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ON G_δ-SPACES

(Preliminary Communication)

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1. A topological space R is called an extension of a space $P \subset R$ if P is dense in R; if, moreover, R is compact, then it is called a compactification of P. We shall call a Hausdorff topological space P a G_δ-space if P is a G_δ-set in every Hausdorff extension of P. In the present article, an "internal" characterization of G_δ-spaces is given, as well as of "Baire spaces", to be defined in the sequel. Proofs are omitted and will be published elsewhere.

The terminology of J. Kelley, General Topology, is used throughout. The letters P, R always denote a topological space; βP denotes the Čech-Stone compactification of a (completely regular) P.

2. Let us recall that, by a well known theorem, a metrizable G_δ-space is characterized "internally" (i. e. without reference to imbedding in larger spaces) as a space homeomorphic with a complete metric space.

Definition. We shall say that an open base $\boldsymbol{B}$ of P has the property (V) (or, shortly, is a V-base) if there exist open bases $\boldsymbol{B}_n \subset \boldsymbol{B}$, $n = 1, 2, \ldots$, such that

(i) $\boldsymbol{B}_1 \supset \boldsymbol{B}_2 \supset \ldots$,

(ii) if $\boldsymbol{G}$ is a family of open sets, $\boldsymbol{G}$ has the finite intersection property, and $\boldsymbol{G} \cap \boldsymbol{B}_n \neq \emptyset$, $n = 1, 2, \ldots$, then $\bigcap \{\overline{G}; G \in \boldsymbol{G}\} \neq \emptyset$.

3. Theorem. *If P is regular and has a V-base, then every G_δ-subset P has a V-base. If R is a Hausdorff extension of P, and P has a V-base, then P is a G_δ-set in R.*

4. Theorem.*) *If P is completely regular, then the following conditions are equivalent:*

(i) *P is a G_δ-space;*

(ii) *P has a V-base;*

(iii) *P is a G_δ-set in βP;*

*) The equivalence of the conditions (i), (iii) and (iv) was proved by E. Čech in his paper *On bicompact spaces*, Annals of Math., Vol 38 (1937), 823—844.

(iv) *P is a G_δ-set in some compactification of P.*

This follows at once from 3, since every open base of a compact space is evidently a V-base.

5. It is quite easy to extend the preceding results in each of the following ways:

(i) we may consider, instead of G_δ-sets, intersectilos of m open sets, m being a fixed infinite cardinal;

(ii) instead of completely regular spaces we may consider regular ones taking Alexandroff's extensions instead of compactifications.

6. Consider a topological property $\boldsymbol{W}$ of subsets of topological spaces (i. e., for every P a family $\boldsymbol{W}(P)$ of subsets of P is given, and if f is a homeomorphism of P_1 onto P_2, then f transforms $\boldsymbol{W}(P_1)$ onto $\boldsymbol{W}(P_2)$). We define: a Hausdorff space P has property $\overline{\boldsymbol{W}}$ (or, belongs to the class $\overline{\boldsymbol{W}}$, written $P \in \overline{\boldsymbol{W}}$), if, for any Hausdorff extension R of P, P belongs to $\boldsymbol{W}(R)$. For instance, in 4 an "internal" characterization is given of spaces belonging to the class $\overline{\boldsymbol{W}} \cap A$ where $\boldsymbol{W}$ is property of being a G_δ-subset, A is the class of completely regular spaces; if $\boldsymbol{W}$ denotes the property of being closed, then $\overline{\boldsymbol{W}}$ is class of all H-closed spaces.

Now, for any P, let $M \in \boldsymbol{W}(P)$ if and only if M satisfies the Baire condition (is a Baire set), i. e. if there is a meager set $N \subset P$ such that $(M - N) \cup \cup (N - M)$ is open in P. If $P \in \overline{\boldsymbol{W}}$, we shall say that P satisfies the absolute Baire condition or, simply, that P is a Baire space. We give now an internal characterization of Baire spaces.

7. Theorem. *The following properties of a completely regular P are equivalent:*

(i) *P is a Baire set in βP;*

(ii) *P is a Baire set in some compactification of P;*

(iii) *P is a Baire set in every compactification of P;*

(iv) *P is a Baire set in every completely regular extension R of P;*

(v) *P is a union of a meager subset and a subset which is a G_δ-space.*

8. Problem. To give an internal characterization of spaces $P \in \overline{\boldsymbol{W}}$, $\boldsymbol{W}$ denoting the property of being a Borel set.

Резюме

О G_δ-ПРОСТРАНСТВАХ

(Предварительное сообщение)

ЗДЕНЕК ФРОЛИК (Zdeněk Frolík), Прага

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Мы называем пространство Хаусдорфа P G_δ-пространством, если P является G_δ-множеством в любом пространстве Хаусдорфа R, в котором P содержится как плотное подмножество, пространством Бэра, если P имеет в любом таком R свойство Бэра (т. е. отличается от некоторого открытого в R множества только на множество первой категории). В статье дается „внутренняя“ характеризация G_δ-пространств и пространств Бэра. Доказательства будут опубликованы отдельно.