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A NOTE TO THE UNICITY OF GENERALIZED DIFFERENTIAL EQUATIONS

IVO VRKOČ, Praha

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It is shown that a certain assumption in the Theorem 1 of the previous paper must not be omitted.

We shall construct an equation, which shows that the assumption $x(\tau) = c$ in Theorem 1 of the previous paper must not be omitted.

Let $\frac{1}{2} < \beta < 1$ and let us put $f(x, t) = 1 - (t - x)^\beta$ for $x < t$, $f(x, t) = 1$ for $x \geqq t$,

$$F(x, t) = \int_0^t f(x, \tau) \, \mathrm{d}\tau \, .$$

$f(x, t)$ is continuous and according to the results of section 2, [1] the solutions of the generalized differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F(x, t) \tag{1}$$

are identical with the ones of the classical equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, t) \, .$$

The functions $x_1(t) = t$ and $x_2(t) = t$ for $t \leqq 0$, $x_2(t) = t - [(1 - \beta)\, t]^{\frac{1}{1-\beta}}$ for $t > 0$ are obviously solutions of (2) and of (1).

We shall prove that

$$F(x, t) \in \mathbf{F}(E_2, \eta, 3\eta^\beta, 1) \subset \mathbf{F}(E_2, 3\eta^\beta, 3\eta^\beta, 1) \, . \tag{3}$$

As $|f(x, t)| \leqq 1$, we have

$$|F(x, t_2) - F(x, t_1)| = \left|\int_{t_1}^{t_2} f(x, \tau) \, \mathrm{d}\tau\right| \leqq |t_1 - t_2| \, . \tag{4}$$

Let us denote by U (V) the set of such points $[x, t]$ that $x \geqq t$ ($x \leqq t$) and by R the rectangle with vertices $[x_2, t_2]$, $[x_2, t_1]$, $[x_1, t_2]$, $[x_1, t_1]$ (where $x_2 > x_1$, $t_2 > t_1$). We put

$$\Delta(R) = F(x_2, t_2) - F(x_2, t_1) - F(x_1, t_2) + F(x_1, t_1) \, .$$

If $R \subset U$, then obviously $\Delta(R) = 0$. If $R \subset V$, then

$$\Delta(R) = \int_{t_1}^{t_2} \{(\tau - x_1)^\beta - (\tau - x_2)^\beta\} \, d\tau = \frac{1}{\beta + 1} \{(t_2 - x_1)^{\beta+1} - (t_1 - x_1)^{\beta+1} -$$

$$- (t_2 - x_2)^{\beta+1} + (t_1 - x_2)^{\beta+1}\} = (x_2 - x_1) \{(t_2 - \xi)^\beta - (t_1 - \xi)^\beta\} ,$$

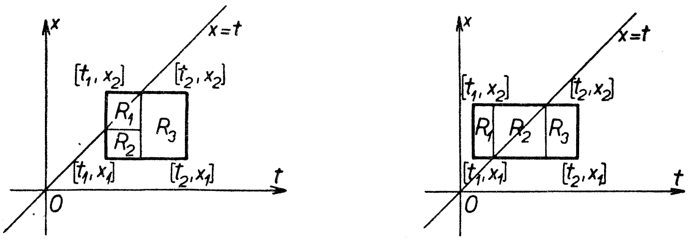
where $x_1 < \xi < x_2$ ($\leqq t_1$). As $(t_2 - \xi)^\beta - (t_1 - \xi)^\beta$ is increasing in ξ for $\xi \leqq t_1$, we obtain

$$0 \leqq \Delta(R) \leqq (x_2 - x_1)(t_2 - t_1)^\beta . \tag{5}$$

Finally let us denote by W the set of rectangles R with $x_1 = t_1$, $x_2 = t_2$. If $R \in W$, then

$$\Delta(R) = \int_{t_1}^{t_2} (\tau - x_1)^\beta \, d\tau = \frac{1}{\beta + 1} (t_2 - x_1)^{\beta+1} ,$$

so that (5) holds again.



Let R be a given rectangle. In the manner indicated on the figure we find that there exist at most three rectangles R_1, R_2, R_3 such that

$$\Delta R = \Delta R_1 + \Delta R_2 + \Delta R_3, \quad R_i \subset R \text{ and } R_i \subset U \text{ or } R_i \subset V \text{ or } R_i \in W$$

for $i = 1, 2, 3$.

Consequently

$$0 \leqq \Delta R \leqq 3(x_2 - x_1)(t_2 - t_1)^\beta \tag{6}$$

and (3) holds according to (4) and (6). It follows that Theorem 1 of the previous paper becomes false if we omit the assumption that the solution $x(\tau)$ is constant.

LITERATURE

[1] *J. Kurzweil*: Generalized Ordinary Differential Equations and Continuous Dependence on a Parameter, Czech. Math. Journal *7 (82)* 1957, 418—449.

Резюме

ЗАМЕТКА О ЕДИНСТВЕННОСТИ РЕШЕНИЙ ОБОБЩЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

ИВО ВРКОЧ (Ivo Vrkoč), Прага

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В этой статье приводится пример, который доказывает, что условие $x(\tau) = c$ в теореме 1 предыдущей статьи Я. Курцвейля „Однозначность решений обобщенных дифференциальных уравнений“, стр. 502, нельзя выпустить.