \in H\}$. We then have $LS = L . \{\mathfrak{R}(L) \cup Z\} = LZ = \bigcup_{\nu \in H} [Lz_\nu]$. The summands on the right need not be all different, but omitting repetitions we get a decomposition $LS = \bigcup_{\nu \in H_1} [Lz_\nu]$ in which $Lz_\mu \neq Lz_\varkappa$ for $\mu \neq \varkappa$ $(\mu, \varkappa \in H_1)$. Since $L \subset LS$ (see Lemma 1,6), there is an z_α, $\alpha \in H_1$, with $L = Lz_\alpha$. Note further: if M is a two-sided ideal and $M \cap Lz_\nu \neq 0$, then, with respect to the minimality of Lz_ν $(\nu \in H_1)$, we have $Lz_\nu \subset M$.

Suppose now that M is any two-sided ideal of S with $0 \neq M \subset LS$ and M does not contain $L = Lz_\alpha$. Then M can be written in the form $M = \bigcup_{\nu \in H_2} Lz_\nu$, where α does not belong to H_2. Since M is a right ideal, we have

$$[\bigcup_{\nu \in H_2} (Lz_\nu)]\, S \subset \bigcup_{\nu \in H_2} (Lz_\nu) \,. \tag{5}$$

For any fixed chosen z_ν $(\nu \in H_2)$ the set $z_\nu S$ cannot contain z_α, since otherwise (5) would not be satisfied. The right ideal $z_\nu S$ contains z_ν (see Lemma 1,6), while $\mathfrak{R}(L)$ does not contain z_ν. Hence, with respect to the maximality of $\mathfrak{R}(L)$, we have $z_\nu S \cup \mathfrak{R}(L) = S$. This constitutes an apparent contradiction, since z_α is contained neither in $z_\nu S$ nor in $\mathfrak{R}(L)$. We have proved that any two-sided subideal $\neq 0$ of LS contains L.

It is now easy to prove that LS is the minimal two-sided ideal of S containing L. Let M_1 be any two-sided ideal of S contained in LS. Since $L \subset M_1$, we have $L \subset M_1 \subset LS$. Multiplying by S to the left we get $LS \subset M_1S \subset LS^2$. But since $S = S^2$, and $M_1S = M_1$ (see Corollary 1,6a), we have $M_1 = LS$. This proves our Lemma.

[4a]) I am indebted to Professor A. H. Clifford for this remark.

Remark. An example given in [11] (p. 255) shows that Lemma 5,1 need not hold in a semigroup which is not dual.

Lemma 5,2. *Let S be a dual semigroup with a nilpotent radical N. Then every minimal left (right, two-sided) ideal of S is contained in $\mathfrak{R}(N) \cap \mathfrak{L}(N)$.*

Proof. Since every minimal left (right) ideal is contained in a minimal two-sided ideal, it is sufficient to prove our Lemma for minimal two-sided ideals.

Let M be a minimal two-sided ideal of S. The set NM is a two-sided ideal of S and $NM \subset M \cap N \subset M$. Since M is minimal we have either $NM = 0$ or $NM = M$. The second possibility cannot hold since $NM = M$ implies $N^{\varrho}M = M$ for every integer $\varrho > 0$. But since $N^{\tau} = 0$ for some $\tau > 0$ this would imply $M = 0$, contrary to the supposition. The relation $NM = 0$ implies $M \subset \mathfrak{R}(N)$. An analogous argument shows that $M \subset \mathfrak{L}(N)$; hence $M \subset \mathfrak{R}(N) \cap \mathfrak{L}(N)$.

Lemma 5,3. *Let be $\{L_{\alpha}^{*} \mid \alpha \in \Lambda_1\}$, $\{R_{\alpha}^{*} \mid \alpha \in \Lambda_2\}$, $\{M_{\alpha}^{*} \mid \alpha \in \Lambda\}$ the set of all maximal left, right, two-sided ideals, respectively, of a dual semigroup with a nilpotent radical N. We then have: $N \subset \bigcap_{\alpha \in \Lambda_1} L_{\alpha}^{*}$, $N \subset \bigcap_{\alpha \in \Lambda_2} R_{\alpha}^{*}$, $N \subset \bigcap_{\alpha \in \Lambda} M_{\alpha}^{*}$.*

Proof. The set $\mathfrak{L}(R_{\alpha}^{*})$ is a minimal left ideal of S. With respect to the foregoing lemma we have $\mathfrak{L}(R_{\alpha}^{*}) \subset \mathfrak{L}(N)$. This implies $N \subset R_{\alpha}^{*}$ for every $\alpha \in \Lambda_2$. Hence $N \subset \bigcap_{\alpha \in \Lambda_2} R_{\alpha}^{*}$. The remaining statements can be proved analogously.

In the following we shall impose to the semigroup S the following minimal condition:

Condition A. *S is a semigroup in which every non-nilpotent two-sided ideal of S contains at least one left and at least one right minimal non-nilpotent ideal of S* (i. e. a left ideal L such that for any left ideal L' with $0 \subseteqq L' \subsetneqq L$ we have $L'^{\varrho} = 0$ for some $\varrho > 0$, and a right ideal R such that for any right ideal $0 \subseteqq R' \subsetneqq R$ we have $R'^{\sigma} = 0$ for some $\sigma > 0$).

The following Lemma is known (see Clifford [4], Theorem 5,2, p. 842):

Lemma 5,4. *Let S be a semigroup with a nilpotent radical N satisfying Condition **A**. Then every left (or right) non-nilpotent ideal of S contains an idempotent element not in N.*

Theorem 5,1. *Let S be a dual semigroup with a nilpotent radical N satisfying Condition **A**. Let $\{R_{\alpha}^{*} \mid \alpha \in \Lambda_2\}$ and $\{M_{\alpha}^{*} \mid \alpha \in \Lambda\}$ be the set of all maximal right and maximal two-sided ideals of S, respectively. Suppose further that every right ideal of S is contained in a maximal right ideal of S. We then have: $N = \bigcap_{\alpha \in \Lambda_2} R_{\alpha}^{*} = \bigcap_{\alpha \in \Lambda} M_{\alpha}^{*}$.*

Proof. a) Denote $P = \bigcap_{\alpha \in \Lambda_2} R_\alpha^*$. To prove $N = P$, it is sufficient (with respect to Lemma 5,3) to show that $P \subset N$.

Suppose $P \not\subset N$. According to Lemma 5,4 the right ideal P contains then an idempotent $e_1 \neq 0$. Consider the left ideal Se_1. The supposition that every right ideal is contained in a maximal right ideal implies that every left ideal contains a minimal left ideal. Denote by L a minimal left ideal of S contained in Se_1, $0 \neq L \subset Se_1$. Since $\mathfrak{R}(L)$ is a maximal right ideal of S, we have $e_1 \in P \subset \mathfrak{R}(L)$. Hence $Le_1 \subset L\mathfrak{R}(L) = 0$. On the other side $L \subset Se_1$ implies $Le_1 = L \neq 0$. This contradiction proves $P \subset N$.

b) We prove now $N = \bigcap_{\alpha \subset \Lambda} M_\alpha^*$. Note first: Since (according to Lemma 5,1) every minimal left ideal is contained in a minimal two-sided ideal, every maximal right ideal R_α^* of S contains a (unique) maximal two-sided ideal $M_{r(\alpha)}^*$ of S. (Different R_α^* may contain the same $M_{r(\alpha)}^*$.) Hence $R_\alpha^* \supset M_{r(\alpha)}^* \supset N$ and $N = \bigcap_{\alpha \in \Lambda_2} R_\alpha^* \supset \bigcap_{\alpha \in \Lambda_2} M_{r(\alpha)}^* \supset \bigcap_{\alpha \in \Lambda} M_\alpha^* \supset N$. This proves $N = \bigcap_{\alpha \in \Lambda} M_\alpha^*$.

Theorem 5,2. *Let S be a dual semigroup with a nilpotent radical satisfying Condition* **A**. *Suppose further that every left and right ideal of S is contained in a maximal left and right ideal of S, respectively. Then we have $\mathfrak{R}(N) = \mathfrak{L}(N)$.*

Proof. Theorem 5,1 implies $\mathfrak{L}(N) = \bigcup_{\alpha \in \Lambda_2} \mathfrak{L}(R_\alpha^*)$. Since $\mathfrak{L}(R_\alpha^*)$ is a minimal left ideal we have (see Lemma 5,2) $\mathfrak{L}(R_\alpha^*) \subset \mathfrak{L}(N) \cap \mathfrak{R}(N)$ for every $\alpha \in \Lambda_2$. Hence $\mathfrak{L}(N) \subset \mathfrak{L}(N) \cap \mathfrak{R}(N)$, which implies $\mathfrak{L}(N) \subset \mathfrak{R}(N)$.

Now supposing that every left ideal of S is contained in a maximal left ideal of S, we can prove analogously as in Theorem 5,1 $N = \bigcap_{\alpha \in \Lambda_1} L_\alpha^*$. We then have further $\mathfrak{R}(N) = \bigcup_{\alpha \in \Lambda_1} \mathfrak{R}(L_\alpha^*) \subset \mathfrak{R}(N) \cap \mathfrak{L}(N)$, hence $\mathfrak{R}(N) \subset \mathfrak{L}(N)$. This proves Theorem 5,2.

It is of some interest to clarify the relation between N and $\mathfrak{R}(N)$ respectively $\mathfrak{L}(N)$.

If $S \neq 0$ and $N = 0$, we have $0 = N \subseteqq \mathfrak{R}(N) = \mathfrak{L}(N) = S$. Example 2 shows that $N \subsetneqq \mathfrak{R}(N)$ is possible also in the case $N \neq 0$. In the most interesting case, namely if in the sense of Theorem 2,2 we have $S' = 0$, we can prove the following theorem:

Theorem 5,3. *Let S be a dual semigroup with a nilpotent radical $N \neq 0$ satisfying Condition* **A**. *Assume further that every two-sided ideal of S different from 0 has a non-zero intersection with N. Then we have $\mathfrak{R}(N) \subset N$.*

Proof. Suppose $\mathfrak{R}(N) \not\subset N$. Since $\mathfrak{R}(N)$ is then non-nilpotent, it contains a left minimal non-nilpotent ideal $L \subset \mathfrak{R}(N)$ (i. e. a left ideal L such that for any left ideal L' with $L' \subsetneqq L$ we have L'^ϱ for some $\varrho > 0$). According to

Lemma 5,4 L contains an idempotent $e \neq 0$. Since $Le \subset L . L \subset L$, further Le is non-nilpotent, and L minimal non-nilpotent, we have $Le = L$.

We have necessarily $L \cap N \neq 0$. We prove it indirectly. $L \cap N = 0$ would imply that L is a minimal left ideal of S (in the usual sense that there is no left ideal $L' \neq 0$ of S with $L' \subsetneqq L$). According to Lemma 5,1 LS would be a minimal two-sided ideal of S. Since $LS \cap N \neq 0$ we would have (with respect to the minimality of LS) $LS \cap N = LS$, hence $LS \subset N$, and $L \subset N$, contrary to the supposition.

Denote $L \cap N = L_1 \neq 0$. The relation $L_1 \subset Le$ implies that every $a \in L_1$ can be written in the form $a = be$, $b \in L$, hence $ae = bee = be = a$, i. e. $L_1e = L_1$. But on the other hand we have $L_1e \subset NL \subset N\mathfrak{R}(N) = 0$. This contradiction proves our Theorem.

Remark. Under the suppositions of Theorem 5,3 $N = \mathfrak{R}(N)$ holds if and only if $N^2 = 0$. For, $\mathfrak{R}(N) = N$ implies $N\mathfrak{R}(N) = N^2 = 0$. On the other hand $N . N = 0$ implies $N \subset \mathfrak{R}(N)$; hence $N = \mathfrak{R}(N)$.

6. THE DUALITY OF THE DIFFERENCE SEMIGROUP S/N

Let S be a semigroup and I a two-sided ideal of S. The difference semigroup $S/I = \bar{S}$ is a semigroup obtained from S by collapsing I into a single zero element $\bar{0}$, while the remaining elements of S retain (up to an isomorphism) their identity. Thus there is a one-to-one correspondence $a \to \bar{a}$ between the elements $a \in S$ not in I and the non-zero elements $\bar{a} \in \bar{S}$ such that $ab \to \bar{a}\bar{b}$ if $ab \notin I$, and $\bar{a}\bar{b} = \bar{0}$ if and only if $ab \in I$.

The homomorphism $S \to \bar{S}$ induces a one-to-one correspondence between the class of all left (right, two-sided) ideals L of S containing I and the class of all left (right, two-sided) ideals $\bar{L}$ of $\bar{S}$.

If S is dual, I any two-sided ideal of S, then S/I need not be dual. This is shown on the following example:

Example 6. Let $S = \{0, a_1 a_2, a_3, a_4\}$ be a semigroup having the following multiplication table:

	0	a_1	a_2	a_3	a_4
0	0	0	0	0	0
a_1	0	0	0	0	a_1
a_2	0	0	a_1	0	a_2
a_3	0	0	0	a_1	a_3
a_4	0	a_1	a_2	a_3	a_4

The lattice of ideals has the graph in Fig. 4.

S is a dual semigroup with the radical $N = (0, a_1, a_2, a_3)$. Choose $I = \mathfrak{R}(N) = (0, a_1)$. Then $S/I = \{\overline{0}, \overline{a}_2, \overline{a}_3, \overline{a}_4\}$ has the following multiplication table:

	$\overline{0}$	$\overline{a}_2$	$\overline{a}_3$	$\overline{a}_4$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{a}_2$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{a}_2$
$\overline{a}_3$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{a}_3$
$\overline{a}_4$	$\overline{0}$	$\overline{a}_2$	$\overline{a}_3$	$\overline{a}_4$

The lattice of ideals has the graph in Fig. 5.

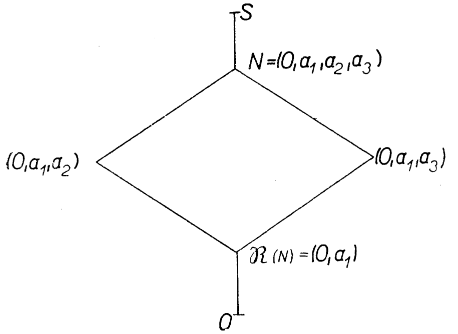

Fig. 4.

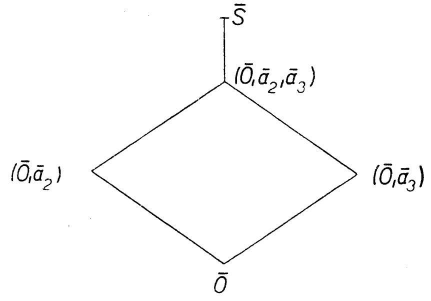

Fig. 5.

The semigroup S/I is clearly not dual. (Note on the other hand that S/N is dual, being a group with zero.)

In what follows we shall study the structure od S/N, where N is the radical of S.

The following Lemma is known (see f. i. Clifford [4], Theorem 5,1, p. 842):

Lemma 6,1. *Let S be a semigroup with a nilpotent radical N. Then S/N is a semigroup without nilpotent ideals.*

The fundamental problem is whether for a dual semigroup S also $S/N = \overline{S}$ remains dual.

Suppose that S is dual and has a nilpotent radical N. According to Lemma 5,3 every maximal two-sided (left, right) ideal of S contains N. This implies that there is a one-to-one correspondence between the maximal ideals of S and the maximal ideals of S/N. If M^* is a maximal two-sided ideal of S, then $\overline{M}^* = M^*/N$ is the corresponding maximal two-sided ideal of S/N. Conversely, if $\overline{M}^*$ is a maximal ideal of S/N the inverse image of $\overline{M}^*$ in the correspondence $S \to S/N$ is a maximal ideal of S.

Remark. The one-to-one correspondence between the maximal ideals of S and S/N is essentially due to the fact that every maximal ideal of S contains N. Example 3 (in which N is not contained in the maximal ideal $J_1 \cup J_2$)

shows that in general such a one-to-one correspondence need not exist. In this example S has three maximal ideals (namely $J_1 \cup N$, $J_2 \cup N$ and $J_1 \cup J_2$), but S/N has only two maximal ideals (which are isomorphic to J_1 and J_2).

Theorem 6,1. *Let S be a dual semigroup with a nilpotent radical N satisfying Condition* **A**. *Suppose that every right and left ideal of S is contained in a maximal right and left ideal of S, respectively. Then S/N is a dual semigroup.*

Proof. A. Since $S \neq N$, S/N contains non-zero elements and it is a semigroup without nilpotent ideals. If $\{M_\alpha^* \mid \alpha \in \Lambda\}$ is the set of all maximal two-sided ideals of S, we have (see Theorem 5,1) $N = \bigcap_{\alpha\in\Lambda} M_\alpha^*$. With respect to the one-to-one correspondence between the maximal two-sided ideals of S and the maximal two-sided ideals of S/N the last relation implies $\bar{0} = \bigcap_{\alpha\in\Lambda} \bar{M}_\alpha^*$, where $\bar{M}_\alpha^*$ runs through all maximal ideals of $\bar{S}$.

Denote by $\bar{\mathfrak{R}}(A)$ and $\bar{\mathfrak{L}}(A)$ the right and left annihilators of A in $\bar{S}$. If card $\Lambda \geqq 2$, we can use Lemma 3,1 according to which: a) $Q_\alpha = \bar{\mathfrak{R}}(\bar{M}_\alpha^*) = \bar{\mathfrak{L}}(\bar{M}_\alpha^*)$ (for every $\alpha \in \Lambda$) is a minimal two-sided ideal of $\bar{S}$ (hence a simple semigroup); b) $\bar{S} = \bigcup_{\alpha\in\Lambda} Q_\alpha$, and $Q_\alpha Q_\beta = Q_\alpha \cap Q_\beta = \bar{0}$ for $\alpha \neq \beta \in \Lambda$. The same result holds (see Remark to Lemma 3,1) if card $\Lambda = 1$. For then N is the unique maximal two-sided ideal of S and S/N is a simple semigroup (with zero $\bar{0}$).

B. Condition **A** implies for the difference semigroup $\bar{S}$ that every Q_α (being a two-sided ideal of $\bar{S}$) contains a minimal left and right ideal of $\bar{S}$ (different from $\bar{0}$). Since Q_α is a minimal two-sided ideal of $\bar{S}$, Q_α is the sum of all minimal left ideals of $\bar{S}$ contained in Q_α (see f. i. Clifford [4], Theorem 2,1, p. 836). Analogously Q_α is the sum of all minimal right ideals of $\bar{S}$ contained in Q_α. Therefore we can write $Q_\alpha = \bigcup_{\gamma\in\Lambda_\alpha} R_\gamma^{(\alpha)} = \bigcup_{\gamma\in H_\alpha} L_\gamma^{(\alpha)}$, where $\{R_\nu^{(\alpha)} \mid \nu \in \Lambda_\alpha\}$ and $\{L_\nu^{(\alpha)} \mid \nu \in H_\alpha\}$ is the set of all minimal right resp. left ideals of $\bar{S}$ contained in Q_α. Since for $\alpha \neq \beta$ we have $Q_\alpha \cdot Q_\beta = \bar{0}$, every $L_\nu^{(\alpha)}$ resp. $R_\nu^{(\alpha)}$ is at the same time a minimal left resp. right ideal of Q_α.

Note for further purposes that $\bar{S}$ can be written as a class sum of all minimal right (left) ideals of $\bar{S}$ in the form $\bar{S} = \bigcup_{\alpha\in\Lambda} \{ \bigcup_{\nu\in\Lambda_\alpha} R_\nu^{(\alpha)} \} = \bigcup_{\alpha\in\Lambda} \{ \bigcup_{\nu\in H_\alpha} L_\nu^{(\alpha)} \}$. This implies that we obtain a maximal left ideal of $\bar{S}$ by taking the sum of all minimal left ideals of $\bar{S}$ except one of them and every maximal left ideal of $\bar{S}$ can be obtained in this manner. Analogously for maximal right ideals.

C. To prove that $\bar{S}$ is dual it is sufficient to prove that Q_α is dual for every $\alpha \in \Lambda$ (see Theorem 3,2). To prove that Q_α is dual we have to prove that for every couple of idempotents $e^{(\alpha)} \neq f^{(\alpha)} \in Q_\alpha$ we have $e^{(\alpha)} f^{(\alpha)} = \bar{0}$ (see Theorem 4,2).[5]

[5]) Since Q_α contains minimal left and right ideals, it contains idempotents and every minimal left ideal of Q_α is generated by an idempotent, i. e. it is of the form $Q_\alpha \cdot e^{(\alpha)}$ with a suitably chosen idempotent $e^{(\alpha)} \in Q_\alpha$.

We return, for a while, to the original semigroup S. Let $e \neq 0$ be any idempotent $\in S$. Consider the left ideal Se of S. Let $\mathfrak{l}$ be a minimal left ideal of S contained in Se.[6]) The relation $\mathfrak{l} \subset Se$ implies $\mathfrak{l}e = \mathfrak{l}$. Now for the maximal right ideal $\mathfrak{R}(\mathfrak{l})$ we have $\mathfrak{l}\mathfrak{R}(\mathfrak{l}) = 0$, hence $\mathfrak{l}e\mathfrak{R}(\mathfrak{l}) = 0$, i. e. $e\mathfrak{R}(\mathfrak{l}) \subset \mathfrak{R}(\mathfrak{l})$. Since $e\mathfrak{R}(\mathfrak{l}) \subset eS$, we have $e\mathfrak{R}(\mathfrak{l}) \subset eS \cap \mathfrak{R}(\mathfrak{l})$. Since further $\mathfrak{R}(\mathfrak{l})$ does not contain e,[7]) the right ideal $eS \cap \mathfrak{R}(\mathfrak{l})$ is a right subideal of S properly contained in eS. But eS is a minimal non-nilpotent right ideal of S.[8]) Hence $eS \cap \mathfrak{R}(\mathfrak{l}) \subset$ $\subset N$. We proved: $e\mathfrak{R}(\mathfrak{l}) \subset N$.

Note now that every non-zero idempotent $e \in S$ is mapped by $S \to \bar{S}$ in a non-zero idempotent $\in \bar{S}$. Returning therefore to the semigroup $\bar{S}$ we can state the following result: If e is any non-zero idempotent $\in \bar{S}$ and $\bar{R}^*$ is the maximal right ideal of $\bar{S}$ which does not contain e, we have $e\bar{R}^* = \bar{0}$. By an analogous argument we get $\bar{L}^*e = \bar{0}$, where $\bar{L}^*$ is the maximal left ideal of $\bar{S}$, which does contain e.

D. Let now be $e^{(\alpha)} \neq \bar{0}$ a fixed chosen idempotent $\in Q_\alpha$ and let $L_{\nu_0}^{(\alpha)}$ and $R_{\mu_0}^{(\alpha)}$ be the minimal left and right ideal, respectively, of Q_α containing $e^{(\alpha)}$, hence $L_{\nu_0}^{(\alpha)} = Q_\alpha e^{(\alpha)}$ and $R_{\mu_0}^{(\alpha)} = e^{(\alpha)}Q_\alpha$. We prove that $L_{\nu_0}^{(\alpha)}$ contains a unique non- zero idempotent ($= e^{(\alpha)}$).

We have $L_{\nu_0}^{(\alpha)} \,.\, R_{\mu_0}^{(\alpha)} = Q_\alpha e^{(\alpha)} \,.\, e^{(\alpha)}Q_\alpha = Q_\alpha$ and $R_{\mu_0}^{(\alpha)}L_{\nu_0}^{(\alpha)} = e^{(\alpha)}Q_\alpha Q_\alpha e^{(\alpha)} \neq \bar{0}$. This implies that $R_{\mu_0}^{(\alpha)} \,.\, L_{\nu_0}^{(\alpha)}$ equals to $R_{\mu_0}^{(\alpha)} \cap L_{\nu_0}^{(\alpha)}$ and it is a group with zero. (See the proof of Theorem 4,1).

Since $\bigcup\limits_{\nu \in \Lambda_\alpha,\, \nu \neq \mu_0} R_\nu^{(\alpha)}$ is contained in the maximal right ideal of $\bar{S}$ which does not contain $e^{(\alpha)}$, we have

$$e^{(\alpha)} \{ \bigcup_{\nu \in \Lambda_\alpha,\, \nu \neq \mu_0} R_\nu^{(\alpha)} \} = \bar{0} \,, \tag{6}$$

and analogously

$$\{ \bigcup_{\nu \in H_\alpha,\, \nu \neq \nu_0} L_\nu^{(\alpha)} \} \, e^{(\alpha)} = \bar{0} \,. \tag{7}$$

The relation (6) implies

$$Q_\alpha e^{(\alpha)} \{ \bigcup_{\nu \in \Lambda_\alpha,\, \nu \neq \mu_0} R_\nu^{(\alpha)} \} = \bar{0} \,, \quad \text{i. e.} \quad L_{\nu_0}^{(\alpha)} \{ \bigcup_{\nu \in \Lambda_\alpha,\, \nu \neq \mu_0} R_\nu^{(\alpha)} \} = \bar{0} \,. \tag{8}$$

Now

$$L_{\nu_0}^{(\alpha)} = L_{\nu_0}^{(\alpha)} \cap Q_\alpha = [L_{\nu_0}^{(\alpha)} \cap R_{\mu_0}^{(\alpha)}] \cup [L_{\nu_0}^{(\alpha)} \cap (\bigcup_{\substack{\nu \in \Lambda_\alpha \\ \nu \neq \mu_0}} R_\nu^{(\alpha)})] \,.$$

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[6]) Such a minimal ideal $\mathfrak{l}$ exists, since according to the supposition S is dual and every right ideal is contained in a maximal right ideal of S. It is, of course, possible that $\mathfrak{l}$ is nilpotent. It can be shown that this is the case if and only if $Se\,S \cap N \neq 0$.

[7]) $e \in \mathfrak{R}(\mathfrak{l})$ would imply $\mathfrak{l}e \subset \mathfrak{l}\mathfrak{R}(\mathfrak{l}) = 0$, i. e. $\mathfrak{l}e = 0$, which is a contradiction to $\mathfrak{l}e = \mathfrak{l} \neq 0$.

[8]) If namely $R \subset eS$ were a non-nilpotent right ideal properly contained in eS, $\bar{R} \neq \bar{0}$ would be a right ideal of $\bar{S}$ properly contained in $\bar{e}\bar{S}$. Now if $\bar{e} \in Q_\alpha$, we have $\bar{e}\bar{S} = \bar{e} \,.$ $. \{ \bigcup\limits_{\alpha \in \Lambda} Q_\alpha \} = \bar{e}Q_\alpha$. But $\bar{e}Q_\alpha$ is a minimal right ideal of Q_α and it cannot contain a proper right subideal $\neq \bar{0}$.

The first summand (being a group with zero) contains a unique non-zero idempotent. The second summand cannot contain an idempotent $\neq \bar{0}$ since otherwise the relation (8) would not hold. This proves that $L_{\nu_0}^{(\alpha)}$ (and hence every minimal left ideal of Q_α) contains a unique non-zero idempotent.

E. Denote the unique non-zero idempotent contained in $L_\nu^{(\alpha)}$ by $e_\nu^{(\alpha)}$. We then have $L_\nu^{(\alpha)} = Q_\alpha e_\nu^{(\alpha)}$ and $Q_\alpha = \bigcup_{\nu \in H_\alpha} L_\nu^{(\alpha)} = \bigcup_{\nu \in H_\alpha} [Q_\alpha e_\nu^{(\alpha)}]$, where $\{e_\nu^{(\alpha)} \mid \nu \in H_\alpha\}$ denotes the set of all different non-zero idempotents $\in Q_\alpha$. Now with respect to (7) we have $[\bigcup_{\substack{\nu \in H_\alpha \\ \nu \neq \lambda}} Q_\alpha e_\nu^{(\alpha)}]\, e_\lambda^{(\alpha)} = \bar{0}$ for every idempotent $e_\lambda^{(\alpha)}$. Hence $e_\nu^{(\alpha)} \,.\, e_\lambda^{(\alpha)} = \bar{0}$ for every couple $e_\nu^{(\alpha)} \neq e_\lambda^{(\alpha)}$. This completes the proof of Theorem 6,1.

7. DUAL SEMIGROUPS WITH A UNIT ELEMENT

Baer proved: A ring S satisfying the minimal condition for descending chains of ideals and in which $x \in Sx$ holds for every $x \in S$ contains a left unit. A number of analogous results concerning rings can be found in his paper [1]. Such theorems do not hold for semigroups. Every dual semigroup satisfies $x \in Sx \cap xS$ for every $x \in S$, but S need not contain (left, right, two-sided) units. This can be shown on the example of the semigroup $S = \{0, a, a^2, b\}$ with the multiplication table:

	0	a	a^2	b
0	0	0	0	0
a	0	a^2	a	0
a^2	0	a	a^2	0
b	0	0	0	b

This semigroup is dual and does not contain a unit element.

It seems to be of some interest to study the influence of the existence of a unit element on the structure of S.

Suppose that S is dual and it has a right unit e_r. Consider the right ideal e_rS. We have $\mathfrak{L}(e_rS) = \{x \mid xe_rS = 0\}$. Since $xe_r = x$, we have $\mathfrak{L}(e_rS) = \{x \mid xS = 0\}$. Now in a dual semigroup $xS = 0$ implies $x = 0$. Therefore $\mathfrak{L}(e_rS) = 0$; hence $\mathfrak{R}[\mathfrak{L}(e_rS)] = e_rS = S$. The relation $e_rS = S$ shows that e_r is also a left unit of S. We have proved:

Lemma 7,1. *A dual semigroup containing a one-sided unit contains a (unique) two-sided unit.*

Let e be the unit element of S. Write in the sense of Theorem 2,2 $S = S' \cup S''$. If $e \in S'$, we have $S'' = S''e \subset S''S' = 0$. If $e \in S''$, we have $S' = S'e \subset S'S'' = 0$. Hence:

Lemma 7,2. *Let S be a dual semigroup with a unit element. Then:*

a) *either is S a semigroup without nilpotent ideals,*

b) *or S has a proper radical N and every two-sided ideal of S has a non-zero intersection with N.*

The case a) can be easily settled.

Theorem 7,1. *A dual semigroup without nilpotent ideals containing a unit element is a group with zero.*

Proof. The existence of a unit element implies that S contains a maximal left ideal L^* with the property that every left ideal of S and different from S is contained in L^*. (See f. i. [12], p. 379.) Analogously there is a unique maximal right ideal R^* and a unique maximal two-sided ideal M^* of S.

Since S is dual, we have, according to the Supplement to Theorem 3,4, $M^* = 0$. Hence S is a simple semigroup (with a zero element). Now $\mathfrak{R}(L^*)$ and $\mathfrak{L}(R^*)$ are minimal right and left ideals of S, respectively. A simple semigroup containing minimal left and right ideals is completely simple. A completely simple semigroup (with zero) containing a unity element is known to be a group with zero. (See Rees [10], p. 394.) This proves Theorem 7,1.

Theorem 7,2. *Let S be a dual semigroup with a nilpotent radical $N \neq 0$ satisfying condition* **A**. *Suppose that S contains a unit element. Then the radical is the unique maximal two-sided ideal of S and we have $S = N \cup G$, $N \cap G = \emptyset$, where G is a group.*

Proof. Analogously as in Theorem 7,1 the existence of a unit element implies the existence of a unique maximal left ideal L^*, a unique maximal right ideal R^* and a unique maximal two-sided ideal M^*.

Theorem 5,1 implies $N = R^* = L^* = M^*$. Now it follows from Theorem 6,1 that S/N is a dual semigroup. Since S/N is a semigroup without nilpotent ideals and it contains a unit element, we conclude from Theorem 7,1 that S/N is a group with zero. Therefore $S = N \cup G$, $N \cap G = \emptyset$, and G is a group. This proves Theorem 7,2.

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Резюме

О ДУАЛЬНЫХ ПОЛУГРУППАХ

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Пусть S — полугруппа с нулем 0, $S \neq 0$. Левым [правым] анулятором непустово множества $A \subset S$ называется множество

$$\mathfrak{L}(A) = \{x \in S \mid xA = 0\} \quad [\mathfrak{R}(A) = \{x \in S \mid Ax = 0\}] .$$

Полугруппа называется дуальной, если для каждого левого идеала L и каждого правого идеала R $\mathfrak{L}[\mathfrak{R}(L)] = L$ и $\mathfrak{R}[\mathfrak{L}(R)] = R$. В этом случае преобразование $L \to \mathfrak{R}(L)$ и $R \to \mathfrak{L}(R)$ определяет взаимно однозначное соответствие между левыми и правыми идеалами полугруппы S, и соотвествующие (полные) структуры левых, соответственно правых, идеалов антиизоморфны.

Целью этой статьи является изучение строения дуальных полугрупп.

Введем ещё одно понятие. Соединение N всех нильпотентных двусторонних идеалов полугруппы S называется радикалом полугруппы S. (В полугруппе без нильпотентных идеалов положим $N = 0$.)

В отделе 2 настоящей работы доказываются следующие теоремы:

Теорема 2.1. *Пусть S-дуальная полугруппа и I-двусторонний идеал из S, для которого $I \cap N = 0$. Тогда I и $\mathfrak{R}(I)$-дуальные полугруппы.*

Кроме того, $\mathfrak{L}(I) = \mathfrak{R}(I)$, $\mathfrak{R}(I) \,.\, I = I \cap \mathfrak{R}(I) = 0$ *и* $I \cup \mathfrak{R}(I) = S$.

Теорема 2.2. *Всякую дуальную полугруппу с радикалом N можно представить как соединение двух двусторонних идеалов в виде $S = S' \cup S''$, где $S'S'' = S''S' = S' \cap S'' = 0$, причем слагаемые обладают следующими*

свойствами: 1. Если $N = 0$, то $S' = S$, $S'' = 0$. 2. Если $N \neq 0$, то S' или 0, или — дуальная полугруппа без нильпотентных идеалов, S''-дуальная полугруппа с радикалом N, в которой всякий двусторонний идеал $\neq 0$ имеет с радикалом пересечение $\neq 0$.

В отделе 3 доказывается следующая теорема:

Теорема 3.3. *Пусть S-полугруппа без нильпотентных идеалов, в которой всякий двусторонний идеал содержит по крайней мере один минимальный двусторонний идеал ($\neq 0$). S является дуальной полугруппой тогда и только тогда, если S-соединение всех минимальных двусторонних идеалов из S и каждый из минимальных двусторонних идеалов является дуальной полугруппой.*

Даются тоже другие формулировки этого результата.

Всякий минимальный двусторонний идеал полугруппы S, необладающей нильпотентными идеалами, есть простая полугруппа (значит, не имеет никаких собственных двусторонних идеалов $\neq 0$). Простая полугруппа, имеющая по крайней мере один минимальный левый и минимальный правый идеал, называется вполне простой.

В теореме 4, 1 и 4, 2 доказывается:

Вполне простая полугруппа (с нулем) является дуальной тогда и только тогда, если для всякой пары идемпотентов $e_\alpha \neq e_\beta \in S$ имеет место соотношение $e_\alpha e_\beta = 0$.

В дальнейшем дается представление вполне простых дуальных полугрупп в виде матричных полугрупп.

Пусть I — множество индексов и $G = G^{(0)} \cup \{0\}$ — группа с внешне присоединенным нулем. Построим множество S всех так называемых „$I \times I$ матриц $\{a_{ik}\}(i, k \in I)$ над G“, причем всякая „матрица“ имеет неболее одного елемента, отличного от нуля. Тогда при естественном определении умножения этих „матриц“ S является дуальной полугруппой. Обратно, каждая вполне простая дуальная полугруппа изоморфна полугруппе, построенной таким образом при удобно избранном множестве I и удобно избранной группе $G^{(0)}$.

В отделе 5 доказываются различние теоремы, касающиеся связи между максимальными идеалами и радикалом.

Скажем, что S удовлетворяет условию **A**, если всякий не-нильпотентный двусторонний идеал имеет по крайней мере один минимальный левый и минимальный правый не-нильпотентный идеал. Имеет место, например, следующая теорема:

Теорема 5.1. *Пусть S-дуальная полугруппа с нильпотентным радикалом, удовлетворяющая условию* **A**. *Пусть $\{R_\alpha^* \mid \alpha \in \Lambda_2\}$ и $\{M_\alpha^* \mid \alpha \in \Lambda\}$ — множества всех максимальных правых, соотвественно, двусторонних идеалов* *из S. Если всякий правый идеал из S содержится в некотором максимальном правом идеале из S, то* $N = \bigcap_{\alpha \in \Lambda_2} R_\alpha^* = \bigcap_{\alpha \in \Lambda} M_\alpha^*$.

Если I-двусторонний идеал полугруппы S, то разностной полугруппой S/I разумеем полугруппу, которую в основном получим отождествлением всех элементов из I с единственным елементом $\bar{0}$, в то время как смысл остальных элементов из S оставим без изменения. В отделе 6 доказывается:

Теорема 6.1. *Пусть S-дуальная полугруппа с нильпотентным радикалом, удовлетворяющая условию* **A**. *Предположим, что каждый левый и правый идеал из S содержится в некотором максимальном левом, соответственно, правом идеале из S. Тогда S/N-дуальная полугруппа.*

В отделе 7 доказаны некоторые теоремы, касающиеся строения дуальных полугрупп с единицей.