Štefan Schwarz

On the structure of the semigroup of measures on a finite semigroup

Czechoslovak Mathematical Journal, Vol. 7 (1957), No. 3, 358–373

Persistent URL: http://dml.cz/dmlcz/100255

ON THE STRUCTURE OF THE SEMIGROUP OF MEASURES ON A FINITE SEMIGROUP

ŠTEFAN SCHWARZ, Bratislava.

(Received August 20, 1956.)

Let $\mathfrak{M}(S)$ be the set of all measures μ defined on a finite semigroup S with $\mu(S) = 1$. Under a suitable definition of multiplication $\mathfrak{M}(S)$ becomes a semigroup. The purpose of this paper is to study the structure of the semigroup $\mathfrak{M}(S)$, especially the role of the right invariant measures on S in it (supposing that such measures exist).

Let S be a finite semigroup. By a measure μ we shall mean an additive, non-negative set function defined on the subsets of S such that $\mu(S) = 1$.

A measure μ is called right invariant if for every subset $E \subseteq S$ and every $x \in S$ $\mu(Ex) = \mu(E)$ holds. In [5] we have found necessary and sufficient conditions for the existence of right invariant measures on a certain type of bicompact semigroups to which also all finite semigroups belong. The knowledge of the results of [5] is presupposed.

The purpose of the present paper is to study the structure of the set of all measures on a given semigroup S and first of all to find the role of the subset of right invariant measures in it.

An analogous problem for bicompact groups has been studied in the paper of Wendel [7]. The present paper has also some contacts with the paper Hewitt-Zuckerman [3], where arithmetic and convergence questions of measures on a certain type of finite commutative semigroups are studied.

In the following S denotes always a finite semigroup. As in [5] it is the non-commutative case in which we shall be principally interested. The symbol $\mathfrak{M}(S)$ will denote the set of all measures on S. The elements of the set $\mathfrak{M}(S)$ will be denoted by small Greek types $\nu, \pi, \mu, \ldots$

Let $S = \{x_1, x_2, \ldots, x_n\}$. A measure $\nu \in \mathfrak{M}(S)$ can be considered in the usual way as a point function on S so that denoting $\nu(x_i) = t_{x_i}$ we have $0 \leqq t_{x_i} \leqq 1$, $\sum_{x_i \in S} t_{x_i} = 1$. Conversely, a point function $f(x)$, where $f(x_i) = t_{x_i}$ and $0 \leqq t_{x_i} \leqq 1$, $\sum_{x_i \in S} t_{x_i} = 1$, can be used to introduce a measure on S by putting $\mu(\{x_{i_1}, \ldots, \ldots, x_{i_k}\}) = t_{x_{i_1}} + \ldots + t_{x_{i_k}}$.

Definition. *Let be $\nu_1, \nu_2 \in \mathfrak{M}(S)$. By the product $\nu_1 * \nu_2$ we shall mean the measure defined by the relation*

$$\nu_1 * \nu_2(x) = \sum_{uv=x} \nu_1(u)\, \nu_2(v)\ .$$

The product $\nu_1 * \nu_2$ is again a measure since we have

$$\sum_{x_i \in S} \nu_1 * \nu_2(x_i) = \sum_{x_i \in S} \sum_{uv = x_i} \nu_1(u)\, \nu_2(v) = \sum_{u \in S} \nu_1(u) \,.\, \sum_{v \in S} \nu_2(v) = 1 \,.\, 1 = 1\ .$$

It is easy to see that the multiplication need not be commutative but is always associative, i. e. $(\nu_1 * \nu_2) * \nu_3 = \nu_1 * (\nu_2 * \nu_3)$. Under this multiplication (convolution) $\mathfrak{M}(S)$ becomes a semigroup.

Let $\mathfrak{A}(S)$ be the semigroup algebra of S, i. e. the set of all formal real linear combinations of elements of S, $\sum_{x_i \in S} t_{x_i} \,.\, x_i$, with termwise addition and scalar multiplication and with the product defined by

$$\Big(\sum_{x_i \in S} t'_{x_i} x_i\Big) \,.\, \Big(\sum_{x_k \in S} t''_{x_k} x_k\Big) = \sum_{x_i \in S} \sum_{x_k \in S} t'_{x_i} t''_{x_k} x_i x_k\ . \tag{1}$$

By $\mathfrak{F}(S)$ we denote the subset of all elements $\in \mathfrak{A}(S)$ with $0 \leqq t_{x_i} \leqq 1$ and $\sum_{x_i \in S} t_{x_i} = 1$. Under the multiplication defined in (1) the set $\mathfrak{F}(S)$ becomes clearly a semigroup.

It is well known and easy to show that $\mathfrak{F}(S)$ and $\mathfrak{M}(S)$ are isomorphic semigroups. Consider to this end the correspondence

$$\nu(x) \in \mathfrak{M}(S) \longleftrightarrow t_{x_1} \,.\, x_1 + \ldots + t_{x_n} \,.\, x_n \in \mathfrak{F}(S)\ , \tag{2}$$

where $t_{x_i} = \nu(x_i)$ for $i = 1, 2, \ldots, n$. This is a one-to-one correspondence between the elements $\in \mathfrak{M}(S)$ and $\mathfrak{F}(S)$. Let

$$\nu_1 \longleftrightarrow t'_{x_1} x_1 + \ldots + t'_{x_n} x_n\ , \quad \nu_2 \longleftrightarrow t''_{x_1} x_1 + \ldots + t''_{x_n} x_n\ .$$

Then the element $\in \mathfrak{F}(S)$ corresponding to the product $\nu_1 * \nu_2$ is

$$\sum_{uv=x_1} \nu_1(u)\, \nu_2(v) \,.\, x_1 + \ldots + \sum_{uv=x_n} \nu_1(u)\, \nu_2(v) \,.\, x_n = \sum_{uv=x_1} t'_u t''_v \,.\, x_1 + \ldots + \sum_{uv=x_n} t'_u t''_v x_n\ .$$

But the last expression is exactly the product $\big(\sum_{x_i \in S} t'_{x_i} \,.\, x_i\big)\big(\sum_{x_k \in S} t'_{x_k} \,.\, x_k\big)$. This proves our assertion.

Remark 1. Denote by $\varepsilon_{x_i}(x)$ the measure defined as follows:

$$\varepsilon_{x_i}(x) = \begin{cases} 0 \text{ for } x \neq x_i\ , \\ 1 \text{ for } x = x_i\ . \end{cases}$$

We have clearly $\varepsilon_{x_i} * \varepsilon_{x_k} = \varepsilon_{x_i x_k}$. Denoting by S' the set of these measures we have $S \cong S'$, i. e. $\mathfrak{M}(S)$ contains a subsemigroup S' isomorphic to S. It follows especially that $\mathfrak{M}(S)$ contains idempotents, namely, at least all measures $\varepsilon_{x_i}(x)$, where x_i is an idempotent $\in S$.

Remark 2. It is possible to introduce in $\mathfrak{M}(S)$ such a topology that $\mathfrak{M}(S)$ becomes a Hausdorff bicompact semigroup. But in the present paper the topological properties of $\mathfrak{M}(S)$ will not be necessary.

1

Definition. *Let be $\nu \in \mathfrak{M}(S)$. The symbol $C(\nu)$ will denote in all the paper the set $\{x_i | x_i \in S, \nu(x_i) \neq 0\}$.*

Lemma 1,1. (See analogously Hewitt-Zuckerman [3], Theorem 4,7.) *If $\nu_1, \nu_2 \in \mathfrak{M}(S)$, then $C(\nu_1) \,.\, C(\nu_2) = C(\nu_1 * \nu_2)$.*

Proof. According to the definition we have $\nu_1 * \nu_2(x_i) = \sum_{uv = x_i} \nu_1(u) . \nu_2(v)$. If $\nu_1(u) > 0$, $\nu_2(v) > 0$, i. e. $u \in C(\nu_1)$, $v \in C(\nu_2)$, we have $\nu_1 * \nu_2(uv) = \sum_{u'v' = uv} \nu_1(u')\, \nu_2(v') \geqq \nu_1(u)\, \nu_2(v) > 0$, therefore $uv \in C(\nu_1 * \nu_2)$ and $C(\nu_1) \,.\, C(\nu_2) \subseteqq C(\nu_1 * \nu_2)$.

Conversely: if for some $x_i \in S$ $\nu_1 * \nu_2(x_i) > 0$, then the sum $\sum_{uv = x_i} \nu_1(u)\, \nu_2(v)$ contains at least one member > 0, i. e. there is a $u \in S$ and $v \in S$ such that $uv = x_i$ and $\nu_1(u) \,.\, \nu_2(v) > 0$. Therefore $C(\nu_1 * \nu_2) \subseteqq C(\nu_1) \,.\, C(\nu_2)$. This proves Lemma 1,1.

An element $\nu \in \mathfrak{M}(S)$ is idempotent if $\nu * \nu = \nu$. In this case Lemma 1,1 implies $C(\nu) \,.\, C(\nu) = C(\nu * \nu) = C(\nu)$. This proves

Lemma 1,2. *If ν is an idempotent $\in \mathfrak{M}(S)$, then $C(\nu)$ is a semigroup.*

In the following we shall need often

Lemma 1,3. *Let μ be a right invariant measure on S. Then $C(\mu)$ and $S - C(\mu)$ are right ideals of S. The set $C(\mu)$ is a left simple semigroup and for every $x \in S$ we have $C(\mu) \,.\, x = C(\mu)$.*

Proof. See [5], Theorem 1,1.

We give now an example on which we shall show later various properties of the semigroup $\mathfrak{M}(S)$.

Example 1,1. Let $S_1 = \{x_1, x_2, x_3, x_4\}$ be a semigroup with the following multiplication table:

	x_1	x_2	x_3	x_4
x_1	x_1	x_2	x_1	x_2
x_2	x_2	x_1	x_2	x_1
x_3	x_3	x_4	x_3	x_4
x_4	x_4	x_3	x_4	x_3 .

To every measure $\nu \in \mathfrak{M}(S_1)$ there corresponds an element $\in \mathfrak{F}(S_1)$:

$$\nu \in \mathfrak{M}(S_1) \longleftrightarrow t_1x_1 + t_2x_2 + t_3x_3 + t_4x_4 \in \mathfrak{F}(S_1)$$

with $\sum_{i=1}^{4} t_i = 1$. If $\nu \in \mathfrak{M}(S_1)$ is idempotent, then the corresponding element $\in \mathfrak{F}(S_1)$ satisfies the relation $(\sum_{i=1}^{4} t_ix_i)(\sum_{i=1}^{4} t_ix_i) = \sum_{i=1}^{4} t_ix_i$. Elementary calculations show that the idempotents $\in \mathfrak{F}(S_1)$ are exactly all elements of the form

$t_1(x_1 + x_2) + t_3(x_3 + x_4)$ and of the form $t_1x_1 + t_3x_3$, where $t_1 \geqq 0$, $t_3 \geqq 0$, $t_1 + t_3 = \frac{1}{2}$.

The measure $\mu \longleftrightarrow t_1(x_1 + x_2) + t_3(x_3 + x_4)$ is right invariant for every couple $t_1 \geqq 0$, $t_3 \geqq 0$ with $t_1 + t_3 = \frac{1}{2}$. The measure $\nu \longleftrightarrow t_1x_1 + t_3x_3$ is not invariant whatever are the numbers t_1, t_3.

Put $\nu_{14} \longleftrightarrow \frac{1}{2}(x_1 + x_4)$, $\nu_{34} \longleftrightarrow \frac{1}{2}(x_3 + x_4)$, $\nu_{13} \longleftrightarrow \frac{1}{2}(x_1 + x_3)$, $\mu \longleftrightarrow \frac{1}{2}(x_1 + x_2 + x_3 + x_4)$. Then in $\mathfrak{M}(S_1)$ following relations hold: $\nu_{14} * \nu_{34} = \nu_{14} * \nu_{14} = \nu_{13} * \nu_{34} = \mu$. We shall need them later.

2

Theorem 2,1. *Let μ be a right invariant measure on S and ν an arbitrary element $\in \mathfrak{M}(S)$. Then $\mu * \nu = \mu$.*

Proof. Since μ is right invariant, we have for every couple $u, v \in S$ $\mu(uv) = \mu(u)$. Therefore

$$\mu * \nu(x_i) = \sum_{uv = x_i} \mu(u)\, \nu(v) = \sum_{uv = x_i} \mu(uv)\, \nu(v) = \mu(x_i) \sum_{uv = x_i} \nu(v)\ .$$

If $x_i \in S - C(\mu)$, we have $\mu(x_i) = 0$; hence $\mu * \nu(x_i) = 0$, i. e. $\mu * \nu(x_i) = \mu(x_i)$. For $x_i \in C(\mu)$ let us calculate $\sum_{uv = x_i} \nu(v)$. According to Lemma 1,3 we have $C(\mu)\,.\,v = C(\mu)$ for every $v \in S$. This means: to every $v \in S$ there exists a unique $u \in C(\mu)$ such that $uv = x_i$. Therefore $\sum_{uv = x_i} \nu(v) = \sum_{v \in S} \nu(v) = \nu(S) = 1$. Hence we have also $\mu * \nu(x_i) = \mu(x_i)$ for every $x_i \in C(\mu)$. This proves Theorem 2,1.

Remark. Let μ be right invariant on S. A relation $\nu_1 * \nu_2 = \mu$ is of course possible also with $\nu_1 \neq \mu$. E. g. in Example 1,1 we have $\nu_{14} * \nu_{34} = \mu$.

Corollary 2,1. *Every right invariant measure on the semigroup S is an idempotent $\in \mathfrak{M}(S)$.*

Theorem 2,2. *Every right invariant measure μ on the semigroup S is a minimal right ideal of the semigroup $\mathfrak{M}(S)$.*

Proof. This follows from the relation $\mu * \mathfrak{M}(S) = \mu$, which itself is a consequence of Theorem 2,1.

Corollary 2,2a. *Let μ be a right invariant measure on S and $\nu \in \mathfrak{M}(S)$. Then $\nu * \mu$ is an idempotent $\in \mathfrak{M}(S)$.*

Proof. With respect to Lemma 2,1 and Corollary 2,1 we have $(\nu * \mu)^2 = \nu * (\mu * \nu) * \mu = \nu * \mu^2 = \nu * \mu$.

Corollary 2,2b. *Let μ be a right invariant measure on S. Then the left ideal $\mathfrak{M}(S) * \mu$ of $\mathfrak{M}(S)$ contains only idempotent elements.*

3

Definition. *Let T be a semigroup and e an idempotent $\in T$. We shall say that e is a primitive idempotent $\in T$ if there does not exist an idempotent $f \neq e$, $f \in T$ such that $ef = fe = f$.*

Example. The idempotents $\nu \in \mathfrak{M}(S_1) \longleftrightarrow t_1(x_1 + x_2) + t_3(x_3 + x_4) \in \mathfrak{F}(S_1)$ ($t_1 \geqq 0$, $t_3 \geqq 0$, $t_1 + t_3 = \frac{1}{2}$) are primitive idempotents $\in \mathfrak{M}(S_1)$. The remaining idempotents $\in \mathfrak{M}(S_1)$ are not primitive idempotents of $\mathfrak{M}(S_1)$.

Theorem 3,1. *Every right invariant measure μ on the semigroup S is a primitive idempotent of the semigroup $\mathfrak{M}(S)$.*

Proof. If μ were not a primitive idempotent $\in \mathfrak{M}(S)$ there would exist an idempotent $\nu \neq \mu$ such that $\mu * \nu = \nu * \mu = \nu$. But Theorem 2,1 implies $\mu * \nu = \mu$. Hence $\mu = \nu$, which is a contradiction.

The following examples[1]) show that the converse of Theorem 3,1 is not true.

Example 3,1. A primitive idempotent $\in \mathfrak{M}(S)$ need not be a right invariant measure on S. Consider the semigroup $S_2 = \{x_1, x_2, x_3\}$ with the multiplication table

	x_1	x_2	x_3
x_1	x_1	x_2	x_3
x_2	x_2	x_2	x_3
x_3	x_3	x_3	x_3 .

It is easy to show that $\mathfrak{M}(S_2)$ contains exactly three idempotents. These are the measures ε_1, ε_2, ε_3 defined by the relation: $\varepsilon_i(x_k) = \delta_{ik}$ ($\delta_{ik} = 1$ for $i = k$, $\delta_{ik} = 0$ for $i \neq k$). The element ε_3 is the unique primitive idempotent $\in \mathfrak{M}(S_2)$. But our semigroup is commutative and not a group. Therefore (see [5], Corollary 3,1) S_2 has no invariant measure at all.

Example 3,2. There exist also semigroups having right invariant measures for which not all primitive idempotents $\in \mathfrak{M}(S)$ are right invariant measures on S. Consider the semigroup $S_3 = \{x_1, x_2, x_3, x_4\}$ with the multiplication table (see [5], Example 3,2)

	x_1	x_2	x_3	x_4
x_1	x_1	x_1	x_1	x_1
x_2	x_1	x_1	x_1	x_1
x_3	x_3	x_3	x_3	x_3
x_4	x_4	x_4	x_4	x_4 .

The measures ν defined by the correspondence

$$\nu \in \mathfrak{M}(S_3) \longleftrightarrow t_1x_1 + t_3x_3 + t_4x_4 \in \mathfrak{F}(S_3)$$

[1]) The construction of various examples is now facilitated by means of complete lists of all semigroups of order 2, 3, 4 published in [4] and [2].

with $t_1 + t_3 + t_4 = 1$ ($t_i \geqq 0$) are exactly all idempotents $\in \mathfrak{M}(S_3)$. Each of them is a primitive idempotent $\in \mathfrak{M}(S_3)$. But right invariant measures on S_3 are only the measures defined by the correspondence

$$\mu \in \mathfrak{M}(S_3) \longleftrightarrow t_3 x_3 + t_4 x_4 \in \mathfrak{F}(S_3) ,$$

where $t_3 + t_4 = 1$ ($t_i \geqq 0$). [This last statement follows immediately from the fact that the decomposition $S = R + (S - R)$, $R = \{a_3, a_4\}$ is the maximal right μ-decomposition of S in the sense of Theorem 3,3 of [5].]

Theorem 3,2. *Suppose that S has at least one right invariant measure. Let π be a primitive idempotent $\in \mathfrak{M}(S)$ and $\nu \in \mathfrak{M}(S)$. Then $\pi * \nu = \pi$.*

Proof. a) Let μ be a right invariant measure on S. We show first that $\pi * \mu = \pi$.

Denote $\tau = \pi * \mu$. According to Corollary 2,2 a τ is an idempotent $\in \mathfrak{M}(S)$. We have further

$$\tau * \pi = \pi * \mu * \pi = \pi * \mu = \tau ,$$
$$\pi * \tau = \pi * \pi * \mu = \pi * \mu = \tau ,$$

hence $\tau * \pi = \pi * \tau = \tau$. With respect to the primitivity of π we have therefore $\tau = \pi$, i. e. $\pi * \mu = \pi$.

b) Let now be ν arbitrary, $\nu \in \mathfrak{M}(S)$. Then

$$\pi * \nu = (\pi * \mu) * \nu = \pi * (\mu * \nu) = \pi * \mu = \pi ,$$

which proves Theorem 3,2.

Remark. The supposition that S has at least one right invariant measure is essential in Theorem 3,2 (and in this following theorems). We show this on

Example 3,3. Let $S_4 = \{x_1, x_2, x_3\}$ be the semigroup with the multiplication table

	x_1	x_2	x_3
x_1	x_1	x_2	x_1
x_2	x_1	x_2	x_2
x_3	x_1	x_2	x_3 .

This semigroup has no right (or left) invariant measure. Elementary calculations show that all idempotents $\in \mathfrak{M}(S_4)$ are a) $\nu \in \mathfrak{M}(S_4) \longleftrightarrow t_1 x_1 + t_2 x_2 \in \in \mathfrak{F}(S_4)$, $t_1 \geqq 0$, $t_2 \geqq 0$, $t_1 + t_2 = 1$, b) $\varepsilon_3 \in \mathfrak{M}(S_4) \longleftrightarrow x_3 \in \mathfrak{F}(S_4)$. Each of the measures ν is a primitive idempotent $\in \mathfrak{M}(S_4)$. Choose $\pi \longleftrightarrow \frac{1}{2}(x_1 + x_2)$, $\nu \longleftrightarrow x_1$. Then $\pi * \nu \longleftrightarrow \frac{1}{2}(x_1 + x_2)\, x_1 = x_1 \in \mathfrak{F}(S_4)$. Hence $\pi * \nu = \nu \neq \pi$.

Theorem 3,3. *Suppose that S has at least one right invariant measure. Then every primitive idempotent $\pi \in \mathfrak{M}(S)$ is a minimal right ideal $\subset \mathfrak{M}(S)$.*

Proof. The statement follows from the relation $\pi * \mathfrak{M}(S) = \pi$, which is itself a consequence of Theorem 3,2.

Theorem 3,4. *Suppose that S has at least one right invariant measure. Then $\mathfrak{M}(S)$ contains one and only one minimal left ideal $\mathfrak{L}$. $\mathfrak{L}$ is exactly the set of all primitive idempotents $\in \mathfrak{M}(S)$. $\mathfrak{L}$ is at the same time the minimal two-sided ideal of $\mathfrak{M}(S)$.*

Proof. a) Let $\mathfrak{l}$ be an arbitrary left ideal of $\mathfrak{M}(S)$, $\nu \in \mathfrak{l}$ and π a primitive idempotent $\in \mathfrak{M}(S)$. Then we have

$$\pi = \pi * \nu \in \pi * \mathfrak{l} \subseteqq \mathfrak{l},$$

i. e. π is contained in $\mathfrak{l}$. Thus the intersection of all left ideals $\in \mathfrak{M}(S)$ is non-vacuous and it contains the set of all primitive idempotents $\in \mathfrak{M}(S)$. There exists therefore a unique minimal left ideal $\mathfrak{L}$ containing all primitive idempotents $\in \mathfrak{M}(S)$.

b) It is known (see e. g. [1], Theorem 2,1) that a semigroup containing at least one minimal left ideal has a kernel, i. e. a minimal two-sided ideal $\mathfrak{N}$ and the kernel is the class sum of all minimal left ideals. Therefore in our case $\mathfrak{L} = \mathfrak{N}$, i. e. $\mathfrak{L}$ is the minimal two-sided ideal of $\mathfrak{M}(S)$.

The semigroup $\mathfrak{L}$ is a left simple semigroup having at least one idempotent. Hence $\mathfrak{L}$ is the sum of disjoint isomorphic groups. The group-components of $\mathfrak{L}$ are the sets $\pi_\alpha * \mathfrak{L}$, where π_α runs through all idempotents $\in \mathfrak{L}$ (See e. g. [6], Theorem 3,3). According to Theorem 3,3 for a primitive idempotent π_α the relation $\pi_\alpha * \mathfrak{L} = \pi_\alpha$ holds. Therefore all group-components of $\mathfrak{L}$ are one point sets and every element $\in \mathfrak{L}$ is an idempotent.

To prove that every π_α is a primitive idempotent it is sufficient to show that $\pi_\alpha * \nu = \nu * \pi_\alpha = \nu$, where ν is an idempotent $\in \mathfrak{M}(S)$, implies $\pi_\alpha = \nu$. The relation $\nu = \pi_\alpha * \nu \in \mathfrak{L} * \nu \subseteqq \mathfrak{L}$ implies first that $\nu \in \mathfrak{L}$. But since $\mathfrak{L}$ is a left simple semigroup we have in $\mathfrak{L}$ $\pi_\alpha * \nu = \pi_\alpha$, hence $\nu = \pi_\alpha$. Theorem 3,4 is completely proved.

Corollary 3,4. *Suppose that S has at least one right invariant measure. Let π be a primitive idempotent $\in \mathfrak{M}(S)$ and $\nu \in \mathfrak{M}(S)$. Then $\nu * \pi$ is a primitive idempotent $\in \mathfrak{M}(S)$.*

Proof. We have $\mathfrak{M}(S) * \pi \subseteqq \mathfrak{M}(S) * \mathfrak{L} = \mathfrak{L}$. Since $\mathfrak{M}(S) * \pi$ is a left ideal of $\mathfrak{M}(S)$ and $\mathfrak{L}$ is minimal we have $\mathfrak{M}(S) * \pi = \mathfrak{L}$. Therefore every element of the form $\nu * \pi$ is a primitive idempotent.

Remark 1. If a semigroup contains also a minimal right ideal, then the kernel is also the sum of all minimal right ideals. Every element $\nu \in \mathfrak{M}(S)$ satisfying $\nu * \mathfrak{M}(S) = \nu$ is clearly a minimal right ideal of $\mathfrak{M}(S)$. Therefore the primitive idempotents $\in \mathfrak{M}(S)$ in Theorem 3,4 are exactly all elements $\in \mathfrak{M}(S)$ satisfying $\nu * \mathfrak{M}(S) = \nu$.

Remark 2. In analogy to Theorem 4.2 of [5] it would be natural to expect that those and only those of the idempotents $\in \mathfrak{M}(S)$ which are contained in $\mathfrak{L} - [\mathfrak{M}(S) - \mathfrak{L}] * \mathfrak{M}(S)$ are primitive. This is not true. We show it on the semigroup S_1 [Example 1,1]. Here ν_{13} is a non-primitive idempotent, therefore $\nu_{13} \in \mathfrak{M}(S_1) - \mathfrak{L}$. Choose further $\nu_{34} \in \mathfrak{M}(S_1)$. Then $\nu_{13} * \nu_{34} = \mu$. But this is a right invariant measure on S_1, hence all right invariant measures need not be contained in the set $\mathfrak{L} - [\mathfrak{M}(S_1) - \mathfrak{L}] * \mathfrak{M}(S_1)$.

Remark 3. Theorems 3,2, 3,4 and Remark 1 show that (under the above suppositions) the set $\mathfrak{L}$ is exactly the set of left zeros of the semigroup $\mathfrak{M}(S)$. (A left zero of a semigroup T is an element $n \in T$ satisfying the relation $nx = n$ for all $x \in T$.) Theorem 3,4 is therefore a special case of the following general statement: Let P be the set of all left zeros of the semigroup T. If $P \neq 0$, then P is the minimal two-sided ideal of T. At the same time P is a left simple semigroup containing only idempotent elements.

Theorem 3,5. *Suppose that S has at least one right invariant measure. Let N be the minimal two-sided ideal of S and π an arbitrary primitive idempotent $\in \mathfrak{M}(S)$. Then*

i) *for all $x \in S$ we have $C(\pi) \,.\, x = C(\pi)$;*

ii) *$C(\pi)$ is a left simple semigroup, which is a sum of maximal subgroups of the semigroup S;*

iii) *$C(\pi) \subseteq N$.*

Proof. i) For every $\nu \in \mathfrak{M}(S)$ we have $\pi * \nu = \pi$, hence $C(\pi) \,.\, C(\nu) = C(\pi)$. Let x be an arbitrary element, $x \in S$. Choose $\nu = \varepsilon_x$ (the point mass at x). Then $C(\nu) = \{x\}$. Therefore $C(\pi) \,.\, x = C(\pi)$.

ii) The assertion i) implies especially that $C(\pi)$ is a right ideal of S and at the same time a left simple semigroup. Hence $C(\pi)$ is a sum of disjoint isomorphic groups. But since $C(\pi)$ is a right ideal of S, $C(\pi)$ contains with every element of a subgroup of S also all elements of the maximal subgroup $\in S$ containing this element.

iii) In the relation $C(\pi)\, x = C(\pi)$ choose especially $x \in N$. Then we have $C(\pi) = C(\pi)\, x \subseteq C(\pi)\, N \subseteq N$. Theorem 3,5 is completely proved.

4

The question arises what is the distinction between the primitive idempotents that are right invariant measures and the remaining primitive idempotents.

In general, i. e. for an arbitrary primitive idempotent π, the set $S - C(\pi)$ is not a right ideal of S. But, if π is at the same time a right invariant measure, we proved in [5] (Theorem 1,1 and Corollary 1,4) that $S - C(\pi)$ is a right ideal of S. In this section we shall show — among other results — that this condition is also sufficient.

Lemma 4,1. *Let μ be an idempotent $\in \mathfrak{M}(S)$. Let $C(\mu) = \sum_{\alpha=1}^{m} \mathfrak{g}_\alpha$ be a left simple semigroup. Then for every α $(\alpha = 1, 2, \ldots, m)$ and all $x, y \in \mathfrak{g}_\alpha$ $\mu(x) = \mu(y)$.*

Proof. Let x_α be such an element $\epsilon\ \mathfrak{g}_\alpha$ in which μ assumes the greatest value. Then (since μ is idempotent) we have

$$\mu(x_\alpha) = \mu * \mu(x_\alpha) = \sum_{uv = x_\alpha} \mu(u)\, \mu(v)\, . \tag{*}$$

Since $\mu[S - C(\mu)] = 0$, it is sufficient to consider in (*) only summands corresponding to those solutions of $\mu v = x_\alpha$ for which $u \in C(\mu)$ and $v \in C(\mu)$. The equation $uv = x_\alpha$ has for a fixed $v \in C(\mu)$ a unique solution $u_v \in C(\mu)$, which is necessarily contained in $\mathfrak{g}_\alpha$ (see [5], section 2). Further, if v runs through all elements $\epsilon\ C(\mu)$, then u_v runs (eventually more times) through all elements $\epsilon\ \mathfrak{g}_\alpha$. Since $\sum_{v \in C(\mu)} \mu(v) = 1$, we have

$$\sum_{v \in C(\mu)} \mu(u_v)\, \mu(v) = \mu(x_\alpha) \sum_{v \in C(\mu)} \mu(v)\, , \quad \sum_{v \in C(\mu)} \mu(v)\, [\mu(x_\alpha) - \mu(u_v)] = 0\, . \tag{3}$$

In (3) every member $\mu(v)$ is > 0 and every member $\mu(x_\alpha) - \mu(u_v)$ is $\geqq 0$. Therefore we have for every u_v $\mu(x_\alpha) - \mu(u_v) = 0$. Since u_v assumes all values $\epsilon\ \mathfrak{g}_\alpha$, we have $\mu(x_\alpha) = \mu(y)$ for all $y \in \mathfrak{g}_\alpha$. This proves our Lemma.

We prove conversely:

Lemma 4,2. *Let S be an arbitrary semigroup and C an arbitrary left simple subsemigroup of S, $C \subseteqq S$. Let ν be an element $\epsilon\ \mathfrak{M}(S)$ satisfying the following conditions: a) $C(\nu) \subseteqq C$, b) in all elements of a group-component of C ν assumes the same values. Then ν is an idempotent $\epsilon\ \mathfrak{M}(S)$.*

Proof. Write $C = \sum_{\alpha=1}^{m} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{x_1^{(\alpha)}, \dots, x_s^{(\alpha)}\}$. (The groups $\mathfrak{g}_\alpha$ are subgroups of S but not necessarily maximal subgroups of S.) The measure ν with the required properties has then the form

$$\nu \in \mathfrak{M}(S) \longleftrightarrow \sum_{\alpha=1}^{m} t_\alpha(x_1^{(\alpha)} + \dots + x_s^{(\alpha)}) \in \mathfrak{F}(S)\, , \quad \sum_{\alpha=1}^{m} s\, .\, t_\alpha = 1\, .$$

Since every $\mathfrak{g}_\alpha$ $(\alpha = 1, \dots, m)$ is a minimal right ideal of C, we have $\mathfrak{g}_\alpha \cdot y = \mathfrak{g}_\alpha$ for every $y \in C$. We have therefore in $\mathfrak{F}(S)$

$$[\sum_{\alpha=1}^{m} t_\alpha(x_1^{(\alpha)} + \dots + x_s^{(\alpha)})]\, .\, y = \sum_{\alpha=1}^{m} t_\alpha(x_1^{(\alpha)} + \dots + x_s^{(\alpha)})\, . \tag{4}$$

This implies at once that ν is an idempotent $\epsilon\ \mathfrak{M}(S)$ as the following calculations show:

$$[\sum_{\alpha=1}^{m} t_\alpha(x_1^{(\alpha)} + \dots + x_s^{(\alpha)})]\, .\, [\sum_{\beta=1}^{m} t_\beta(x_1^{(\beta)} + \dots + x_s^{(\beta)}] = [\sum_{\alpha=1}^{m} t_\alpha(x_1^{(\alpha)} + \dots + x_s^{(\alpha)})]\, .$$
$$.\sum_{\beta=1}^{m} \sum_{i=1}^{s} t_\beta x_i^{(\beta)} = \sum_{\alpha=1}^{m} t_\alpha(x_1^{(\alpha)} + \dots + x_s^{(\alpha)})\, .\, \sum_{\beta=1}^{m} s t_\beta = \sum_{\alpha=1}^{m} t_\alpha(x_1^{(\alpha)} + \dots + x_s^{(\alpha)})\, .$$

Theorem 4,1. *Let μ be an idempotent $\epsilon\ \mathfrak{M}(S)$. Suppose that $C(\mu)$ is a left simple semigroup. Then μ is a right invariant measure on $C(\mu)$.*

Proof. Write again $C(\mu) = \sum_{\alpha=1}^{m} \mathfrak{g}_\alpha$. Since the suppositions of Lemma 4,1 are satisfied, μ assumes the same values in all elements of a group-component $\mathfrak{g}_\alpha$.

Let E be a subset of $C(\mu)$ and x an element $\epsilon\, C(\mu)$. We have to show that $\mu(Ex) = \mu(E)$.

If e_α is the unit element of the group $\mathfrak{g}_\alpha$ we have clearly $E \cap \mathfrak{g}_\alpha = (E \cap \mathfrak{g}_\alpha)\, e_\alpha$. Hence we have also $(E \cap \mathfrak{g}_\alpha)\, x = (E \cap \mathfrak{g}_\alpha)\, e_\alpha x$. Since $e_\alpha x \in \mathfrak{g}_\alpha$, the sets $E \cap \mathfrak{g}_\alpha$ and $(E \cap \mathfrak{g}_\alpha)\, x$ have the same number of different elements and we have $\mu[(E \cap \mathfrak{g}_\alpha)] = \mu[(E \cap \mathfrak{g}_\alpha)\, x]$. Therefore

$$\mu(Ex) = \mu\{[\sum_\alpha (E \cap \mathfrak{g}_\alpha)]\, x\} = \mu\{\sum_\alpha (E \cap \mathfrak{g}_\alpha)\, x\} = \sum_\alpha \mu(E \cap \mathfrak{g}_\alpha) =$$
$$= \mu[\sum_\alpha (E \cap \mathfrak{g}_\alpha)] = \mu(E)\,.$$

This proves Theorem 4,1.

Remark 1. If μ is idempotent, $C(\mu)$ need not be a left simple semigroup. E. g. in Example 3,3 the measure $\nu \in \mathfrak{M}(S_4) \longleftrightarrow \frac{1}{2}(x_1 + x_2) \in \mathfrak{F}(S_4)$ is an idempotent, but $C(\nu) = \{x_1, x_2\}$ is not a left simple semigroup.

Remark 2. The measure μ from Theorem 4,1 need not be a right invariant measure on the whole semigroup S. Consider for instance the semigroup S_1 (see Example 1,1) and the measure $\mu \in \mathfrak{M}(S_1) \longleftrightarrow \frac{1}{2}(x_1 + x_3) \in \mathfrak{F}(S_1)$. This is a right invariant measure on $C(\mu) = \{x_1, x_3\}$. But μ is not right invariant as a measure considered on the whole semigroup S. (We have e. g. $\mu(x_1 x_3) = \mu(x_2) = 0 \neq \mu(x_1)$.)

Theorem 4,1 and Theorem 3,5 imply

Theorem 4,2. *Suppose that S has at least one right invariant measure. Let π be a primitive idempotent $\in \mathfrak{M}(S)$. Then π is a right invariant measure on $C(\pi)$.*

Theorem 4,3. *Suppose that S has at least one right invariant measure. Let π be a primitive idempotent $\in \mathfrak{M}(S)$. Then π assumes the same values in all points of a maximal subgroup of S.*

Proof. According to Theorem 3,5 $C(\pi)$ is a left simple semigroup which a class sum of maximal subgroups of S. Write $S = \sum_{\alpha \in A} G_\alpha$, where G_α are some maximal subgroups of S. Lemma 4,1 implies that $\pi(x) = \pi(y)$ for all $x, y \in G_\alpha$. Since (according to Theorem 3,5) $C(\pi)$ is a right ideal of S, every maximal group of S is either contained in $C(\pi)$ or has an empty intersection with $C(\pi)$. Our assertion is true also for maximal groups contained in $S - C(\pi)$, since for all elements $x \in S - C(\pi)$ we have $\pi(x) = 0$.

Theorem 4.4. *Suppose that S has at least one right invariant measure. Let $N = \sum_{\alpha=1}^{k} G_\alpha$ be the minimal two-sided ideal of S. Let ν be a measure on S having the following properties: a) $C(\nu) \subseteq N$, b) ν assumes the same value in all points of the* *group* G_α $(\alpha = 1, \ldots, k)$. *Then* ν *is a primitive idempotent* $\in \mathfrak{M}(S)$. *Moreover, all primitive idempotents* $\in \mathfrak{M}(S)$ *are obtained in this manner.*

Proof. a) It follows from Theorems 3,5 and 4,3 that every primitive idempotent $\in \mathfrak{M}(S)$ has these properties.

b) Let be $G_\alpha = \{x_1^{(\alpha)}, \ldots, x_r^{(\alpha)}\}$. Construct an arbitrary element $\pi \in \mathfrak{M}(S)$ of the following form

$$\pi \in \mathfrak{M}(S) \longleftrightarrow \sum_{\alpha=1}^{k} t_\alpha(x_1^{(\alpha)} + \ldots + x_r^{(\alpha)}) \in \mathfrak{F}(S)\,, \quad \sum_{\alpha=1}^{k} rt_\alpha = 1\,.$$

It follows from Lemma 4,2 that π is an idempotent $\in \mathfrak{M}(S)$.

To prove that π is a primitive idempotent $\in \mathfrak{M}(S)$ it is sufficient to prove that there does not exist an idempotent $\nu \neq \pi$ such that $\pi * \nu = \nu * \pi = \nu$. The equation $\nu * \pi = \nu$ implies that $C(\nu) = C(\nu)\, C(\pi) \subseteqq C(\nu)\, N \subseteqq N$, i. e. $C(\nu) \subseteqq N$. Therefore we can write

$$\nu \in \mathfrak{M}(S) \longleftrightarrow \sum_{\alpha=1}^{k} \sum_{i=1}^{r} t_{\alpha i} x_i^{(\alpha)} \in \mathfrak{F}(S)\,, \quad \text{where} \quad \sum_{\alpha=1}^{k} \sum_{i=1}^{r} t_{\alpha i} = 1\,. \tag{5}$$

The relation $\pi * \nu = \nu$ written in the corresponding elements $\in \mathfrak{F}(S)$ requires

$$\sum_{\alpha=1}^{k} t_\alpha(x_1^{(\alpha)} + \ldots + x_r^{(\alpha)}) \cdot \sum_{\gamma=1}^{k} \sum_{i=1}^{r} t_{\gamma i} x_i^{(\gamma)} = \sum_{\gamma=1}^{k} \sum_{i=1}^{r} t_{\gamma i} x_i^{(\gamma)}\,.$$

With respect to the relations (4) and (5) the left hand side is again $\sum_{\alpha=1}^{k} t_\alpha(x_1^{(\alpha)} + \ldots$ $\ldots + x_r^{(\alpha)})$. Hence $\sum_{\alpha=1}^{k} t_\alpha(x_1^{(\alpha)} + \ldots + x_r^{(\alpha)}) = \sum_{\gamma=1}^{k} \sum_{i=1}^{r} t_{\gamma i} x_i^{(\gamma)}$, i. e. $\pi = \nu$, which completes the proof.

We take now into consideration the set $R = N - (S - N)\, S$, where N is the minimal two-sided ideal of S. In [5] we proved: if S has at least one right invariant measure, then $R \neq \emptyset$ and the decomposition $S = R + (S - R)$ is the maximal right μ-decomposition of the semigroup S in the sense of Theorem 3,3 of [5].

Theorem 4,5. *Suppose that S has a right invariant measure. A primitive idempotent $\pi \in \mathfrak{M}(S)$ is a right invariant measure on S if and only if $C(\pi) \subseteqq R$.*

Proof. a) The condition is necessary since for every right invariant measure π $C(\pi) \subseteqq R$. (See [5], Theorem 3,3.)

b) Suppose conversely that π is a primitive idempotent $\in \mathfrak{M}(S)$ with $C(\pi) \subseteqq R$. According to Theorem 3,5 $C(\pi)$ is a sum of maximal groups of S: $C(\pi) = \sum_{\alpha \in A} G_\alpha$. Each of these groups is contained in R and is one of the group-components of the left simple semigroup $C(\pi)$. According to Theorem 4,3 π has the same value in all points of the group G_α. According to Theorem 5,2 of [5] a measure having these properties is a right invariant measure on the semigroup S.

This result can be reformulated in the following manner:

Theorem 4,6. *Suppose that S has a right invariant measure. A primitive idempotent $\pi \in \mathfrak{M}(S)$ is a right invariant measure on the semigroup S if and only if $S - C(\pi)$ is a right ideal of S.*

Proof. a) In the introduction to section 4 we have mentioned yet that this condition is necessary.

b) We show that it is also sufficient. Suppose that $S - C(\pi)$ is a right ideal of S. With respect to Theorem 3,5 the decomposition $S = C(\pi) + [S - C(\pi)]$ is then a right μ-decomposition of S. Hence according to Theorem 3,3 of [5] we have necessarily $C(\pi) \subseteqq R$. Therefore according to Theorem 4,5 π is a right invariant measure on S.

5

In this section we show the role of left simple semigroups among all semigroups having at least one right invariant measure.

Lemma 5,1. *Let S be a left simple semigroup and μ an idempotent $\in \mathfrak{M}(S)$. Then μ is right invariant on $C(\mu)$.*

Proof. According to Lemma 1,2 $C(\mu)$ is a semigroup. Since every subsemigroup of a left simple semigroup is itself left simple, $C(\mu)$ is a left simple semigroup. The proof follows now from Theorem 4,1.

Theorem 5,1. *Let S be a left simple semigroup. An element $\mu \in \mathfrak{M}(S)$ is a right invariant measure on S if and only if μ is a primitive idempotent $\in \mathfrak{M}(S)$.*

Proof. a) The necessity of this condition follows from Theorem 3,1.

b) Let conversely π be a primitive idempotent $\in \mathfrak{M}(S)$. For a left simple semigroup we have always $R = S$. Hence $C(\pi) \subseteqq S = R$. Therefore according to Theorem 4,5 π is a right invariant measure on S.

Theorem 5,2. *Let S be a finite semigroup having at least one right invariant measure. Then the set of all primitive idempotents $\in \mathfrak{M}(S)$ is identical with the set of all right invariant measures on S if and only if S is a left simple semigroup.*

Proof. According to Theorem 4,4 we obtain all primitive idempotents $\in \mathfrak{M}(S)$ in the following manner. Construct the decomposition $N = \sum_{\alpha=1}^{k} G_\alpha$. Construct the measure ν_α defined as follows: $\nu_\alpha(G_\alpha) = 1$, $\nu_\alpha(S - G_\alpha) = 0$, where ν_α assumes in all points $\in G_\alpha$ the same value. Then all primitive idempotents $\in \mathfrak{M}(S)$ are of the form $t_1\nu_1 + \ldots + t_k\nu_k$, where $t_i \geqq 0, t_1 + \ldots + t_k = 1$.

According to Theorem 5,2 of [5] we get all right invariant measures in the following manner. Construct the decomposition $R = N - (S - N) S = = \sum_{i=1}^{m} G_{\alpha_i}$, where $\alpha_1, \alpha_2, \ldots, \alpha_m$ is a suitable chosen subset of the set of indices $\{1, 2, \ldots, k\}$. Then all right invariant measures on S are of the form $t'_1\nu_{\alpha_1} + \ldots \ldots + t'_m\nu_{\alpha_m}$ $(t'_i > 0,\ t'_1 + \ldots + t'_m = 1)$.

These two sets of measures are identical if and only if $N = N - (S - N) S$. Theorem 3,4 of [5] says that this is the case if and only if $S = N$, i. e. S is a left simple semigroup.

6

In this last section we prove some further results and we find a new expression for the set of all right invariant measures on S.

Lemma 6.1. *Let μ be a right invariant measure on S. Let be $\nu_1, \nu_2 \in \mathfrak{M}(S)$ and $\nu_1 * \nu_2 = \mu$. Then $C(\nu_1) \subseteqq C(\mu)$. Hereby $C(\nu_1)$ has a non-vacuous intersection with every group-component of $C(\mu)$.*

Proof. Lemma 1,1 implies $C(\nu_1) C(\nu_2) = C(\mu)$. Denote $C(\nu_1) \cap C(\mu) = A$ and suppose that $B = C(\nu_1) \cap (S - C(\mu)) \neq \emptyset$. Then

$$(A + B) C(\nu_2) = C(\mu) ,$$

$$A \,.\, C(\nu_2) + B \,.\, C(\nu_2) = C(\mu) . \tag{6}$$

But (see Lemma 1,3) $B \,.\, C(\nu_2) \subseteqq [S - C(\mu)] C(\nu_2) \subseteqq S - C(\mu)$. This is a contradiction with (6). Hence $B = \emptyset$, i. e. $C(\nu_1) \subseteqq C(\mu)$.

Let $C(\mu) = \sum_{\alpha \in \Lambda} \mathfrak{g}_\alpha$ be the decomposition of $C(\mu)$ into the group-components. Suppose that $C(\nu_1) \cap \mathfrak{g}_\gamma = \emptyset$. Since $\mathfrak{g}_\alpha$ $(\alpha \in \Lambda)$ is a right ideal of S (see [5], Lemma 3,2) we have $\mathfrak{g}_\alpha C(\nu_2) \subseteqq \mathfrak{g}_\alpha$ for every $\alpha \in \Lambda$. Hence the product $C(\nu_1) \,.\, .\, C(\nu_2)$ would not contain the group $\mathfrak{g}_\gamma$. This is a contradiction to $C(\nu_1) C(\nu_2) = = C(\mu)$.

Remark 1. In Lemma 6.1 the equality $C(\nu_1) = C(\mu)$ need not hold. This can be shown on the semigroup S_1 of Example 1,1. We have $\nu_{14} * \nu_{34} = \mu$, hence $C(\nu_{14}) C(\nu_{34}) = C(\mu)$, but $C(\nu_{14}) \neq C(\mu)$, since $C(\nu_{14}) = \{x_1, x_4\} \neq C(\mu) = = S$.

Remark 2. An analogous lemma for primitive idempotents is in general not true. This means: if $\nu_1 * \nu_2 = \pi$ is a primitive idempotent, $C(\nu_1) \subseteqq C(\pi)$ need not hold. Choose in Example 3,2 $\nu_1 \longleftrightarrow \frac{1}{2}(x_1 + x_2)$, $\nu_2 = \varepsilon_{x_1} \longleftrightarrow x_1$. Then $\nu_1 * \nu_2 = e_{x_1}$, but $C(\nu_1) = \{x_1, x_2\} \supset C(\varepsilon_{x_1}) = \{x_1\}$.

Lemma 6,2. *Suppose that S has at least one right invariant measure. Let π be a primitive idempotent $\in \mathfrak{M}(S)$. Then $\nu * \pi$ is a right invariant measure if and only if $C(\nu) \subseteqq R$.*

Proof. a) We know from Corollary 3,4 that $\nu * \pi = \mu$ is a primitive idempotent $\in \mathfrak{M}(S)$. If μ is a right invariant measure, it must hold (according to Lemma 6,1) $C(\nu) \subseteqq C(\nu * \pi) = C(\mu) \subseteqq R$. (We use Theorem 4,5 according to which for a right invariant measure $C(\mu) \subseteqq R$.)

b) Let be conversely $C(\nu) \subseteqq R$. Then for the primitive idempotent $\mu = \nu * \pi$ we have $C(\mu) = C(\nu)\, C(\pi) \subseteqq R \,.\, C(\pi) \subseteqq R$. Hence (see again Theorem 4,5) μ is a right invariant measure on S. This completes the proof.

Suppose that S has at least one right invariant measure. Denote by $\mathfrak{M}_R(S)$ the set of all $\nu \in \mathfrak{M}(S)$ with $C(\nu) \subseteqq R$. Then we have:

Lemma 6,3. *The semigroup $\mathfrak{M}(S)$ can be written as a sum of two disjoint right ideals of $\mathfrak{M}(S)$ in the form $\mathfrak{M}(S) = \mathfrak{M}_R(S) + [\mathfrak{M}(S) - \mathfrak{M}_R(S)]$.*

Proof. Let be $\nu_1 \in \mathfrak{M}_R(S)$, $\nu \in \mathfrak{M}(S)$. Then $C(\nu_1 * \nu) = C(\nu_1) \,.\, C(\nu) \subseteqq R \,.\, S \subseteqq R$, i. e. $\nu_1 * \nu \in \mathfrak{M}_R(S)$. Hence $\mathfrak{M}_R(S)$ is a right ideal of $\mathfrak{M}(S)$.

If $R = S$, the second assertion is trivial. Suppose therefore $\emptyset \neq R \neq S$. Let be $\nu_2 \in \mathfrak{M}(S) - \mathfrak{M}_R(S)$, i. e. $C(\nu_2) \cap (S - R) \neq \emptyset$ and $\nu \in \mathfrak{M}(S)$. Then

$$C(\nu_2 * \nu) = C(\nu_2)\, C(\nu) = [C(\nu_2) \cap R]\, C(\nu) + [C(\nu_2) \cap (S - R)]\, C(\nu) \,.$$

We have further $\emptyset \neq [C(\nu_2) \cap (S - R)]\, C(\nu) \subseteqq (S - R)\, C(\nu) \subseteqq S - R$, hence $C(\nu_2 * \nu)$ has a non-vacuous intersection with $S - R$. Therefore $\nu_2 * \nu \in \mathfrak{M}(S) - \mathfrak{M}_R(S)$.

Remark. The semigroup $\mathfrak{M}_R(S)$ is — in general — not left simple. This is shown by Example 1,1 in which $\mathfrak{M}_R(S) = \mathfrak{M}(S)$. Choose e. g. $\nu \in \mathfrak{M}(S_1) \longleftrightarrow \frac{1}{2}(x_1 + x_2) \in \mathfrak{F}(S_1)$. Then

$$\mathfrak{M}(S_1) * \nu \longleftrightarrow \Big(\sum_{i=1}^{4} t_i x_i\Big) \cdot \tfrac{1}{2}(x_1 + x_2) = \tfrac{1}{2}(t_1 + t_2)(x_1 + x_2) + \tfrac{1}{2}(t_3 + t_4)(x_3 + x_4)$$

with $\sum_{i=1}^{4} t_i = 1$. Hence $\mathfrak{M}(S_1) * \nu$ does not contain all elements $\in \mathfrak{M}(S_1)$.

Theorem 6,1. *Suppose that S has at least one right invariant measure. Then $\mathfrak{M}_R(S) \cap \mathfrak{L} = \mathfrak{M}_R(S) \,.\, \mathfrak{L}$ and each of these sets is exactly the set of all right invariant measures on S.*

Proof. Theorem 4,5 implies that $\mathfrak{M}_R(S) \cap \mathfrak{L}$ is exactly the set of all right invariant measures on S. Lemma 6,2 implies that all elements $\in \mathfrak{M}_R(S) \,.\, \mathfrak{L}$ and only these elements are right invariant measures on S. Therefore $\mathfrak{M}_R(S) \cap \mathfrak{L} = \mathfrak{M}_R(S) \,.\, \mathfrak{L}$, which proves our assertion.

REFERENCES

[1] *A. H. Clifford:* Semigroups containing minimal ideals, Amer. J. of Math., *70* (1948), 521—526.

[2] *G. E. Forsythe:* SWAC computes all 126 distinct semigroups of order 4, Proc. Amer. Math. Soc. *6* (1955), 443—447.

[3] *E. Hewitt-H. S. Zuckerman:* Arithmetic and limit theorems for a class of random variables, Duke Math. Journal, *22* (1955), 595—616.

[4] *E. Hewitt-H. S. Zuckerman:* Finite dimensional convolution algebras, Acta Mathematica, *93* (1955), 67—119.

[5] *Št. Schwarz:* О существовании инвариантных мер на некоторых типах бикомпактных полугрупп, Чех. мат. ж., *7* (82), 1957, 165—182.

[6] *Št. Schwarz:* On the structure of simple semigroups without zero, Czech. Math. J., *1* (*76*) (1951), 41—53.

[7] *J. G. Wendel:* Haar measure and the semigroup of measures on a compact group, Proc. Amer. Math. Soc. *5* (1954), 923—929.

Резюме

О СТРУКТУРЕ МНОЖЕСТВА МЕР КОНЕЧНОЙ ПОЛУГРУППЫ

ШТЕФАН ШВАРЦ (Štefan Schwarz), Братислава.

(Поступило в редакцию 20/VIII 1956 г.)

Пусть S — конечная полугруппа. Мерой μ называем неотрицательную аддитивную множественную функцию, определенную на подмножествах S, для которой $\mu(S) = 1$. Обозначим символом $\mathfrak{M}(S)$ множество всех мер полугруппы S. Пусть $\nu_1, \nu_2 \in \mathfrak{M}(S)$. Произведением $\nu_1 * \nu_2$ будем называть меру, определенную сверткой, т. е. меру, которая для любого $x \in S$ удовлетворяет соотношению $\nu_1 * \nu_2(x) = \sum_{uv=x} \nu_1(u) \, . \, \nu_2(v)$. По отношению к определенному таким образом умножению множество $\mathfrak{M}(S)$ образует полугруппу. (Если S содержит более одного элемента, то полугруппа $\mathfrak{M}(S)$ бесконечна.)

Меру μ назовем справа инвариантной, если для любого подмножества $E \subseteq S$ и любого $x \in S$ $\mu(Ex) = \mu(E)$. В работе [5] мы нашли необходимое и достаточное условие существования справа инвариантных мер на некоторых типах бикомпактных полугрупп, к которым относятся и все конечные полугруппы. Цель настоящей работы — изучить структуру полугруппы $\mathfrak{M}(S)$ и исследовать прежде всего, какую роль в ней играют справа инвариантные меры полугруппы S.

Идемпотент e полугруппы T называем примитивным идемпотентом, если не существует ни одного идемпотента $f \in T$, $f \neq e$, удовлетворяющего соотношению $e \, . \, f = f \, . \, e = f$. Известно, что каждая конечная (и даже каждая хаусдорфова бикомпактная) полугруппа обладает хоть одним примитивным идемпотентом.

Пусть S имеет хоть одну справа инвариантную меру. Тогда полугруппа $\mathfrak{M}(S)$ обладает одним единственным минимальным левым идеалом $\mathfrak{L}$. Мно-

жество $\mathfrak{L}$ тождественно с множеством всех примитивных идемпотентов $\in \mathfrak{M}(S)$ и является в то же время минимальным двусторонним идеалом полугруппы $\mathfrak{M}(S)$.

Каждая справа инвариантная мера полугруппы S является примитивным идемпотентом полугруппы $\mathfrak{M}(S)$, но не наоборот.

Пусть $C(\nu) = \{x_i | x_i \in S,\ \nu(x_i) \neq 0\}$. Пусть, далее, N — минимальный двусторонний идеал полугруппы S и пусть $R = N - (S - N)\,S$. Тогда имеет место утверждение: если S обладает хоть одной справа инвариантной мерой, то примитивный идемпотент $\pi \in \mathfrak{M}(S)$ будет справа инвариантной мерой полугруппы S тогда и только тогда, если $C(\pi) \subseteqq R$. В частности, множество примитивных идемпотентов полугруппы $\mathfrak{M}(S)$ тождественно множеству всех справа инвариантных мер полугруппы S тогда и только тогда, если S — слева простая полугруппа.

В предположении, что S обладает хоть одной справа инвариантной мерой, в работе описано построение всех примитивных идемпотентов полугруппы $\mathfrak{M}(S)$ и выведен ряд дальнейших свойств полугруппы $\mathfrak{M}(S)$.