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TRANSFORMATION OF m-DIMENSIONAL LEBESGUE INTEGRALS

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A substitution theorem is proved for arbitrary mappings with continuous derivatives of the first order.

Lemma 1. *Let K be an m-dimensional cube (i. e. a cartesian product of m closed intervals of equal finite and positive length). Let F be a function, which is defined on the family of all cubes $I \subset K$ and let F have the following property: If $I_1, \ldots, I_n$, I are cubes and $\bigcup_{i=1}^{n} I_i = I \subset K$, then $\sum_{i=1}^{n} F(I_i) \geqq F(I)$. Let ε be a positive number. Suppose that for every point $b \in K$ there exists a neighbourhood U of b such that for every cube I, where $b \in I \subset K \cap U$, we have $F(I) \leqq \varepsilon\mu(I)$.*[1] *Then $F(K) \leqq \varepsilon\mu(K)$.*

Proof. Let $F(K)$ be greater than $\varepsilon\mu(K)$. We divide K in an obvious way into $2^m = r$ smaller cubes $I_1, \ldots, I_r$. The relations $\varepsilon\mu(I_i) \geqq F(I_i)$ $(i = 1, \ldots, r)$ would imply $\varepsilon\mu(K) = \sum_{i=1}^{r} \varepsilon\mu(I_i) \geqq \sum_{i=1}^{r} F(I_i) \geqq F(K)$; it follows that $\varepsilon\mu(I_i) < F(I_i)$ for some i. We put $I_i = K_1$. In a similar way we find a cube $K_2 \subset K_1$ such that $\varepsilon\mu(K_2) < F(K_2)$ and so on. Let $b \in \bigcap_{n=1}^{\infty} K_n$. By assumption, $F(K_n) \leqq \varepsilon\mu(K_n)$ for some n; we arrive at a contradiction.

Definition. *We say that a mapping φ of an open set $G \subset E_m$*[2]*) into E_m is of the class C_1, if $\varphi(x) = [\varphi_1(x), \ldots, \varphi_m(x)]$, where the functions $\varphi_1, \ldots, \varphi_m$ have continuous derivatives of the first order in G. We denote by $D\varphi(x)$ the functional determinant of φ in the point $x \in G$.*

Lemma 2. *Let φ be a mapping of the class C_1 of the open set $G \subset E_m$ into E_m. Suppose that $b \in G$ and $D\varphi(b) = 0$. Let $\varepsilon > 0$. Then there exists a neighbourhood U of b such that $\mu(\varphi(K)) \leqq \varepsilon\mu(K)$*[1] *for every cube K, where $b \in K \subset U$.*

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[1]) μ is the m-dimensional Lebesgue measure (volume).

[2]) E_m is the m-dimensional euclidean space.

Proof. Suppose, for instance, that the m-th row of the matrix $\left(\frac{\partial \varphi_i(b)}{\partial x_k}\right)$ can be expressed as a linear combination of the other rows, i. e.

$$\frac{\partial \varphi_m(b)}{\partial x_k} = \sum_{i=1}^{m-1} \alpha_i \frac{\partial \varphi_i(b)}{\partial x_k} \quad (k = 1, \ldots, m) .$$

If $y \in E_m$, $y = [y_1, \ldots, y_m]$, we put

$$l(y) = [y_1, \ldots, y_{m-1}, y_m - \sum_{i=1}^{m-1} \alpha_i y_i] .$$

Then l is a linear mapping, $Dl(y) = 1$; let $\psi(x) = l(\varphi(x))$ $(x \in G)$. We have $\mu(A) = \mu(l(A))$ for every measurable set A, therefore $\mu(\varphi(A)) = \mu(l(\varphi(A))) = \mu(\psi(A))$ for every compact set $A \subset G$. Let K_1 be a cube with center b, $K_1 \subset G$. There exists a finite positive constant C such that $\left|\frac{\partial \psi_i(x)}{\partial x_j}\right| \leqq C$ for every $x \in K_1$ and for all i, j. But $\frac{\partial \psi_m(b)}{\partial x_k} = 0$ for all k; consequently, there exists a cube $K_2 \subset K_1$ with center b such that

$$\left|\frac{\partial \psi_m(x)}{\partial x_k}\right| \leqq \frac{\varepsilon}{(2m)^m \cdot C^{m-1}}$$

for every $x \in K_2$ and all k. Now let K be a cube such that $b \in K \subset K_2$ and let x be an arbitrary point of K. The segment with the end-points b, x contains points $c^{(i)}$ such that

$$\psi_i(x) - \psi_i(b) = \sum_{j=1}^{m} \frac{\partial \psi_i(c^{(i)})}{\partial x_j} (x_j - b_j)$$
$$(i = 1, \ldots, m)$$

(where $[x_1, \ldots, x_m] = x$, $[b_1, \ldots, b_m] = b$). Since $|x_j - b_j| \leqq \eta$, where $\eta^m = \mu(K)$, we have

$$|\psi_i(x) - \psi_i(b)| \leqq mC\eta \quad (i = 1, \ldots, m-1) ,$$

$$|\psi_m(x) - \psi_m(b)| \leqq m \cdot \frac{\varepsilon}{(2m)^m \cdot C^{m-1}} \cdot \eta .$$

Consequently, the set $\psi(K)$ is contained in an m-dimensional interval of the volume

$$(2mC\eta)^{m-1} \cdot 2m \cdot \frac{\varepsilon}{(2m)^m \cdot C^{m-1}} \eta = \varepsilon\eta^m .$$

It follows that $\mu(\psi(K)) \leqq \varepsilon\eta^m = \varepsilon\mu(K)$, which completes the proof.

Lemma 3. *Let φ be a mapping of the class C_1 of the open set $G \subset E_m$ into E_m. Let $B = E[x; D\varphi(x) = 0]$. Then $\mu(\varphi(B)) = 0$.*

Proof. First of all, let A be a compact subset of G and $D\varphi(x) = 0$ for every $x \in A$. Let the cube K contain the set A. If I is a cube, $I \subset K$, put $F(I) = \mu(\varphi(A \cap I))$. If $I_1, \ldots, I_n, I$ are cubes, $\bigcup_{i=1}^{n} I_i = I \subset K$, then

$$\bigcup_{i=1}^{n} A \cap I_i = A \cap I\,, \quad \bigcup_{i=1}^{n} \varphi(A \cap I_i) = \varphi(A \cap I)\,,$$

whence

$$F(I) = \mu(\varphi(A \cap I)) \leqq \sum_{i=1}^{n} \mu(\varphi(A \cap I_i)) = \sum_{i=1}^{n} F(I_i)\,.$$

Let $\varepsilon > 0$, $b \in K$. If b non $\in A$, we have $F(I) = 0 < \varepsilon\mu(I)$ for every sufficiently small cube I, where $b \in I \subset K$. Let now $b \in A$. It follows from lemma 2 that there exists a neighbourhood U of the point b such that $\mu(\varphi(I)) \leqq \varepsilon\mu(I)$ for every cube I, where $b \in I \subset U$. If I is a cube such that $b \in I \subset K \cap U$, we have therefore

$$F(I) = \mu(\varphi(A \cap I)) \leqq \mu(\varphi(I)) \leqq \varepsilon\mu(I)\,.$$

By lemma 1, $\mu(\varphi(A)) = \mu(\varphi(A \cap K)) = F(K) \leqq \varepsilon\mu(K)$; ε being an arbitrary positive number, we obtain $\mu(\varphi(A)) = 0$.

Let now $F_1, F_2, \ldots$ be compact, $G = \bigcup_{n=1}^{\infty} F_n$. Then $B = \bigcup_{n=1}^{\infty} (B \cap F_n)$. Since the sets $B \cap F_n$ are closed in G, they are closed in F_n; hence they are compact. It follows $\mu(\varphi(B \cap F_n)) = 0$ for $n = 1, 2, \ldots$, $0 \leqq \mu(\varphi(B)) \leqq \sum_{n=1}^{\infty} \mu(\varphi(B \cap F_n)) = 0$, which proves this lemma.

Definition. *Let N be an arbitrary set of indices; let a_n be a non-negative number for every $n \in N$. We put $\sum_{n \in N} a_n = \sup_F \sum_{n \in F} a_n$, where F is a finite subset of N.*

If a_n are real numbers ($n \in N$) and if at least one of the values

$$\sum_{n \in N} (a_n)_+\,, \quad \sum_{n \in N} (a_n)_-\ ^{3)}$$

is finite, we put

$$\sum_{n \in N} a_n = \sum_{n \in N} (a_n)_+ - \sum_{n \in N} (a_n)_-$$

and say that the sum $\sum_{n \in N} a_n$ exists.

Theorem. *Let G be open in E_m. Let φ be a mapping of the class C_1 of G into E_m. Let f be a function on G such that the Lebesgue integral*

$$I = \int_G |D\varphi(t)|\, f(t)\, \mathrm{d}t$$

[3]) $b_+ = \max(b, 0)$, $b_- = \max(-b, 0)$.

exists. If $x \in \varphi(G)$, let $N(x)$ be the set of all $t \in G$ such that $\varphi(t) = x$. Then the sum $g_f(x) = \sum_{t \in N(x)} f(t)$ exists for almost all $x \in \varphi(G)$ and

$$\int_{\varphi(G)} g_f(x) \, \mathrm{d}x = I .$$

Proof. I. First, suppose that $D\varphi(t) \neq 0$ for every $t \in G$. If $t_0 \in G$, there exists a bounded open neighbourhood U of t_0 ($\bar{U} \subset G$) such that $\varphi(t_1) \neq \varphi(t_2)$ for $t_1, t_2 \in U$, $t_1 \neq t_2$. For $x \in \varphi(U)$ put $\psi(x) = t$, where $\varphi(t) = x$, $t \in U$. Let f be a bounded measurable function on G such that $f(t) = 0$ for t non $\in U$. Let g be a function on $\varphi(G)$, which is defined as follows: $g(x) = f(\psi(x))$ for $x \in \varphi(U)$, $g(x) = 0$ otherwise. Evidently $g(x) = \sum_{t \in N(x)} f(t) = g_f(x)$ for every $x \in \varphi(G)$; if $t \in U$, we have $f(t) = g(\varphi(t))$. Since $\int_U |D\varphi(t)| \, g(\varphi(t)) \, \mathrm{d}t = \int_{\varphi(U)} g(x) \, \mathrm{d}x$ (see Jarník, Integrální počet II, p. 219, theorem 103), we have

$$\int_G |D\varphi(t)| \, f(t) \, \mathrm{d}t = \int_U |D\varphi(t)| \, g(\varphi(t)) \, \mathrm{d}t = \int_{\varphi(U)} g(x) \, \mathrm{d}x = \int_{\varphi(G)} g_f(x) \, \mathrm{d}x .$$

Now let K be a compact subset of G. For every $v \in K$ there exists a neighbourhood U_v with the following property: If f is a bounded measurable function on G such that $f(t) = 0$ for t non $\in U_v$, then

$$\int_G |D\varphi(t)| \, f(t) \, \mathrm{d}t = \int_{\varphi(G)} g_f(x) \, \mathrm{d}x . \tag{1}$$

There exist $v_1, \ldots, v_n$ such that $K \subset U_{v_1} \cup \ldots \cup U_{v_n}$. Let $V_i = U_{v_i} - \bigcup_{j<i} U_{v_j}$ ($i = 1, \ldots, n$). Let f be a bounded and measurable function on G such that $f(t) = 0$ for $t \in G - K$. Let $f_i(t) = f(t)$ for $t \in V_i$, $f_i(t) = 0$ otherwise ($i = 1, \ldots, n$). Then the relations

$$\int_G |D\varphi(t)| \, f_i(t) \, \mathrm{d}t = \int_{\varphi(G)} g_{f_i}(x) \, \mathrm{d}x \tag{2}$$

hold for $i = 1, \ldots, n$. Evidently $\sum_{i=1}^{n} f_i = f$, $\sum_{i=1}^{n} g_{f_i} = g_f$. If we add the equalities (2), we obtain a relation of the form (1).

Let now f be an arbitrary non-negative measurable function on G. There exist compact sets $K_n \subset G$ and bounded non-negative measurable functions f_n such that $f_n(t) = 0$ for $t \in G - K_n$ and $f = \sum_{n=1}^{\infty} f_n$. Adding the relations (2) for $i = 1, 2, \ldots$, we obtain (1) again. If $\int_G |D\varphi(t)| \, f(t) \, \mathrm{d}t < \infty$, then $g_f(x) < \infty$ almost everywhere in $\varphi(G)$. If f is an arbitrary function on G such that $\int_G |D\varphi(t)| \, . \, f(t) \, \mathrm{d}t$ exists, we apply the proved results to the functions $(f(t))_+$ and $(f(t))_-$. Thus the theorem is proved for the case $D\varphi(t) \neq 0$ on G.

II. Let now φ be an arbitrary mapping of the class C_1; let the integral $\int_G |D\varphi(t)| \, f(t) \, \mathrm{d}t$ exist. Let G_1 be the set of all $t \in G$, where $D\varphi(t) \neq 0$; let $N_1(x)$

be the set of all $t \in G_1$, where $\varphi(t) = x$ $(x \in \varphi(G_1))$. By the part I of our proof, the sum $g_f^{(1)}(x) = \sum_{t \in N_1(x)} f(t)$ exists for almost all $x \in \varphi(G_1)$ and

$$\int_{G_1} |D\varphi(t)|\, f(t)\, dt = \int_{\varphi(G_1)} g_f^{(1)}(x)\, dx\,.$$

Put $Z = \varphi(B)$, where $B = G - G_1$. Evidently $N(x) = N_1(x)$ for every $x \in \in \varphi(G) - Z$ and

$$\varphi(G) - Z \subset \varphi(G_1) \subset \varphi(G)\,.$$

By lemma 3, $\mu(Z) = 0$, whence $\int_{\varphi(G_1)} g_f^{(1)}(x)\, dx = \int_{\varphi(G)} g_f(x)\, dx$. Thus we obtain

$$\int_G |D\varphi(t)|\, f(t)\, dt = \int_{G_1} |D\varphi(t)|\, f(t)\, dt = \int_{\varphi(G_1)} g_f^{(1)}(x)\, dx = \int_{\varphi(G)} g_f(x)\, dx\,,$$

which proves the theorem.

Резюме

ПРЕОБРАЗОВАНИЯ m-МЕРНЫХ ИНТЕГРАЛОВ ЛЕБЕГА

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Теорема. *Пусть φ — отображение открытого множества $G \subset E_m$ в E_m; пусть отображение φ имеет непрерывные производные 1-ого порядка. Пусть $D\varphi(t)$ — функциональный определитель отображения φ в точке $t \in G$. Пусть f — функция на множестве G такая, что существует интеграл Лебега*

$$I = \int_G f(t)\, |D\varphi(t)|\, dt\,.$$

Тогда для почти всех $x \in \varphi(G)$ имеет смысл сумма $g(x) = \sum_{\varphi(t)=x} f(t)$ и

$$I = \int_{\varphi(G)} g(x)\, dx\,.$$