Antonín Špaček

Note on K. Menger's probabilistic geometry

*Czechoslovak Mathematical Journal*, Vol. 6 (1956), No. 1, 72–74

Persistent URL: `http://dml.cz/dmlcz/100179`

Чехословацкий математический журнал, т. 6 (81) 1956, Прага

# NOTE ON K. MENGER'S PROBABILISTIC GEOMETRY

ANTONÍN ŠPAČEK, Praha.

(Received June 28, 1955.)

The purpose of this note is to establish a number of completely elementary results in Menger's probabilistic geometry from the point of view of the theory of random processes.

Following roughly the definition of K. MENGER in [1] the probabilistic geometry is a theory of random distance functions in an abstract space $X \neq 0$. It will be considered here as a probability measure in a properly chosen $\sigma$-algebra of random events in the space $F$ of all real functions defined in the Cartesian power $X^2 = X \times X$. This $\sigma$-algebra $\mathfrak{F}$ is defined to be the smallest $\sigma$-algebra of subsets of $F$ containing the class

$$\{\{f : f(x, y) < r\} : x, y \in X, r \in R\} ,$$

where $R$ denotes the space of all real numbers. If $\mu$ is a probability measure in $\mathfrak{F}$ then, according to the definition in [2], $(F, \mathfrak{F}, \mu)$ is a random function.

For $A \subset X$ let us denote by $T(A)$ the set of all functions from $F$ which are distance functions in $A$. *The random function* $(F, \mathfrak{F}, \mu)$ *is said to be a distance function or a metric with probability one, if* $\bar{\mu}(T(X)) = 1$, where $\bar{\mu}$ denotes the outer measure induced by $\mu$. This definition seems to be the natural one.

Clearly, the class $\mathfrak{D}$ of all denumerable subsets of $X$ satisfies the conditions (2) and (3) in [2] and the transform $T$ satisfies the conditions (6), (7) and (8) in [2]. The property (5) of $T$ in [2] follows from the obvious fact that if $A \subset X$ and $\varrho$ is a metric in $A$ then there exists a metric $\delta$ in $X$ such that $\delta$ and $\varrho$ coincide on $A$. Using theorem 1 in [2] we obtain

**Theorem 1.** *A necessary and sufficient condition for a random function* $(F, \mathfrak{F}, \mu)$ *to be a metric with probability one is that*

$$\mu\{f : f(x, x) = 0\} = 1 \quad \textit{for} \quad x \in X , \tag{1}$$

$$\mu\{f : f(x, y) > 0\} = 1 \quad \textit{for} \quad x, y \in X,\ x \neq y , \tag{2}$$

$$\mu\{f : f(x, y) = f(y, x)\} = 1 \quad \textit{for} \quad x, y \in X , \tag{3}$$

$$\mu\{f : f(x, y) + f(y, z) \geq f(x, z)\} = 1 \quad \textit{for} \quad x, y, z \in X . \tag{4}$$

If the power of $X$ does not exceed $2^{\aleph_0}$, if $A \subset X$ is denumerable and $\varrho$ is a metric in $A$ then there exists a metric $\delta$ in $X$ such that the space $X$ is separable with respect to $\delta$ and $\delta$ coincides with $\varrho$ on $A$; hence, using in addition the fact that the separability is a hereditary property, we can state

**Theorem 2.** *If the power of $X$ does not exceed $2^{\aleph_0}$ and the random function $(F, \mathfrak{F}, \mu)$ is a metric with probability one then it is a separable metric with probability one.*

Using theorem 1 we can easily verify that *a necessary condition for a random function $(F, \mathfrak{F}, \mu)$ to be a metric with probability one is that*

$$\mu\{f : f(x, x) < r\} = 1 \text{ for } x \in X,\ r > 0\,, \tag{5}$$

$$\mu\{f : f(x, y) < r\} = 0 \text{ for } x, y \in X,\ r \leqq 0\,, \tag{6}$$

$$\mu\{f : f(x, y) < r\} = \mu\{f : f(y, x) < r\} \text{ for } x, y \in X,\ r \in R\,, \tag{7}$$

$$\left\{\begin{array}{l} \mu\{f : f(x, y) + f(y, z) < r\} \leqq \mu\{f : f(x, z) < r\} \\ \quad \text{for } x, y, z \in X,\ r \in R\,. \end{array}\right. \tag{8}$$

But unfortunately *it is not sufficient* as will be shown by the following example:

Let $X = \{a, b, c\}$ be a set consisting of three points. Then there is a probability measure $\mu_0$ in $\mathfrak{F}$ such that

$$\mu_0\{f : f(a, a) = 0\} = \mu_0\{f : f(b, b) = 0\} = \mu_0\{f : f(c, c) = 0\} = 1\,,$$

$$\mu_0\{f : f(a, b) < t\} = \mu_0\{f : f(a, c) < t\} = \mu_0\{f : f(b, c) < t\} =$$
$$= 1 - e^{-t} \text{ or } 0 \text{ according as } t \geqq 0 \text{ or } t < 0\,,$$

$$\mu_0(\{f : f(a, b) < t_1\} \cap \{f : f(a, c) < t_2\} \cap \{f : f(b, c) < t_3\}) =$$
$$= \mu_0\{f : f(a, b) < t_1\}\, \mu_0\{f : f(a, c) < t_2\}\, \mu_0\{f : f(b, c) < t_3\}\,,$$

hence (5), (6), (7) and (8) are satisfied, but for example

$$\mu_0\{f : f(a, b) = f(b, a)\} = 0\,,$$
$$\mu_0\{f : f(a, b) + f(b, c) \geqq f(a, c)\} = \tfrac{3}{4}\,,$$

i. e. (3) and (4) do not hold and $\mu_0(T(X)) = 0$. Essentially the same example can be constructed if $X$ is of arbitrary power. *We can state that the random function $(F, \mathfrak{F}, \mu_0)$ is a metric with probability zero or that it is almost never a metric.*

We see from [1] that (5), (6), (7), (8) correspond exactly to the definition of a random metric given by Menger, hence, Menger's conditions do not suffice for the characterization of a random metric in the sense of our definition.

## BIBLIOGRAPHY


[1] *K. Menger*: Probabilistic geometry. Proc. Nat. Acad. Sci. USA, vol. *37* (1951), pp. 226—229.

[2] *A. Špaček*: Regularity properties of random transforms. Czechoslowak Mathematical Journal, vol. *80* (1955), pp. 143—151.

Резюме

## ЗАМЕТКА К ВЕРОЯТНОСТНОЙ ГЕОМЕТРИИ К. МЕНГЕРА

АНТОН ШПАЧЕК (Antonín Špaček), Прага.
(Поступило в редакцию 28/VI 1955 г.)

Если мы определим приведенным в статье натуральным способом случайную метрику, то мы сможем выразить необходимые и достаточные условия для того, чтобы случайная функция двух переменных, определенная в абстрактном пространстве, была почти наверно метрикой. Если мощность этого пространства не превосходит $2^{\aleph_0}$ и если приведенное случайное преобразование является почти наверно метрикой, то оно почти наверно будет сепарабельной метрикой. Указано, что аксиомы Менгера недостаточны для определения случайной метрики.