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STOCHASTIC BRANCHING PROCESSES WITH CONTINUOUS STATE SPACE

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The paper is concerned with stochastic branching processes the state space of which is the whole non-negative part of the n-dimensional Euclidean space. Existence theorems and fundamental properties are proved and several kinds of degeneration are studied.

1. General properties. In the last few years the theory of stochastic branching processes with discrete valued realisations has been developed.*) The values of the random variables describing the states of these processes are vectors with non-negative integral coordinates and the n-th coordinate usually means the number of particles of the n-th kind. As the quantity of particles can sometimes be expressed by other means than by counting, it seems reasonable to consider branching stochastic processes with more general states. It is the purpose of the present paper to give the definition and to study some properties of stochastic branching processes the state space of which is the whole non-negative part of the Euclidean space. Throughout the whole paper n will denote the number of different particles and, accordingly, the dimension of the state space.

The following notation will be used in the sequel. If a is a n-dimensional row vector, a_i will denote the i-th coordinate of a and we shall write $a = (a_1, \dots, a_n)$. The corresponding transposed vector will be denoted by $\tilde{a}$ and thus we can write $a\tilde{b}$ instead of $\sum_{i=1}^{n} a_i b_i$. To denote special vectors we shall write $\bar{0} = (0, \dots, 0)$, $\bar{1} = (1, \dots, 1)$ and $d^{(j)} = (d_1^{(j)}, \dots, d_n^{(j)})$, where $d_i^{(j)} = 0$ if $i \neq j$ and $d_j^{(j)} = 1$. More generaly $\bar{0}$ will denote any zero matrix. The relations $<, \leqq, =$ between two vectors means that the relation holds between all corresponding coordinates. The same rule applies to the operations $+, -, \int, \frac{\partial}{\partial t}$ etc. Thus, if $f_i(x), g_i(x), h_i(x), (i = 1, \dots, n)$ are functions of n variables $(x_1, \dots, x_n) = x$, the relation $f(x) = g(h(x))$ means that $f_i(x_1, \dots, x_n) = g_i(h_1(x_1, \dots, x_n), \dots,$

*) For the definition of these processes we refer to [1] or [2].

$\ldots, h_n(x_1, \ldots, x_n))$ for every $i = 1, 2, \ldots, n$. For $f(x) = f(x_1, \ldots, x_n)$ we shall write $f^{(i)}(x) = \frac{\partial}{\partial x_i} f(x_1, \ldots, x_n)$, $f^{(i,j)}(x) = \frac{\partial^2}{\partial x_i \, \partial x_j} f(x_1, \ldots, x_n)$. Since $A \pm a$ will be used in sec. 3 as a symbol for the shift of the set A by a, we shall denote the set difference by $\setminus$.

X, X_0 and X_1 will denote the n-dimensional Cartesian power of the set of all complex numbers with non-positive real part, of the non-positive part of the real line and of the interval $\langle 0, 1 \rangle$ respectively.

Let T be any set of non-negative numbers such that $0 \in T$, let E be the n-dimensional Cartesian power of the non-negative part of the real line, and let $\mathscr{E}$ be the system of all Borel subsets of E. T will represent the parameter set and E the state space of the process. We shall denote the characteristic function of $A \subset E$ by $d(a, A)$ and, more generally, if A is any subset of the k-dimensional Cartesian product $E \times \ldots \times E$, by $d(a^{(1)}, \ldots, a^{(k)}, A)$.

Any function $P(s, a, t, A)$ defined for $s \in T$, $a \in E$, $s \leqq t \in T$, $A \in \mathscr{E}$ will be called *a stochastic branching process with continuous state space* or, more briefly, *a* B-*process*, if the following conditions are fulfilled:

C 1. *$P(s, ., t, A)$ is a $\mathscr{E}$-measurable function on E for every $s \in T$, $s \leqq t \in T$, $A \in \mathscr{E}$.*

C 2. *$P(s, a, t, .)$ is a σ-additive and non-negative measure on $\mathscr{E}$ for every $s \in T$, $s \leqq t \in T$, $a \in E$.*

C 3. *$P(s, a, t, E) = 1$ for every $s \in T$, $a \in E$, $s \leqq t \in T$.*

C 4. *$P(t, a, t, A) = d(a, A)$ for every $t \in T$, $a \in E$, $A \in \mathscr{E}$.*

C 5. *$P(t_1, a, t_3, A = \int_E P(t_2, b, t_3, A)\, P(t_1, a, t_2, \mathrm{d}b)$ holds for all $t_i \in T$ $(i = 1, 2, 3)$ such that $t_1 \leqq t_2 \leqq t_3$ and all $a \in E$, $A \in \mathscr{E}$.*

C 6. $P(s, a^{(1)} + a^{(2)}, t, A) =$
$= \int_E \int_E d(b^{(1)} + b^{(2)}, A)\, P(s, a^{(1)}, t, \mathrm{d}b^{(1)})\, P(s, a^{(2)}, t, \mathrm{d}b^{(2)})$.

C 5 is the well known Chapman-Kolmogorov equation and C 6 is the characteristic property of-branching processes. From C 6 if follows that

$$\int_E f(b)\, P(s, a^{(1)} + a^{(2)}, t, \mathrm{d}b) = \int_E \int_E f(b^{(1)} + b^{(2)})\, P(s, a^{(1)}, t, \mathrm{d}b^{(1)})\, P(s, a^{(2)}, t, \mathrm{d}b^{(2)}) \tag{1.1}$$

for every bounded and measurable function f and using this relation we easily obtain

$$P(s, \sum_{j=1}^{k} a^{(j)}, t, A) = \int_E \ldots \int_E d(\sum_{j=1}^{k} b^{(j)}, A) \prod_{j=1}^{k} P(s, a^{(j)}, t, \mathrm{d}b^{(j)})$$

for every natural k. In particular, if $a \in E$ has integral coordinates, the last relation gives

$$P(s, a, t, A) = \int_E \dots \int_E d(\sum_{i=1}^{n} \sum_{j=1}^{a_i} b^{(i,j)}, A) \prod_{i=1}^{n} \prod_{j=1}^{a_i} P(s, d^{(i)}, t, \mathrm{d}b^{(i,j)}) .$$

This is a direct generalisation of the relation defining branching processes with discrete state space (see for example [1] (12) on page 53). Similarly to (1.1), it follows from C 5 that

$$\int_E f(c)\, P(t_1, a, t_2, \mathrm{d}c) = \int_E \int_E f(c)\, P(t_2, b, t_3, \mathrm{d}c)\, P(t_1, a, t_2, \mathrm{d}b) . \tag{1.2}$$

Ξ will denote the Cartesian power E^T and $\mathcal{A}$ the corresponding σ-algebra generated by the well known Kolmogorov procedure. Every $\xi \in \Xi$ is a function on T with values in Ξ and thus, $\xi(t) = (\xi_1(t), \dots, \xi_n(t))$ is its value at time t. Let P_0 be an arbitrary probability measure on E, corresponding to the probability distribution at time 0. Using P_0 and P we can always define a probability measure $\Pi_{P_0,P}$ on $\mathcal{A}$ such that

$$\Pi_{P_0,P}(A \times E^{T \setminus \{t_1,\dots,t_k\}}) = \int_E \dots \int_E d(a^{(1)}, \dots, a^{(k)}, A)\, P(t_{k-1}, a^{(k-1)}, t_k, \mathrm{d}a^{(k)}) \dots$$
$$\dots P(0, a^{(0)}, t_1, \mathrm{d}a^{(1)})\, P_0(\mathrm{d}a^{(0)})$$

holds for every natural k, $t_1 \leqq t_2 \leqq \dots \leqq t_k \in T$ and $A \in \mathscr{E} \times \dots \times \mathscr{E}$. This probability measure satisfies the condition for Markov processes and we have $\Pi_{P_0,P}(A \times E^{T \setminus \{t\}} | \xi(s) = a) = P(s, a, t, A)$ almost everywhere. For any random variable η on Ξ, $\mathbf{E}_{P_0,P}(\eta)$ will denote its expected value $\int_\Xi \eta(\xi)\, \Pi_{P_0,P}(\mathrm{d}\xi)$.

In accordance with methods of discrete space processes we define for $s \in T$, $a \in E$, $s \leqq t \in T$, $x \in X_1$ the generating function

$$F(s, a, t, x) = \int_E x_1^{b_1} x_2^{b_2} \dots x_n^{b_n}\, P(s, a, t, \mathrm{d}b)$$

and we write, in particular, $F_i(s, t, x) = F(s, d^{(i)}, t, x)$, $F(s, t, x) = (F_1(s, t, x), \dots, F_n(s, t, x))$. The following theorem states that the fundamental iteration rule for generating functions still holds.

1.1 *For every $x \in X^{(1)}$ and $t_1 \leqq t_2 \leqq t_3 \in T$ we have*

$$F(t_1, t_3, x) = F(t_1, t_2, F(t_2, t_3, x)) .$$

Proof. From (1.1) it follows that $F(s, a + b, t, x) = F(s, a, t, x) \,.\, F(s, b, t, x)$. Having in mind that $F(s, a, t, x)$ is measurable with respect to a we infer from the last functional equation that

$$F(s, a, t, x) = \prod_{i=1}^{n} [F_i(s, t, x)]^{a_i} . \tag{1.3}$$

Combining (1.2) and (1.3) we complete the proof.

Although generating functions have proved themselves to be a useful tool for studying branching stochastic processes, we shall replace them in almost all cases by complex Laplace transforms. There are several reasons for this, the main being that the generating function does not define here uniquely the corresponding transition probability. The ordinary characteristic functions would not be appropriate because of the important relation (1.5). The definition of X and E enables us to define for every $s \in T$, $a \in E$, $s \leqq t \in T$, $x \in X$

$$\Phi(s, a, t, x) = \int_E e^{x\tilde{b}} P(s, a, t, \mathrm{d}b) \tag{1.4}$$

and we write again $\Phi_i(s, t, x) = \Phi(s, d^{(i)}, t, x)$, $\Phi(s, t, x), \ldots, \Phi_n(s, t, x))$. Clearly, $\Phi(s, a, t, x)$ is continuous in x, $\mathscr{E}$-measurable in a and $|\Phi(s, a, t, x)| \leqq 1$, $\Phi(s, a, t, \bar{0}) = 1$.

1.2 *There exists exactly one function* $\Psi(s, a, t, x)$ *which is for every* $s \leqq t \in T$ *and every* $a \in E$ *continuous in* $x \in X$ *and such that* $\Psi(s, a, t, \bar{0}) = 0$, $\Phi(s, a, t, x) = \exp[\Psi(s, a, t, x)]$. *Setting* $\Psi_i(s, t, x) = \Psi(s, d^{(i)}, t, x)$ *and* $\Psi(s, t, x) = (\Psi_1(s, t, x), \ldots, \Psi_n(s, t, x))$ *we have*

$$\Phi(s, a, t, x) = \exp[\Psi(s, t, x)\, \tilde{a}] \tag{1.5}$$

and if $t_1 \leqq t_2 \leqq t_3 \in T$, *then*

$$\Psi(t_1, t_3, x) = \Psi(t_1, t_2, \Psi(t_2, t_3, x)) . \tag{1.6}$$

Proof. The first part (existence and uniqueness) of the theorem can be considered as a generalisation of a weakened form of the theorem on characteristic functions of infinitly divisible laws. Nevertheless, because of the fundamental importance of the theorem, we shall sketch how the theorem can be proved directly in our case. First of all we deduce from (1.1) for all $x \in X$ that

$$\Phi(s, a + b, t, x) = \Phi(s, a, t, x)\, \Phi(s, b, t, x) . \tag{1.7}$$

Further we remark that (1.4) has meaning for all $x \in X_0$ and for these real x (1.5) can be proved by the method used in the proof of 1.1. But (1.5) being proved for $x \in X_0$ implies $\lim_{a \to \bar{0}+} P(s, a, t, \{b \in E : b \geqq \varepsilon\, .\, \bar{1}\}) = 0$ and from this and (1.7) it follows that $\Phi(s, a, t, x)$ is continuous in both $a \in E$ and $x \in X$, and different from zero. Hence, we deduce that there exists exactly one function $\Psi(s, a, t, x)$ which is continuous in both $a \in E$, $x \in X$ and such that $\Phi(s, a, t, x) = \exp[\Psi(s, a, t, x)]$ and $\Psi(s, \bar{0}, t, \bar{0}) = 0$. All other properties of Ψ follow from the uniqueness. In particular, to deduce (1.6) we combine (1.2) and (1.5) and we obtain $\Phi(t_1, a, t_3, x) = \int_E {}^{x\tilde{b}} P(t_1, a, t_3, \mathrm{d}b) =$

$$= \int_E \int_E e^{x\tilde{b}} P(t_2, c, t_3, \mathrm{d}b)\, P(t_1, a, t_2 . \mathrm{d}c) = \int_E \Phi(t_2, c, t_3, x)\, P(t_1, a, t_2, \mathrm{d}c) =$$
$$= \int_E \exp[\Psi(t_2, t_3, x)\, \tilde{c}]\, P(t_1, a, t_2, \mathrm{d}c) = \Phi(t_1, a, t_2, \Psi(t_2, t_3, x)).$$

2. B-processes with discrete parameter. In this section we assume that the parameter set T is formed by non-negative integers, i. e., $T = \{0, 1, 2, \ldots\}$. Any function $P(s, a, t, A)$ satisfying C 1—C 6 with respect to this T will be called *a stochastic branching process with continuous state space and discrete parameter* or briefly, *a* B_d*-process*. According to C 6, $P(s, a, t, .)$ is for every s, a, t an infinitely divisible probability measure on $\mathscr{E}$. It is the aim of the following theorem to show that, on the other hand, any sequence of infinitely divisible probability measures on $\mathscr{E}$ defines uniquely a B_d-process. This proves the existence of B_d-processes because infinitely divisible probability measures on $\mathscr{E}$ exist; as an example, we can instance for $n = 1$ the Poisson distribution or, among absolutely continuous distributions, the Γ distribution, defined by the density function

$$\frac{\mu^\lambda}{\Gamma(\lambda)} e^{-\mu a} a^{\lambda-1} \quad (\mu > 0\,, \quad \lambda > 0)\,. \tag{2.1}$$

2.1. *Suppose that* $P_i(t, A)$ *are infinitely divisible probability measures on* $\mathscr{E}$ *for all* $i = 1, 2, \ldots, n$ *and all* $t \in T$. *Then there exists exactly one* B_d*-process such that* $P(t, d^{(i)}, t + 1, A) = P_i(t, A)$ *for all* $i = 1, 2, \ldots, n$.

Proof. We denote for all $x \in X$ the logarithm of the complex Laplace transform of $P_i(t, .)$ by $\Psi_i(t, x)$ and we define for all $t \in T$ an all $a \in E$ the function $P(t, a, t + 1, A)$ as the probability measure on $\mathscr{E}$ the Laplace transform of which is $\exp\,[\sum_{i=1}^{n} \Psi_i(t, x)\, a_i]$. Further we set $P(t, a, t, A) = d(a, A)$ and define by recurrence

$$P(t, a, t + s, A) = \int_E P(t + s - 1, b, t + s, A)\, P(t, a, t + s - 1, \mathrm{d}b)\,. \tag{2.2}$$

It is well known that the function $P(s, a, t, A)$ defined in this way satisfies C 1—C 5. To prove C 6 we assume $a^{(1)}, a^{(2)} \in E$, $s \in T$ to be fixed. C 6 holds for $t = s + 1$ and for all $A \in \mathscr{E}$ by definition and suppose, by induction, that C 6 holds for some $t > s$ and all $A \varepsilon \mathscr{E}$. For every function $f(c^{(1)}, c^{(2)})$ which is bounded and $\mathscr{E} \times \mathscr{E}$ measurable we define

$$I(f) = \int_E \int_E [\int_E \int_E f(b^{(1)}, b^{(2)})\, P(t, b^{(1)}, t + 1, \mathrm{d}c^{(1)})\, P(t, b^{(2)}, t + 1, \mathrm{d}c^{(2)})]\,.$$

$$.\, P(s, a^{(1)}, t, \mathrm{d}b^{(1)})\, P(s, a^{(2)}, t, \mathrm{d}b^{(2)})\,.$$

Starting with $f(c^{(1)}, c^{(2)}) = d(c^{(1)}, A^{(1)})\, d(c^{(2)}, A^{(2)})$ and continuing by the usual procedure we deduce from C 6

$$I(f) = \int_E \int_E f(b^{(1)}, b^{(2)})\, P(s, a^{(1)}, t + 1, \mathrm{d}b^{(1)})\, P(s, a^{(2)}, t + 1, \mathrm{d}b^{(2)})\,. \tag{2.3}$$

By (1.1), (2.2) and our assumption we have

$$P(s, a^{(1)} + a^{(2)}, t + 1, A) = \int_E P(t, b, t + 1, A)\, P(s, a^{(1)} + a^{(2)}, t, \mathrm{d}b) =$$

$$= \int_E \int_E P(t, b^{(1)} + b^{(2)}, t + 1, A)\, P(s, a^{(1)}, t, \mathrm{d}b^{(1)})\, P(s, a^{(2)}, t, \mathrm{d}b^{(2)})\,.$$

Since $P(t, a, t + 1, A)$ satisfies C 6, the last relation proves that $P(s, a^{(1)} + a^{(2)}, t + 1, A) = I(f)$ with $f(c^{(1)}, c^{(2)}) = d(c^{(1)} + c^{(2)}, A)$. Inserting into (2.3) we complete the proof.

Corollary. If $\Psi^{(1)}(x)$, $\Psi^{(2)}(x)$ are logarithms of two Laplace transforms of infinitely divisible probability measures on $\mathscr{E}$, then the same holds for the composite function $\Psi^{(2)}(\Psi^{(1)}(x))$.

In the rest of this section we shall consider homogeneous B_d-processes only, i. e. we shall assume $P(s, a, t, A) = P(0, a, t - s, A)$ for all $s \leqq t \in T$. We shall write $P(a, t, A)$ instead of $P(s, a, s + t, A) = P(0, a, t, A)$, $P_i(t, A)$ instead of $P(d^{(i)}, t, A)$ and $P_i(A)$ instead of $P_i(1, A)$. According to 2.1, the homogeneous B_d-process is determined by n infinitely divisible probability measures $P_i(A)$ $i = 1, 2, \ldots, n$. Similarly to the notation just described we shall write $\Psi(a, t, x)$ instead of $\Psi(s, a, s + t, x)$, $\Psi_i(t, x)$ instead of $\Psi(d^{(i)}, t, x)$, $\Psi_i(x)$ instead of $\Psi_i(1, x)$ and further we shall write $\Psi(x) = (\Psi_1(x), \ldots, \Psi_n(x))$. The same rule will hold for Φ and F. To denote the first moments of $P(t, a, .)$ we shall write $M_j(a, t) = \int_E b_j P(a, t, \mathrm{d}b)$. For simplicity reasons, $M_j(d^{(i)}, t)$ will be replaced by $M_{ij}(t)$ and $M(t)$ will denote the square matrix $(M_{ij}(t))$, $i, j = 1, \ldots, n$. In particular, we shall write M_{ij}, M instead of $M_{ij}(1)$, $M(1)$. Just as in the case of branching processes with discrete states it can be proved that, if all M_{ij} $(i, j = 1, \ldots, n)$ are finite, then all $M_{ij}(t)$ $(i, j = 1, \ldots, n,$ $t = 1, 2, \ldots)$ are finite too and

$$M(t) = M^t\,, \tag{2.4}$$

$$M_{ij}(t) = \Phi_i^{(j)}(t, 0) = \Psi_i^{(j)}(t, 0)\,. \tag{2.5}$$

We shall assume throughout the rest of this section 2 that all M_{ij} are finite. The matrix M is non-negative and we shall denote its maximal characteristic number by R.

We write $\Lambda_t = \{\xi \in \Xi : \xi(t) = 0\}$ and $\Lambda = \bigcup_{t=1}^{\infty} \bigcap_{\tau=t}^{\infty} \Lambda_\tau$. From (1.4) we have $P(\bar{0}, t, \{\bar{0}\}) = 1$ and accordingly $\Pi_{P_0,P}(\bigcap_{\tau=t}^{s} \Lambda_\tau) = \int_E \ldots \int_E \prod_{\tau=t}^{s} d(a^{(\tau)}, \{\bar{0}\})\,.$

$$. P(a^{(s-1)}, 1, \mathrm{d}a^{(s)}) \ldots P(a, t, \mathrm{d}a^{(t)})\, P_0(\mathrm{d}a) = \int_E P(a, t, \{\bar{0}\})\, P_0(\mathrm{d}a)$$

and

$$\Pi_{P_0,P}(\Lambda) = \lim_{t\to\infty} \Pi_{P_0,P}(\Lambda_t) = \lim_{t\to\infty} \int_E P(a, t, \{\bar{0}\})\, P_0(\mathrm{d}a)\,. \tag{2.6}$$

The homogeneous B_d-process $P(a, t, A)$ will be said to *converge*

a) *strictly to zero,* if for every P_0 we have $\Pi_{P_0,P}(\Lambda) = 1$; in other words, if $\Pi_{P_0,P}$ almost all realisations are equal to $\bar{0}$, beginning from certain t.

b) *strongly to zero,* if for every P_0 and every $i = 1, 2, \ldots n$ $\xi_i(t)$ converges to zero $\Pi_{P_0,P}$ — almost surely.

c) *weakly to zero,* if for every P_0 and every $o = 1, 2, \ldots, n$ $\xi_i(t)$ converges to zero in $\Pi_{P_0,P}$ — probability.

Clearly, the strict convergence implies the strong convergence, and the strong convergence implies the weak convergence.

From [1], Theorem 9 it is apparent that in the case of branching processes with discrete state space and with finite first moments all three kinds of the convergence to zero are equivalent. In our case the equivalence ceases to hold. More precisely, the strict convergence to zero is not equivalent to the strong convergence to zero, as it follows from the remark to 2.2. On the other hand, according to 2.6 the strong convergence to zero is under sufficiently general conditions equivalent to the weak convergence to zero. The author does not know, whether this equivalence holds generally.

2.2. *The homogeneous B_d-process converges strictly to zero if and only if the system of n equations*

$$F(x) = x \tag{2.7}$$

has in the domain X_1 no solution, except for $x = \bar{1}$.

Proof. We omit the details of the proof, which would be the imitation of the methods developed for branching processes with discrete state space. The proof lies on the following three facts a) according to (2.6), the process converges strictly to zero if and only if $\lim_{t\to\infty} F(t, \bar{0}) = \bar{0}$; b) according to 1.1 $\lim_{t\to\infty} F(t, \bar{0}) = y$ always exists and satisfies (2.7); c) if $z \in X_1$ satisfies (2.7), then $z \geqq y$.

Remark to 2.2. The same statement holds for branching processes with discrete state space and is implicitely obtained in the proof of Theorem 9, [1]. However, the mentioned theorem gives the necessary and sufficient conditions in a different form, namely: a) $R \leqq 1$ b) there are no final groups. In our case, these conditions would not be true for strict convergence. This can be shown by the following example. We take $n = 1$ and define the fundamental probability measure $P_1(A)$ by means of the density function (2.1) with $\lambda < \mu$. Then both a) and b) are satisfied, but $F(0) = 0$ and this implies, according to 1.1, $F(t, 0) = 0$ for all t. On the other hand, conditions similar to a) and b) are necessary and sufficient for the weak convergence and, under one additional, condition, for the strong convergence to zero (see 2.4, 2.6).

Before we prove these theorems, we introduce some new notation. Every subset $I = \{i_1, \ldots, i_m\}$ of $\{1, 2, \ldots, n\}$ will be called an index set. For such a set I and for every $x = (x_1, \ldots, x_n) \in X_0$ we shall write $x^{(I)} = (x_{i_1}, \ldots, x_{i_m})$. Further, $M_I(t)$ will denote the matrix $(M_{ij}(t))_{ij\in I}$ and we shall write M_I instead of $M_I(1)$. R_I will denote the maximal characteristic number of M_I. The index set I will be said to be irreducible, if there exists no decomposition $I = I_1 \cup I_2$ into two disjoint sets I_1, I_2 such that $P_i(\{b \in E : b_j = 0) = 1$ for each $i \in I_1$, $j \in I_2$. Clearly, a necessary and sufficient condition that I be irreducible is that M_I be irreducible. The index set I will be called a final group, if I is irreducible, $R_I = 1$ and $P_i(\bigcap_{j\in I}\{b : b_j = M_{ij}\}) = 1$ for each $i \in I$. Although in the following theorems the domain of the Laplace transform is restricted to X_0, i. e. to real x only, the logarithms ψ_i could not be replaced by the generating functions F_i.

2.3. *Under the assumption that the index set I is irreducible and $R_I \leqq 1$, the following three statements hold*:

a) *If there exists a vector $\bar{0} \neq x \in X_0$ such that $x_i = 0$ for i non $\in I$ and*

$$P_i(\{b \in E : \tilde{x}b = x_i\}) = 1 \quad \textit{for all} \quad i \in I\,, \tag{2.8}$$

then I is a final group.

b) *If there exists a vector $\bar{0} \neq x \in X_0$ such that $x_i = 0$ for i non $\in I$ and $\Psi_i(x) = x_i$ for $i \in I$, then I is a final group.*

c) *If I is not a final group, then for any $\eta < 0$ there exists a vector $x \in X_0$ such that $x_i = 0$ for i non $\in I$, and $\eta < x_i < 0$, $\psi_i(x) > x_i$ for $i \in I$.*

Proof of a). From (2.8) we get $M_I\tilde{x}^I = \tilde{x}^I$ and consequently 1 must be the characteristic number of M_I. This shows, according to the assumption $R_I \leqq 1$, that $R_I = 1$. Further, M_I being irreducible, we have $x_I < \bar{0}$. For each $i, j \in I$ and for every Borel set $A \subset \langle 0, \infty)$ we define $Q_{ij}(A) = P_i(\{b \in E : b_j \in A)$. Because of C 6, Q_{ij} is an infinitely divisible probability measure and from (2.8) we deduce $Q_{ij}\left(< 0, \frac{x_i}{x_j} >\right) = 1$. But it is proved in [3] that there exists no non-trivial infinitely divisible probability distribution on a bounded interval and hence $P_i(\{b \in E : b_j = M_{ij}\}) = 1$ for each $i, j \in I$.

Proof of b). Let J be the set of all indices, for which $x_i < 0$. Note that always $J \subset I$. According to the Taylor theorem and (2.5) we have

$$\Psi_i(x) = \sum_{j\in I} M_{ij}x_j + \sum_{j,k\in J} \Psi_I^{(j,k)}(y)\, x_j x_k\,, \tag{2.9}$$

where $y \in X_0$ is a vector such that $y_i = 0$ for i non $\in J$ and $x_i < y_i < 0$ for $i \in J$. It is easy to show that

$$\Psi_i^{(j,k)}(y) = [\Phi_i(y)]^{-1} \int_E (b_j - \Psi_i^{(j)}(y))(b_k - \Psi_i^{(k)}(y))\, e^{y\tilde{b}} P_i(\mathrm{d}b)$$

for each $j, k, \in J$ and consequently

$$\sum_{j,k \in J} \Psi_i^{(j,k)}(y)\, x_j x_k = [\Phi_i(y)]^{-1} \int_E [\sum_{j \in J}(b_j - \Psi_i^{(j)}(y))\, x_j]^2\, e^{y\tilde{b}} P_i(\mathrm{d}b) \geqq 0\,. \quad (2.10)$$

Since $\Psi_i(x) = x_i$ for all $i \in I$, we have by (2.9) and (2.10)

$$\sum_{j \in I} M_{ij}(-x_j) = -x_j + \sum_{j,k \in J} \Psi_i^{(j,k)}(y)\, x_j x_k \geqq -x_i \geqq R_I(-x_i) \qquad (2.11)$$

for all $i \in I$. Since $x^{(I)} \neq 0$, (2.11) proves that $(-x^{(I)})$ is an extremal vector of M_I, i. e. an eigen-vector belonging to R_I. But then the sign of equality must hold in (2.11) and hence $R_I = 1$, $\sum_{j \in J} M_{ij} x_j = x_i$ and $\sum_{j,k \in J} \Psi_i^{(j,k)}(y)\, x_j x_k = 0$ for all $i \in I$. By (2.10), it now follows that $P_i(\{b \in E : \sum_{j=1}^n b_j x_j = \sum_{j=1}^n \Psi_i^{(j)}(y)\, x_j\}) = 1$ for all $i \in I$, and integrating we obtain $x_i = \sum_{j=1}^n M_{ij} x_j = \sum_{j=1}^n \Psi_i^{(j)}(y)\, x_j$. This proves that $P_i(\{b \in E : x\tilde{b} = x_i\}) = 1$ and the assertion of b) follows from a).

Proof of c). Let V be the set of all $x \in X_0$ such that $\eta < x_i < 0$ for all $i \in I$ and $x_i = 0$ for all i non $\in I$. M_I being irreducible, there exists $x^{(0)} \in V$ such that $\sum_{j \in I} M_{ij} x_j^{(0)} = R_I x_j^{(0)} \geqq x_i$. By (2.9) and (2.10), we have $\Psi(x^{(0)}) \geqq x^{(0)}$ and, according to b), there exists at least one index $i_0 \in I$ such that $\Psi_{i_0}(x^{(0)}) > x_{i_0}^{(0)}$. Suppose that a couple $(J, x^{(1)})$ has already been found in such a way that $J \subset I$, $x^{(1)} \in V$,

$$\sum_{j \in I} M_{ij} x_j^{(1)} \geqq x_i^{(1)} \quad \text{for all } i \in I \setminus J \text{ and } \Psi_i(x^{(1)}) > x_i^{(1)} \text{ for all } i \in J \quad (2.12)$$

$(\{i_0\}, x^{(0)})$ is an example of such a couple. If $J \neq I$, there exist $i_1 \in I - J$, $j_1 \in J$ such that $M_{i_1 j_1} > 0$ because of the irreducibility of M_I. Ψ being continuous, we can find $x^{(2)} \in V$ in such a way that $x_i^{(2)} = x_i^{(1)}$ for all $i \neq j_1$, $x_{j_1}^{(1)} < x_{j_1}^{(2)} < 0$, $\Psi_i(x^{(2)}) > x_i^{(2)}$ for all $i \in J$. Since $M_{i_1 j_1} > 0$, we have $\sum_{j \in I} M_{i_1 j} x_j^{(2)} > x_{i_1}^{(2)}$ (2.12.) and then, according to (2.9) and (2.10), $\Psi_{i_1}(x^{(2)}) > x_{i_1}^{(2)}$. Finally, the inequality $\sum_{j \in I} M_{ij} x_j^{(2)} \geqq x_i^{(2)}$ continues to be valid for $i \in I \setminus (J \cup \{i_1\})$ because $j_1 \in J$. This proves that (2.12) continues to hold, if the couple $(J, x^{(1)})$ is replaced by $(J \cup \{i_1\}, x^{(2)})$. Continuing in this way we obtain finally a couple (I, x) and this completes the proof of c).

2.4. *A homogeneous* B_d*-process with finite first moments converges weakly to zero if and only if*

a) $R \leqq 1$, b) *there are no final groups.*

Remark. If $R < I$, then b) is always satisfied because $R_I \leqq R$ for each index set I.

Proof of the theorem. We can assume without loss of generality that the indices are ordered in such a way that

$$M = \begin{bmatrix} M^{(1)} & \bar{0} & \bar{0} & . & . & \bar{0} \\ M^{(1,2)} & M^{(2)} & \bar{0} & . & . & . \\ . & & & & . & . \\ . & & & & \bar{0} & \bar{0} \\ . & & & & M^{(s-1)} & \bar{0} \\ M^{(s,1)} & . & . & . & M^{(s,s-1)} & M^{(s)} \end{bmatrix} \tag{2.13}$$

where $M^{(j)}$ $(j = 1, 2, \ldots, s)$ are irreducible square matrices. Let $M^{(j)}$ be of dimension $m_j \times m_j$, so that $\sum_{j=1} m_j = n$. The index sets $I_j = \{\sum_{i=1}^{j-1} m_i + 1, \ldots, \ldots, \sum_{i=1}^{j} m_i\}$ are irreducible and from the form of M we see that, if $i \in T_j$, the function $\Psi_i(x)$ does not depend on x_l with $l \in \bigcup_{k>j} I_k$.

We prove first the sufficiency. Assuming that the conditions a), b) are satisfied, we shall prove that to each $j = 1, 2, \ldots, s$ there exists a vector $x^{(j)} \in X_0$ such that

$$\begin{gathered} x_i^{(j)} = 0 \quad \text{for all} \quad i \in \bigcup_{k<j} I_k\,, \\ x_i^{(j)} < 0 \quad \text{and} \quad \Psi_i(x^{(j)}) > x_i^{(j)} \quad \text{for all} \quad i \in \bigcup_{k \geqq j} I_k\,. \end{gathered} \tag{2.14}$$

This is certainly true for $i = s$ because of 2.3c). Suppose, by induction, that (2.14) holds for some $j > 1$. Since Ψ_i is continuous, we can find $\eta < 0$ in such a way that for an arbitrary vector $y \in X_0$ satisfying $y_i = x_i^{(j)}$ for $i \in \bigcup_{k \neq j-1} I_k$ and $\eta < y_i \leqq 0$ for $i \in I_{j-1}$ we have

$$\Psi_i(y) > y_i \quad \text{for} \quad i \in \bigcup_{k \geqq j} I_k\,. \tag{2.15}$$

According to 2.3c), there exists $z \in X_0$ such that

$$\begin{gathered} z_i = 0 \quad \text{for} \quad i \in \bigcup_{k \neq j-1} I_k\,, \\ \eta < z_i < 0 \quad \text{and} \quad \Psi_i(z) > z_i \quad \text{for} \quad i \in I_{j-1}\,. \end{gathered} \tag{2.16}$$

Let $x^{(j-1)} \in X_0$ be the vector the coordinates of which are $x_i^{(j-1)} = x_i^{(j)}$ for $i \in \bigcup_{k \neq j-i} I_k$ and $x_i^{(j-1)} = z_i$ for $i \in I_{j-1}$. Then by (2.14), $x_i^{(j-1)} = 0$ for $i \in \bigcup_{k<j-i} I_k$ and (2.15) proves that $\Psi_i(x^{(j-i)}) > x_i^{(j-1)}$ for $i \in \bigcup_{k \geqq j} I_k$. Finally, (2.16) implies $\Psi_i(x^{(j-1)}) > x_i^{(j-1)}$ for $i \in I_{j-1}$, because $x_i^{(j-1)} = z_i$ for $i \in \bigcup_{k<j-1} I_k$ and because Ψ_i with $i \in I_{j-1}$ does not depend on coordinates x_l with $l \in \bigcup_{k \geqq j} I_k$. This proves (2.14)

for $j - 1$ and, consequently, for all $j = 1, 2, \ldots, s$. In particular, taking $j = 1$ we obtain a vector $x^{(1)} < \bar{0}$ such that $\Psi_i(x^{(1)}) = x_i^{(1)}$ for all i. Since Ψ is non-decreasing in the domain X_0, the last inequality yields, according to (1.6), the inequality $\Psi(t-1, x^{(1)}) \leqq \Psi(t, x^{(1)})$ for all t. Hence $\lim_{t\to\infty} \Psi(t, x^{(1)}) = x^{(0)}$ exists, and using (1.6) we obtain $\Psi(x^{(0)}) = x^{(0)}$. We now prove $x^{(0)} = \bar{0}$. Let $y \in X_0$ be the vector the coordinates of which are $y_i = x_i^{(0)}$ for $i \in I_1$ and $y_i = 0$ for $i \in \bigcup_{k>1} I_k$. Then $\Psi_i(y) = y_i$ for $i \in I$, because Ψ_i with $i \in I$ does not depend on x_l with $l \in \bigcup_{k>1} I_k$. But this shows, by 2.3b), that $y = \bar{0}$ and, consequently, $x_i^{(0)} = 0$ for all $i \in I_1$. This procedure is to be continued step by step for I_2, $I_3, \ldots, I_s$, and finally we obtain $x_i^{(0)} = 0$ for all i. Summarizing our results we see that there exists a vector $x^{(1)} < \bar{0}$ such that $\lim_{t\to\infty} \Psi(t, x^{(1)}) = \bar{0}$. Then, by (1.5), $\lim_{t\to\infty} \Phi(\mathrm{a}, \mathrm{t}, x^{(1)}) = 1$ for all $a \in E$. This proves that

$$\lim_{t\to\infty} P(a, t, \{b \in E : b_i > \varepsilon\}) = 0$$

for all a, i and $\varepsilon > 0$, because of the inequality

$$P(a, t, \{b \in E : b_i > \epsilon\}) \leqq \frac{1 - \Phi(a, t, x^{(1)})}{1 - \varepsilon^{\varepsilon x_i^1}} .$$

The assertion of the theorem now follows from the relation

$$\Pi_{P_0,P}(\{\xi \in \Xi : \xi_i(t) > \varepsilon\xi) = \int_E P(a, t, \{b \in E : b_i > \varepsilon\})\, P_0(\mathrm{d}b) .$$

To prove the necessity of the conditions we assume first that the condition a) is not satisfied, that is, we suppose $R > 1$. Then, according to the well known properties of non-negative matrices, there exist $t_0 \in T$ and an index j such that $M_{jj}(kt_0) > 1$ for every natural k. But this implies $P_j(kt_0, \{b \in E : b_j = 0\}) < 1$ for all k and consequently

$$\Psi_j(kt_0, d^{(j)}) < 0 \quad \text{for all } k . \tag{2.17}$$

If $G(x) = \Psi_j(t_0, x) - x_j$, then $G(\bar{0}) = 0$, $G^{(j)}(\bar{0}) = M_{jj}(t_0) - 1 > 0$ and accordingly there exists $\delta < 0$ such that

$$\Psi_j(t_0, x) < x_j \quad \text{for all } x \text{ with } \delta < x_j < 0 . \tag{2.18}$$

Suppose that the process converges weakly to zero. Then, in particular, $\lim_{t\to\infty} P_j(t, \{b \in E : b_j > \varepsilon\}) = 0$ for all $\varepsilon > 0$, and consequently

$$\lim_{t\to\infty} \Psi_j(kt_0, d^{(j)}) = 0 . \tag{2.19}$$

From (2.17) and (2.19) it then follows that there exists K such that

$$\delta < \Psi_j(kt_0, d^{(j)}) < 0 \quad \text{for all } k \geqq K . \tag{2.20}$$

Finally we see from (1.6) that $\Psi_j((k+1)\,t_0, x) = \Psi_j(t_0, \Psi(kt_0, x))$, and applying this relation to (2.18) and (2.20) we obtain $\Psi_j(kt_0, d^{(j)}) < \Psi_j(Kt_0, d^{(j)}) < 0$ for all $k > K$. But this contradicts (2.19).

It remains to consider the case where there exists a final group I and $R = 1$. In fact, we see from the remark before the proof of this theorem that the eventuality $R < 1$ is impossible under the existence of a final group. Suppose again that the matrix M is of the normal form (2.13). Since the final group I is irreducible, there exists an index k $(1 \leqq k \leqq s)$ such that $I \subset I_k$. If I were a proper subset of I_k, then, by the well known properties of irreducible matrices, the inequality $R_I < R_{I_k} \leqq R$ would hold. But this is impossible because $R_I = R = 1$, and consequently $I = I_k$. Let $-x^{(I_k)}$ be the eigen-vector of the matrix M_{I_k} corresponding to R_{I_k} and let $x^{(0)} \in X_0$ be the vector with the coordinates $x_i^{(0)} = 0$ for i non $\in I_k$ and $x_i^{(0)} = x_i^{(I_k)}$ for $i \in I_k$. Then

$$P_i(\{b \in E : \sum_{j \in I_k} b_j x_j^{(0)} = x_i^{(0)}\}) = 1 \quad \text{for all } i \in I_k\,. \tag{2.21}$$

The function Ψ_i with $i \in \bigcup_{j<k} I_j$ does not depend on x_l with $l \in I_k$ and consequently, by (2.21),

$$\Psi_i(x^{(0)}) = x_i^{(0)} \quad \text{for all} \quad i \in \bigcup_{j \leqq k} I_j\,. \tag{2.22}$$

The matrix $M(t) = M^t$ being again of the same quasi-triangular form as M, the function $\Psi_i(t, x)$ with $i \in I_k$ does not depend on x_l with $l \in \bigcup_{j>k} I_j$. Hence, by (1.6) and (2.22), $\Psi_i(t+1, x^{(0)}) = \Psi_i(t, \psi(x^{(0)})) = \Psi_i(t, x^{(0)})$ for all $i \in I_k$, and from this and (2.22) if follows that $\Psi_i(t, x^{(0)}) = x^{(0)}$ for all t and all $i \in I_k$. But this proves that $\Psi_i(t, x^{(0)})$ cannot converge to 0 because $x_i^{(0)} < 0$ for $i \in I_k$, and the proof is completed.

2.5. *Each homogeneous B_d-process with finite first moments and with $R < 1$ converges strongly to zero.*

Proof. We first suppose that the initial distribution P_0 has finite first moments, that is, $\bar{M}_i = \int_E b_i P_0(\mathrm{d}b) < \infty$ for all $i = 1, 2, \ldots, n$. Then

$$\mathbf{E}_{P_0,P}(\xi_i(t)) = \int_\Xi \xi_i(t)\, \Pi_{P_0,P}(\mathrm{d}\xi) = \int_E \int_E b_i P(a, t, \mathrm{d}b)\, P_0(\mathrm{d}a) =$$

$$= \int_E M(a, t)\, P(\mathrm{d}a) = \int_E \sum_{j=1}^n a_j M_{ji}(t)\, P_0(\mathrm{d}a) = \sum_{j=1}^n \bar{M}_j M_{ji}(t)\,.$$

Since $M_{ij}(t)$ are members of the matrix $M(t) = M^t$, we have $M_{ij}(t) = O(R_0^t)$ for any R_0 such that $R < R_0 < 1$. Hence, $\mathbf{E}_{P_0,P}(\xi_i(t)) = O(R_0^t)$ and consequently $\sum_{t=1} \mathbf{E}_{P_0,P}(\xi_i(t)) < \infty$ for all i. Since $\xi_i(t) \geqq 0$, we see by a well known theorem that $\xi_i(t) \to 0$ with $\Pi_{P_0,P}$ probability 1. To remove the assumption that the first moments of P_0 are finite, it is sufficient to represent the probability measure P_0 as a limit of probability measures whose first moments are finite.

2.6. *Suppose that a homogeneous* B_d*-process has finite first moments and let the eigen-vector of the matrix* M*, corresponding to the maximal characteristic number* R*, be positive. Then the process converges strongly to zero if and only if*

a) $R \leqq 1$, b) *there are no final groups.*

Proof. In view of the theorems 2.4 and 2.5, it is sufficient to prove that the process converges strongly to zero, if both conditions $R = 1$ and b) are satisfied. As in the proof of 2.5 we can assume that all first moments $\overline{M}_i$ of P_0 are finite. Let $l = (l_1, \ldots, l_n)$ be a positive eigen-vector of M corresponding to $R = 1$ and define $\eta(t) = \xi(t)\,\tilde{l}$. Using the method of [2] page 313 we can prove that the sequence $\eta(t)$ forms a bounded martingale process and consequently $\lim\limits_{t\to\infty} \eta(t) = \eta$ exists with $\Pi_{P_0,P}$-probability 1 [See [4], Chap. VII, Theorem 4.1]. But according to 2.5 $\xi_i(t)$ converges to zero in $\Pi_{P_0,P}$-probability and this implies $\eta = 0$ with $\Pi_{P_0,P}$-probability 1. Finally, since $\xi_i(t) \geqq 0$ and $l_i > 0$, we have $\lim\limits_{t\to\infty} \xi_i(t) = 0$ with $\Pi_{P_0,P}$-probability 1.

Remark. As follows from the theorems 2.4 and 2.6, the weak and the strong convergence are equivalent, if there exists a positive vector of the matrix M belonging to the characteristic number R. This last condition is always satisfied, if $n = 1$ or, more generally, if the matrix M is irreducible.

3. B-processes with continuous state space. In this section we allow the time parameter to assume any non-negative value, that is, we suppose $T = \langle 0, \infty.)$ We shall consider purely discontinuous processes only,*) that is, we shall suppose that the B-process satisfies in addition the condition

C 7. *There exist finite limits*

$$\lim_{t\to s-} \frac{P(t, a, s, A) - (Ps, a, s, A)}{t - s} = \lim_{t\to s+} \frac{P(s, a, t, A) - P(s, a, s, A)}{t - s} =$$
$$= p(s, a, A) \tag{3.1}$$

for all $s \in T$, $a \in E$, $A \in \mathscr{E}$.

The function $p(s, a, A)$ defined by (3.1) is sometimes called a transition intensity. Every B-process with $T = \langle 0, \infty)$ which satisfies (3.1) will be called *a* B_c*-process*. Each B_c-process determines uniquely the corresponding transition intensity $p(s, a, A)$. The essential part of this section is devoted to the inverse problem, that is, to the construction of the B_c-process, if its transition intensity is given. To be able to do so, we must first examine the function $p(s, a, A)$. We write $E_i(\delta) = \{b \in E : b_i < \delta\}$ for all $\delta < 0$ and all $i = 1, 2, \ldots, n$. If $A \in \mathscr{E}$ and $a \in E$, $A \pm a$ will denote the set $\{b \in E : b \mp a \in A\}$.

*) See for example [5].

3.1. *Let $p(s, a, A)$ be the transition intensity function of a B_c-process. Then*

a) *$p(s, ., A)$ is a $\mathscr{E}$-measurable function for all $s \in T$, $A \in \mathscr{E}$.*

b) *$p(s, a, .)$ is a finite and σ-additive set function for all $s \in T$, $a \in E$.*

c) *$p(s, a, \{a\}) \leqq 0$, $p(s, a, E) = 0$.*

d) *$p(s, a, A) \geqq 0$ if $A \subset E \setminus \{a\}$.*

e) *$p(s, d^{(i)}, E_i(1)) = 0$.*

f) *$p(s, a, A) = \sum_{i=1}^{n} a_i p(s, d^{(i)}, A + d^{(i)} - a)$.*

Proof. The assertions a)—d) express well known properties of a transition intensity function and are easily derived from C 1—C 5 and (3.1). To derive the others we define $\varphi(s, a, x) = \int_E e^{x\tilde{b}} p(s, a, \mathrm{d}b)$ and $\psi(s, a, x) = e^{-x\tilde{a}} \varphi(s, a, x)$ for all $x \in X$. We shall write again $\varphi_i(s, x) = \varphi(s, d^{(i)}, x)$, $\varphi(s, x) = (\varphi_1(s, x), \ldots, \varphi_n(s, x))$, and the same for ψ. Since, by (3.1), $\varphi(s, a, x)$ is the derivative of $\Phi(s, a, t, x)$ with respect to t in the point $t = s$, we have according to (1.7) $\varphi(s, a^{(1)} + a^{(2)}, x) = \varphi(s, a^{(1)}, x)\, e^{x\tilde{a}^{(2)}} + \varphi(s, a^{(2)}, x)\, e^{x\tilde{a}^{(1)}}$ and consequently $\psi(s, a^{(1)} + a^{(2)}, x) = \psi(s, a^{(1)}, x) + \psi(s, a^{(2)}, x)$. The function ψ is regular enough to be then necessarily of the form $\psi(s, a, x) = \psi(s, x)\, \tilde{a}$ and hence,

$$\varphi(s, a, x) = \sum_{i=1}^{n} a_i \varphi_i(s, x)\, e^{x(\widetilde{a - d^{(i)}})} . \tag{3.2}$$

This proves f). To prove e), consider $0 < \delta < 1$. Then, by c) and f), $0 = p(s, \delta d^{(i)}, E) = \delta p(s, d^{(i)}, E + (1 - \delta)\, d^{(i)}) = -\delta p(s, d^{(i)}, E_i(1 - \delta)$ and hence we deduce c) by the relation $E_i(1) = \bigcup_{k=1}^{\infty} E_i\left(1 - \frac{1}{k}\right)$.

Remark. As it appears from d) and f), the assertion e) can easily be generalised to the following proposition. If, for a given $a = (a_1, \ldots, a_n) \in E$, $E(a)$ denotes the union of all $E_i(a_i)$ with $a_i > 0$, then $p(s, a, A) = 0$ for all $A \subset E(a)$.

3.2. *Let $p_i(s, A)$ $(i = 1, 2, \ldots, n\}$ be functions on the domain $T \times \mathscr{E}$. Then, in order that there be on the domain $T \times E \times \mathscr{E}$ a function $p(s, a, A)$ satisfying the conditions a)—f) of 3.1 and such that $p(s, d^{(i)}, A) = p_i(s, a)$ for all i, it is necessary and sufficient that the functions $p_i(s, A)$ satisfy the following three conditions:*

c 1. *$p_i(s, .)$ is a finite and σ-additive set function on $\mathscr{E}$ for all $i = 1, \ldots, n$ and all $s \in T$.*

c 2. $p_i(s,A) \geqq 0$ for all $A \subset E \setminus \{d^{(i)}\}$.

c 3. $p_i(s, \{d^{(i)}\}) \geqq 0$, $p_i(s, E) = 0$, $p_i(s, E_i(1)) = 0$.

If these conditons are satisfied, then the function $p(s, a, A)$ is determined uniquely by our requirements.

Proof. Suppose that c 1—c 3 hold. From f) we see that $p(s, a, A) = \sum_{i=1}^{n} a_i p_i(s, A + d^{(i)} - a)$ is the only possible way of defining the function $p(s, a, A)$. If $p(s, a, A)$ is defined in this way, all properties required by the theorem are obvious except perhaps for $p(s, a, E) = 0$. To prove this last, we distinguish two cases. If $a_i \geqq 1$, then $E + d^{(i)} - a = E$, and if $a_i < 1$, then $E + d^{(i)} - a = E \setminus E_i(1 - a_i)$. We have always $p_i(s, E) = 0$ and, in the case $a_i < 1$, $p_i(s, E_i(1 - a_i)) = 0$ by c 2 and c 3. Consequently, $p_i(s, E + d^{(i)} - a) = 0$ in both cases and $p(s, a, E) = 0$ follows from the definition.

Because some regularity condition concerning the variable will be necessary in the sequel, we shall suppose that

c 4. $p_i(., A)$ is *continuous on* $T = \langle 0, \infty)$ *for each* i *and each* $A \in \mathscr{E}$.

A vector function $p(s, A) = (p_1(s, A), \ldots, p_n(s, A))$ will be called *a* b-*function*, if all $p_i(s, A)$ satisfy c 1—c 4.

Let $p(s, A)$ be a b-function and let $p(s, a, A)$ be the corresponding function defined in 3.2. Write $p_i(s) = p_i(s, \{d^{(i)}\})$, $p(s) = (p_1(s), \ldots, p_n(s))$, $q(s, a, A) = p(s, a, A \setminus \{a\})$, $q_i(s, A) = q(s, d^{(i)}, A)$. Then, by 3.1f), we have

$$p(s, a, \{a\}) = p(s)\, \tilde{a}\,, \tag{3.3}$$

$$q(s, a^{(1)} + a^{(2)}, A) = q(s, a^{(1)}, A - a^{(2)}) + q(s, a^{(2)}, A - a^{(1)})\,. \tag{3.4}$$

Finally, let us define in accordance with [5]

$$J(s, t) = \int_t^s p(\tau)\, \mathrm{d}\tau, \quad P^{(0)}(s, a, t, A) = d(a, A) \exp\,[J(s, t)\, \tilde{a}]\,,$$

$$P^{(k)}(s, a, t, A) = \int_s^t \exp\,[J(s, \sigma)\, \tilde{a}] \int_E P^{(k-1)}(\sigma, b, t, A)\, q(\sigma, a, \mathrm{d}b)\, \mathrm{d}\sigma$$

$$\text{and } P(s, a, t, A) = \sum_{k=0}^{\infty} P^{(k)}(s, a, t, A) \text{ for all } s \leqq t \in T,\ a \in E,\ A \in \mathscr{E}.$$

Let a b-function $p(s, A)$ be given. Then the function $P(s, a, t, A)$ defined by the procedure just decribed will be said to be the corresponding P(b)-*function*. A B_c-process whose intensity function $p(s, a, A)$ satisfies $p(s, d^{(i)}, A) = p_i(s, A)$ will be called *a* B_c-*process generated by the* b-*function* $p(s, A)$. The following theorems prove that under certain conditions the P(b)-function is the B_c-process generated by $p(s, A)$. This insures the existence of B_c-processes because the existence of b-functions is obvious.

3.3. *Let* $p(s, A)$ *be a* b-*function. Then the corresponding* P(b)-*function satisfies* C 1, C 2, C 4—C 7 (*with respect to* $p(s, a, A)$ *defined in* 3.2) *and we have*

$$0 \leqq P(s, a, t, A) \leqq 1 \quad \textit{for all} \quad s \leqq t,\ a,\ A\,. \tag{3.5}$$

Proof. The truth of C 1, C 2, C 4, C 5, C 7 and (3.5) is the assertion of Theorems 1, 2 of [5]. Therefore it remains to prove C 6. According to (3.4) we have for any bounded and measurable function f and all $a^{(0)}, a^{(1)} \in E$

$$\int_E f(b)\, q(s, a^{(0)} + a^{(1)}, \mathrm{d}b) = \sum_{i=0}^{1}{}^{*} \int_E f(a^{(i+1)} + b)\, q(s, a^{(i)}, \mathrm{d}b)\,, \tag{3.6}$$

where the symbol $\sum^*$ means that the indices of the members of the sum are to be taken mod 2. We now prove

$$\begin{gathered} P^{(k)}(s, a^{(0)} + a^{(1)}, t, A) = \\ = \sum_{j=0}^{k} \int_E \int_E d(b^{(0)} + b^{(1)}, A)\, P^{(j)}(s, a^{(0)}, t, \mathrm{d}b^{(0)})\, P^{(k-j)}(s, a^{(1)} t, \mathrm{d}b^{(1)}) \end{gathered} \tag{3.7}$$

for all $k = 0, 1, \ldots$ The relation (3.7) holds for $k = 0$ by definition, and suppose, by induction, that it holds for some k. Using (3.6) we get

$$\begin{gathered} P^{(k+1)}(s, a + a, t, A) = \\ = \int_s^t \exp\left[J(s, \sigma)\widetilde{(a^{(0)} + a^{(1)})}\right] \int_E P^{(k)}(\sigma, b, t, A)\, q(\sigma, a^{(0)} + a^{(1)}, \mathrm{d}b)\, \mathrm{d}\sigma = \\ = \sum_{i=0}^{1}{}^{*} \int_s^t \exp\left[J(s, \sigma)\widetilde{(a^{(0)} + a^{(1)})}\right] \int_E P^{(k)}(\sigma, a^{(i+1)} + b, t, A)\, q(\sigma, a^{(i)}, \mathrm{d}b)\, \mathrm{d}\sigma = \\ = \sum_{j=0}^{k} \sum_{i=0}^{1}{}^{*} \int_s^t \exp\left[J(s, \sigma)\widetilde{(a^{(0)} + a^{(1)})}\right] \int_E \int_E \int_E d(c^{(0)} + c^{(1)}, A)\, P^{(j)}(\sigma, a^{(i+1)}, t, \mathrm{d}x^{(0)})\,. \\ .\, P^{(k-j)}(\sigma, b, t, \mathrm{d}c^{(1)}) q(\sigma, a^{(1)}, \mathrm{d}b)\, \mathrm{d}\sigma = \\ = \sum_{i=0}^{1}{}^{*} \int_E \int_E d(c^{(0)} + c^{(1)}, A) (\exp\left[J(s, t)\, \tilde{a}^{(i+1)}\right] d(a^{(i+1)}, \mathrm{d}c^{(0)})\,. \\ \cdot \int_s^t (\exp\left[J(s, \sigma)\, \tilde{a}^{(i)}\right] \int_E P^{(k)}(\sigma, b, t, \mathrm{d}c^{(1)})\, q(\sigma, a^{(i)}, \mathrm{d}b)\, \mathrm{d}\sigma) + \\ + \sum_{j=1}^{k} \sum_{i=0}^{1}{}^{*} \int_E \int_E d(c^{(0)} + c^{(1)}, A) \int_s^t \int_\sigma^t \exp\left[J(s, \sigma)\, \tilde{a}^{(i)} + J(s, \tau)\, \tilde{a}^{(i+1)}\right] . \\ \cdot (\int_E P^{(j-1)}(\tau, b^{(0)}, t, \mathrm{d}c^{(0)})\, q(\tau, a^{(i+1)}, \mathrm{d}b^{(0)}))(\int_E P^{(k-j)}(\sigma, b^{(1)}, t, \mathrm{d}c^{(1)})\, q(\sigma, a^{(i)}, \mathrm{d}b^{(1)}))\, \mathrm{d}\tau\, \mathrm{d}\sigma\,. \end{gathered}$$

In the last step, we used a rule for interchanging the order of integration which can be proved by usual methods mentioned in the proof of 2.1. Interchanging once more the order of integration with respect to σ and τ in the terms with $i = 0$ and $j > 0$, we get finally

$$\begin{gathered} P^{(k+1)}(s, a^{(0)} + a^{(1)}, t, A) = \\ = \sum_{i=0}^{1}{}^{*} \int_E \int_E d(c^{(0)} + c^{(1)}, A)\, P^{(0)}(s, a^{(i+1)}, t, \mathrm{d}c^{(0)})\, P^{(k+1)}(s, a^{(i)}, t, \mathrm{d}c^{(1)}) + \\ + \sum_{j=1}^{k} \int_E \int_E d(c^{(0)} + c^{(1)}, A) (\int_s^t \exp\left[J(s, \tau)\, \tilde{a}^{(0)}\right] \int_E P^{(j-1)}(\tau, b^{(0)}, t, \mathrm{d}c^{(0)})\, q(\tau, a^{(0)}, \mathrm{d}b^{(0)})\, \mathrm{d}\tau)\,. \\ .\, (\int_s^t \exp\left[J(s, \sigma)\, \tilde{a}^{(1)}\right] \int_E P^{(k-j)}(\sigma, b^{(1)}, t, \mathrm{d}c^{(1)})\, q(\sigma, a^{(1)}, \mathrm{d}b^{(1)})\, \mathrm{d}\sigma) = \\ = \sum_{j=0}^{k+1} \int_E \int_E d(c^{(0)} + c^{(1)}, A)\, P^{(j)}(s, a^{(0)}, t, \mathrm{d}c^{(0)})\, P^{(k+1-j)}(s, a^{(1)}, t, \mathrm{d}c^{(1)})\,. \end{gathered}$$

This proves (3.7) for all k and C 6 follows from the relation

$$P(s, a^{(0)} + a^{(1)}, t, A) = \sum_{k=0}^{\infty} P^{(k)}(s, a^{(0)} + a^{(1)}, t, A) =$$

$$= \int_E \int_E d(b^{(0)} + b^{(1)}, A) \sum_{k=0}^{\infty} \sum_{j=0}^{k} P^{(j)}(s, a^{(0)}, t, \mathrm{d}b^{(0)})\, P^{(k-j)}(s, a^{(1)}, t, \mathrm{d}b^{(1)}) =$$

$$= \int_E \int_E d(b^{(0)} + b^{(1)}, A)\Big(\sum_{k=0}^{\infty} P^{(k)}(s, a^{(0)}, t, \mathrm{d}b^{(0)})\Big)\Big(\sum_{l=0}^{\infty} P^{(l)}(s, a^{(1)}, \mathrm{d}b^{(1)})\Big) .$$

It is well known that the "transition probabilities" constructed by the procedure we have used here need not satisfy C 3 and in general, all known conditions that insure the validity of C 3 are either too complicated or too restrictive. Even the relation (3.4), which is essential for our theory, does not remove generally this undesirable fact, as it appears from the example which will be given later on. On the other hand, it enables us to derive simple and sufficiently general conditions for C 3.

If $p(s, A)$ is a b-function, we shall write, in accordance with the notation used in the proof of 3.1, $\psi_i(s, x) = \int_E e^{x\tilde{b}} p_i(s, \mathrm{d}b)$, $\psi(s, x) = (\psi_1(s, x), \ldots, \psi_n(s, x))$. The first moments of $p(s, A)$ will be denoted by $m_{ij}(s) = \int_E b_j p_i(s, \mathrm{d}b)$.

3.4. *Let $p(s, A)$ be a* b*-function and suppose that, for every $t > 0$, the system of n differential equations*

$$y'(s) = -\psi(s, y(s)) \tag{3.8}$$

has in the interval $\langle 0, t\rangle$ exactly one non-positive solution $y(s)$ such that $y(t) = \bar{0}$. Then there exists exactly one B_c*-process generated by $p(s, A)$ and it is equal to the* P(b)*-function. Moreover, the corresponding function $\Psi(s, t, x)$ defined in sec 1 satisfies as a function of s the system* (3.8) *with the initial condition $\Psi(t, t, x) = x$ for all $t > 0$, $x \in X$. The condition of the theorem is always satisfied, if $m_{ij}(s) < \infty$ for all $s \in T$ and all i, j.*

Proof. In view of 3.3, C 3 is the only missing condition for P(b) to be a B_c-process. According to 3.3, the P(b)-function $P(s, a, t, A)$ satisfies C 6 and hence, in particular, $P(s, a^{(1)} + a^{(2)}, t, E) = P(s, a^{(1)}, t, E)\, P(s, a^{(2)}, t, E)$. As it appears from the definition of the P(b)-function, $P(s, a, t, E) > 0$ and consequently there exists $Q(s, t) = (Q_1(s, t), \ldots, Q_n(s, t))$ such that $P(s, a, t, E) = \exp [Q(s, t)\, \tilde{a}]$. Then

$$Q_i(s, t) = \lim_{h \to 0+} \frac{P(s, hd^{(i)}, t, E) - 1}{h} . \tag{3.9}$$

Moreover, $Q(s, t) \leqq 0$ by 3.5), and $Q(t, t) = \bar{0}$ by the definition of $P(t, a, t, E)$. From the definition of $P(s, a, t, A)$ we have

$$P(s, a, t, A) =$$

$$= d(a, A) \exp [J(s, t)\, \tilde{a}] + \int_s^t \exp [J(s, \sigma)\, \tilde{a}] \int_E P(\sigma, b, t, A)\, q(\sigma, a, \mathrm{d}b)\, \mathrm{d}\sigma \tag{3.10}$$

and, in particular, $P(s, a, t, E) =$

$$= \exp[J(s, t)\,\tilde{a}] + \int_s^t \exp[J(s, \sigma)\,\tilde{a}] \int_E \exp[Q(\sigma, t)\,\tilde{b}]\, q(\sigma, a, \mathrm{d}b)\, \mathrm{d}\sigma =$$

$$= \exp[J(s, t)\,\tilde{a}] + \int_s^t \exp[J(s, \sigma)\,\tilde{a}](\varphi(\sigma, a, Q(\sigma, t)) - \exp[Q(\sigma, t)\,\tilde{a}]\, p(\sigma)\,\tilde{a}]\, \mathrm{d}\sigma\,.$$

Then, according to (3.9), $Q_i(s, t) = \int_s^t \varphi_i(\sigma, Q(\sigma, t)) \exp[-Q_i(\sigma, t)]\, \mathrm{d}\sigma = \int_s^t \psi_i(\sigma, Q(\sigma, t))\, \mathrm{d}\sigma$. Differentiating with respect to s, we see that $Q(s, t)$ considered as a function of s only is a non-positive solution of (3.8) with the initial condition $Q(t, t) = \bar{0}$ for all $t > 0$. But then, according to the assumption, $Q(s, t)$ is the only solution and consequently $Q(s, t) = \bar{0}$ for all $s \leqq t$. This implies $P(s, a, t, E) = 1$ for all a and $s \leqq t$, that is, C 3 holds. Concerning the unicity we remark that Markov processes for which C 7 holds always satisfy the backward integro-differential equation and it is well known that the B_c-function satisfying C 3 is the only Markov process which is the solution of this equation. The assertion that $\Psi(s, t, x)$ satisfies (3.8) can be deduced from the relation $\Psi_i(s, t, x) = \lim_{h \to 0+} \frac{1}{h} [\Phi(s, hd^{(i)}, t, x) - 1]$ and from (3.10) by the method used in the beginning of the proof. If $m_{ij}(s) < \infty$, then $0 \leqq \psi_i^{(j)}(s, x) \leqq \int_E (b_j - d_j^{(i)})\, p_i(s, \mathrm{d}b) \leqq m_{ij}s) < \infty$ and the Lipschitz condition for (3.8) is fulfilled.

3.5. *If $n = 1$, that is, if $E = \langle 0, \infty)$, then the first condition of* 3.4 *is not only sufficient but also necessary for the* $\mathsf{P}(\mathsf{b})$*-function to satisfy* C 3.

Proof. Suppose that the $\mathsf{P}(\mathsf{b})$-function satisfies C 3. From the preceding proof we see that $\Psi(s, t, x)$ as a function of s satisfies (3.8) with the initial condition $\Psi(t, t, x) = x$ for all $t > 0$ and all $x \leqq 0$. Further, $0 \leqq \Psi^{(1)}(s, x) = \int_E (b - 1)\, e^{x(b-1)}\, p(s, \mathrm{d}b) < \infty$ for $x < 0$, and this implies that the system (3.8) has for any $t > 0$ and $x < 0$ in the neighbourhood (t, x) exactly one non-positive solution $y(s)$ such that $y(t) = x$. Using these two facts we can easily prove that (3.8) has exactly one non-positive solution with $y(t) = 0$ for a given $t > 0$.

The author does not know whether the condition is necessary in the general case $n > 1$.

Remark. If $n = 1$ and if the b-function $p(s, A)$ does not depend on s, that is, if $p(s, A) = p(A)$ and $\psi(s, x) = \psi(x)$ for all s, then according to a well known theorem on differential equations the necessary and sufficient condition for $\mathsf{P}(\mathsf{b})$ to satisfy C 3 is that

$$\int_{x_0}^{0} \frac{1}{\varphi(x)}\, \mathrm{d}x = \int_{x_0}^{0} \frac{1}{\psi(x)}\, \mathrm{d}x = -\infty$$

where $x_0 < 0$ and $\frac{1}{\varphi(x)} = \frac{1}{\psi(x)} = -\infty$ if $\varphi(x) = \psi(x) = 0$. This condition is closely related to Theorem 7 of [5].

Example. Suppose $n = 1$ and take an arbitrary $\varepsilon \in (0, 1\rangle$. Let us define a measure $q(A)$ on $\mathscr{E}$ by means of its Radon-Nikodym derivative $\frac{\mathrm{d}q}{\mathrm{d}\lambda}$ where λ is the Lebesgue measure, $\frac{\mathrm{d}q}{\mathrm{d}\lambda}(a) = \frac{1}{a^{1+\varepsilon}}$ for $a > 1$ and $\frac{\mathrm{d}q}{\mathrm{d}\lambda}(a) = 0$ for $0 \leqq a \leqq 1$. If $p = -q(E) = -\int\limits_0^\infty a^{-(1+\varepsilon)}\,\mathrm{d}a$ and $p(A) = q(A) + d(1, A)\,p$, then $p(A)$ is a b-function and $\varphi(x) = (-x)^\varepsilon(-\varepsilon)^{-1}\int\limits_{-x}^\infty e^{-u}u^{-\varepsilon}\,\mathrm{d}u$. In the case $\varepsilon < 1$, the integral $\int\limits_{x_0}^\infty \frac{1}{\varphi(x)}\,\mathrm{d}x$ is finite and according to the preceding remark the corresponding P(b)-function does not satisfy C 3. In the case $\varepsilon = 1$, $m_{11} = \int\limits_0^\infty ap(\mathrm{d}a) = \infty$ but $\int\limits_{x_0}^\infty \frac{1}{\varphi(x)}\,\mathrm{d}x = -\infty$ and consequently, C 3 holds. This shows that the condition of finite first moments $m_{ij}(s)$ in 3.4 is not necessary.

We suppose in the rest of this section that the B_c-process is homogeneous and we write again $P(a, t, A)$ instead of $P(s, a, s + t, A)$. Then the transition intensities and the b-functions do not depend on s and we shall denote them by $p(a, A)$ and $p(A) = (p_1(A), \ldots, p_n(A))$ respectively. The first moment of $p(A)$ will be denoted by $m_{ij} = \int\limits_E b_j p_i(\mathrm{d}b)$ and their matrix by m. As in the case of branching processes with discrete states, all $M_{ij}(t)$ are finite, if all m_{ij} are finite, and then $M(t) = e^{mt}$. We shall denote the maximal characteristic roof of $M(1)$ and m by R and r respectively. Then, $R = e^r$.

The weak, strong and strict convergence to zero of B_c-processes can be defined in the same way as for B_d-processes, the strong and strict ones under the assumption that the process is separable, of course. But we have the following theorem:

3.6. *None of the three kinds of convergence to zero exists for B_c-processes with finite first moments m_{ij}.*

Proof. It is sufficient to prove that weak convergence cannot exist. Suppose, on the contrary, that the process converges weakly to zero. Then also the B_d-process $P(a, t, A)$ $[t = 0, 1, \ldots]$ converges weakly to zero and hence, by 2.4, $R \leqq 1$ and there are no final groups. But then $r \leqq 0$ and $m_{ii} \leqq 0$ for all i and hence, according to the property c 3, $p_i(\{b \in E : b_i \neq 1\}) = 0$ for all i. By

3.4, the B_c-process is identical with the corresponding $\mathsf{P}(\mathsf{b})$-function and we see from the definition of the $\mathsf{P}(\mathsf{b})$-function that $P_i(\{b \in E : b_i = 1\}) = 1$. This proves that all $\{i\}$ are final groups and the process cannot converge to zero.

4. The extension of branching processes with discrete state space to B-processes. In this last section $\overline{E}$ will denote the set of all vectors $a \in E$ with integral coordinates and $\overline{\mathscr{E}}$ will denote the σ-algebra of all subsets of $\overline{E}$. If we replace in the definition of a B-process the sets E and $\mathscr{E}$ by $\overline{E}$ and $\overline{\mathscr{E}}$ respectively, we obtain the definition of a branching stochastic process with discrete state space. We shall call it *a* $\overline{\mathsf{B}}$*-process* and we shall denote the corresponding transition probability function by $\overline{P}(s, a, t, A)$. Generally, any symbol supplied with a bar will denote an object the definition of which we obtain if we replace in the preceding theory the sets E and $\mathscr{E}$ by $\overline{E}$ and $\overline{\mathscr{E}}$ respectively. Thus, we obtain the definition of $\overline{\mathsf{B}}_d$*-processes*, $\overline{\mathsf{B}}_c$*-processes*, transition intensity functions $\overline{p}(s, a, A)$ and $\overline{\mathsf{P}}(\overline{\mathsf{b}})$*-functions*. The only exception concerns the $\overline{\mathsf{b}}$-function, which will be defined later on. Each probability measure $\overline{P}$ on $\overline{\mathscr{E}}$ induces in an obvious way a probability measure P on $\mathscr{E}$. If this P is infinitely divisible, then $\overline{P}$ will be said to be infinitely divisible, too. A B-process $P(s, a, t, A)$ will be said to be an extension of a $\overline{\mathsf{B}}$-process $\overline{P}(s, a, t, A)$, if $P(s, a, t, A) = \overline{P}(s, a, t, A)$ for all $a \in \overline{E}$, $s \leqq t \in T$ and $\overline{A} \in \overline{\mathscr{E}}$. Clearly, the extension is always unique if it exists.

We suppose first $T = \{0, 1, 2, \ldots\}$. The following extension theorem follows easily from C 6 and 2.1.

4.1. *Let $\overline{P}(s, a, t, A)$ be a $\overline{\mathsf{B}}_d$-process. Then the extension to a B_d-process exists if and only if the probability measures $\overline{P}(s, d^{(i)}, s + 1, A)$ are infinitely divisible for all $s \in T$ and all $i = 1, 2, \ldots, n$.*

We now suppose $T = \langle 0, \infty)$. It can be easily verified that the transition intensity $\overline{p}(s, a, A)$ has all the properties stated in 3.1 with the exception of e). In accordance with this last fact, we define a $\overline{\mathsf{b}}$*-function* as a vector function $\overline{p}(s, A) = (\overline{p}_1(s, A), \ldots, \overline{p}_n(s, A))$ whose domain of definition is $T \times \overline{\mathscr{E}}$ and which satisfies the following two conditions:

a) $\overline{p}_i(., A)$ *is continuous for all $A \in \overline{\mathscr{E}}$ and all i,*

b) $\overline{p}_i(s, .)$ *is a finite and σ-additive set function such that $p_i(s, \{a\}) \geqq 0$ if $a \neq d^{(i)}$, $p_i(s, \{d^{(i)}\}) \leqq 0$, $p_i(s, \overline{E}) = 0$ for all i and all $s \in T$.*

We remark that the corresponding $\overline{\mathsf{P}}(\overline{\mathsf{b}})$-function need not satisfy C 3 just as in the case of continuous states. To remove this fact, conditions similar to those of 3.4 could be given.

4.2. *Let* $\bar{p}(s, A) = (\bar{p}_1(s, A), \ldots, \bar{p}_n(s, A))$ *be a* $\bar{\mathsf{b}}$*-function such that the corresponding* $\bar{\mathsf{P}}(\bar{\mathsf{b}})$*-function* $\bar{P}(s, a, t, A)$ *is a* $\bar{\mathsf{B}}_c$*-process. Then the* $\bar{\mathsf{B}}_c$*-process generated by* $\bar{p}(s, A)$ *can be extended to a* B_c*-process if and only if* $\bar{p}_i(s, \{b \in \bar{E} : b_i = 0\}) = 0$ *for all* i.

Proof. The necessity of the condition follows from 3.1 e). Suppose conversely that the condition holds and define $p(s, A) = \bar{p}(s, A \cap \bar{E})$ for all $A \in \mathscr{E}$. Then $p(s, A)$ is a b-function and it is easily seen that the corresponding $\mathsf{P}(\mathsf{b})$-function is an extension of the $\bar{\mathsf{P}}(\bar{\mathsf{b}})$-function. Moreover, it is a $\bar{\mathsf{B}}_c$-process because $\bar{P}(s, d^{(i)}, t, E) = 1$ by hypothesis. The following theorem is a trivial but surprising consequence of 4.2 and C 6.

4.3. *Let the assumption of* 4.2 *be satisfied and let* $\bar{p}_i(s, \{b \in \bar{E} : b_i = 0\}) = 0$ *for all* i. *Then all transition probabilities* $\bar{P}(s, a, t, A)$ *are infinitely divisible.*

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Резюме

ВЕТВЯЩИЕСЯ СЛУЧАЙНЫЕ ПРОЦЕССЫ С НЕПРЕРЫВНЫМ ПРОСТРАНСТВОМ СОСТОЯНИЙ

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Обозначим через E множество всех n-мерных векторов $a = (a_1, \ldots, a_n)$ с неотрицательными компонентами, через $\mathscr{E}$-систему всех борелевских подмножеств пространства E и через T — некоторое множество неотрицательных чисел такое, что $0 \in T$. Множество E представляет пространство состояний процесса, а T — множество значений параметра. Всякую функцию $P(s, a, t, A)$, определенную для $a \in E$, $s \leqq t \in T$, $A \in \mathscr{E}$ будем называть ветвящимся случайным процессом с непрерывным пространством состояний, или коротко B-процессом, если она выполняет условия C1—C6. Вместо производящих функций $F(s, a, t, x)$, удобных для исследования процессов с дискретными состояниями, пользуемся в статье логарифмами $\Psi(s, a, t, x)$ комплексных и действительных преобразований Лапласа, для которых опять имеет место фундаментальное соотношение (1.6) — (Теорема 1.2).

B-процесс с множеством параметров $T = \{0, 1, 2, \ldots\}$ называется B_d-процессом (Часть 2.). Всякий B_d-процесс можно построить при помощи произвольной последовательности безгранично делимых вероятностных мер $P(s, d^{(i)}, s + 1, A)$ $[s = 0, 1, \ldots; i = 1, \ldots, n]$, где $d^{(i)} = (0, \ldots, 0, 1, 0, \ldots, 0)$ с единицей на i-том месте. (Теорема 2.1.) Скажем, что B_d-процесс сходится к нулю

а) строго, если почти все выборочные функции равны, начиная с некоторого места, нулю,

б) сильно, если почти все выборочные функции сходятся к нулю,

в) слабо, если выборочные функции сходятся к нулю по вероятности для всякого начального распределения.

Процесс сходится строго к нулю тогда и только тогда, когда существует в замкнутом единичном кубе только одно решение системы (2.7) — (Теорема 2.2). Условия, высказанные в [1] — Теорема 9, являются здесь необходимыми и достаточными для слабой сходимости к нулю, если удобно определим понятие финального класса (Теорема 2.4). Они также необходимы и достаточны для сильной сходимости к нулю, если максимальному характеристическому числу матрицы первых моментов соответствует положительный вектор (Теорема 2.6).

B-процесс с множеством $T = \langle 0, \infty)$ называется B_c-процессом, если выполняет добавочное условие C 7 (Часть 3). Функция $p(s, A) = (p_1(s, A), \ldots, \ldots, p_n(s, A))$, определенная на $T \times \mathscr{E}$, называется b-функцией, если выполнены условия c 1—c 4. Доказывается, что при некоторых довольно общих условиях регулярности существует точно один B_c-процесс, для которого имеет место C 7 с $a = d^{(i)}$ и $p(s, d^{(i)}, A) = p_i(s, A)$ (Теорема 3.4). Ни один из приведенных видов сходимости к нулю не существует для B_c-процессов (Теорема 3).

В последней части (4) изучается расширение ветвящихся процессов с дискретными состояниями на соответствующие B-процессы. Самым интересным и в некотором смысле поразительным результатом является следующее утверждение (Теорема 4.3): Если для ветвящегося процесса с дискретными состояниями невозможны переходы из состояния $d^{(i)}$ в состояние $(\ldots, a_{i-1}, 0, a_{i+1}, \ldots)$, то все вероятности перехода такого процесса безгранично делимы.