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ON THE THEOREM OF STAUDT IN MOUFANG PLANE

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In this paper the author generalizes the Staudt theorem on mappings which preserve the harmonic conjugacy of point-quadruplets. The validity of this theorem is known in the case of Desarguesian plane. The author extends the theorem to the projective plane in which the Theorem of Complete Quadrilateral holds.

§ 1. Introduction.

We shall investigate the projective plane π in which the Theorem of Complete Quadrilateral holds universally. We take the symbols introduced in the paper [3].

Definition 1. *a) Let p, p' be lines and S a point not collinear with p, p'. The perspectivity between p, p' mediated by S is the mapping $A \longleftrightarrow A'$ of p on p' so that for every $A \in p - p'$ is $|S, A, A'|$ and for every $A \in p \cap p'$ is $A = A'$.*

b) The composition of finite number of perspectivities is projectivity.

Definition 2. *Staudt projectivity is one-one point-mapping $A \longleftrightarrow A'$ of the line p on the line p', which preserves harmonic quadruplets* ([2], *introducing chapter*).

Lemma 1. *Projectivity is the special case of Staudt projectivity.*

Proof (fig. 1). Given perspectivity $p \longleftrightarrow p'$ mediated by S. Let $(ABCD)$ be valid on p. Without losing the generality we may suppose that $A' \neq p \cap p' \neq C$ (because $(ABCD) \Rightarrow (CDAB)$).

Choose $p_0 = A'C$; then S mediates perspectivities $p \longleftrightarrow p_0 \longleftrightarrow p'$ for which $(ABCD)$, $(A'B_0CD_0)$, $(A'B'C'D')$ hold ([3], lemma 2a). The composition of mentioned perspectivities is the given perspectivity, thus from definition 1b we infer the assertion of the lemma.

Later we shall prove the main theorem of this paper namely that Staudt projectivity may be composed from some projectivity and some semiautomorphism (cf. definition 4).

Loo Keng Hua and Л. А. Скорняков have proved this theorem for Desarguesian planes; for Pascalian planes cf. e. g. [4], p. 275, theorem I. In the case of Pascalian plane the problem is simple because projectivity is uniquely determined by three pairs of corresponding points (this assertion is not valid in Desarguesian plane, cf. [5], theorem 4; [4], p. 275, theorem I). We shall prove the main theorem in algebraic way. We use the results of Loo Keng Hua (see [5]); further we use the introduction of coordinates by Ruth Moufang ([6], § 2; [7], § 2; [4], § 1).

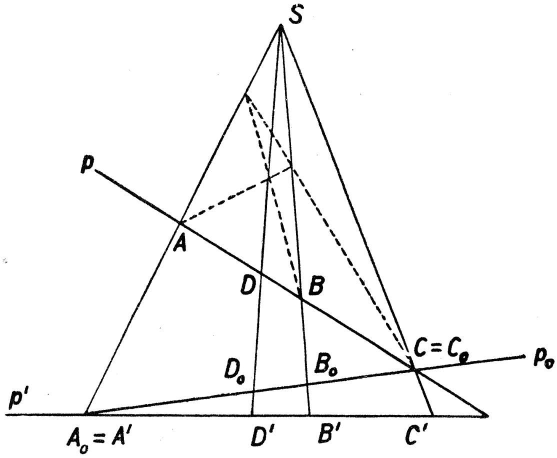

Fig. 1.

§ 2. Generalisation of results of Loo Keng Hua.

Definition 3. *The alternative ring (with unit) is a non-void set with binary operations* $+, .$ *so that all elements form a commutative group with respect to addition, all elements distinct from zero form an alternative grupoid (with a unit element) with respect to multiplication and both distributive laws for multiplication over addition are satisfied.*

We note that in alternative grupoid the weak associative law holds ($x^2y = x(xy)$, $(xy)x = x(yx)$, $xy^2 = (xy)y$).

Definition 4. *The mapping* $x \to x'$ *of an alternative ring* A *into an alternative ring* A' *is called a) homomorphism, if the identities*

$$(a + b)' = a' + b' , \tag{1}$$

$$(ab)' = a'b' \tag{2}$$

hold,

b) *antihomomorphism, if the identities* (1) *and*

$$(ab)' = b'a' \tag{2'}$$

hold,

c) *semihomomorphism, if the identities* (1) *and*

$$(aba)' = a'b'a' \,, \tag{3}$$

$$(a^2)' = (a')^2 \tag{3'}$$

are valid,

d) *Jordan homomorphism, if the identities* (1) *and*

$$(ab + ba)' = a'b' + b'a' \tag{4}$$

hold.

For associative rings Loo Keng Hua has proved in cited paper this theorem:

Theorem 2. *Every semihomomorphism between two rings without zero-divisors is either homomorphism or antihomomorphism.*

This theorem holds for alternative rings too. We follow the proof of Loo Keng Hua using the identity $((ab)c - (cb)a)' - (a'b')c' - (c'b')a' = (a(bc) + + c(ba))' = a'(b'c') - c'(b'a')$.

We shall prove this identity. We shall rewrite the expression $v = (a + c)\,.\,. b(a + c) - aba - cbc$; we obtain $v = (ab + cb)(a + c) - aba - cbc = aba - - (cb)a - (ab)c + cbc - aba - cbc = (cb)a + (ab)c$. Similarly $v = (a + c)(ba + + bc) - aba - cbc = a(bc) + c(ba)$. By (1), (4) and by preceding equations is $v' = (a' + c')\,b'(a' + c') - a'b'a' = a'(b'c') - c'(b'a') = (a'b')\,c' + (c'b')\,a'$. In the other parts of the proof of Hua that is no need of changing anything.

Further Loo Keng Hua has proved this theorem for associative rings:

Theorem 3. *If the characteristic of the rings is not 2, then semihomomorphism is equivalent with Jordan homomorphism.*

The proof is also valid for alternative rings because the weak associative law for multiplication holds.

§ 3. Auxiliary lemmas.

The concept of alternative division ring is introduced in [3], § 2. Now we introduce the point-coordinates of line p on π ([6], § 2, p. 766). Let zero correspond to the point O, unit to the point E and the symbol ∞ to the point U, where $|O, E, U|$ on p. Then every point $X \neq U$ of p may be characterized through an element of an alternative division ring D with a characteristic distinct from 2. Except for the symbol $(ABCD)$ we shall use the equivalent symbols $(abcd)$ or $(abc\infty)$ respectively.

Lemma 2. *There is*

$$(abcd) \Leftrightarrow -(a-c)^{-1}(b-c) = (a-d)^{-1}(b-d) \tag{5,1}$$

$$(abc\infty) \Leftrightarrow c = \tfrac{1}{2}(a+b) \tag{5,2}$$

for various $a, b, c, d \in D$.

The proof is contained in [3], theorem 2, lemma 5.

Lemma 3. *Let* $\alpha, \beta \in D$, $\alpha \neq 0$. *Then the mappings*

$$x' = \alpha x + \beta\,, \tag{6,1}$$

$$x' = x\alpha + \beta \tag{6,2}$$

preserve the harmonic quadruplets.

Proof. First of all we note that (6,1) is one-one because from (6,1) it follows $x = \alpha^{-1}(x' - \beta)$. The equation

$$2(a-b)^{-1} = (a-c)^{-1} - (a-d)^{-1} \tag{7}$$

is equivalent with $(abcd)$ for various $a, b, c, d \in D$ ([3], lemma 3). We put $x = a, b, c, d$ in (7) and we obtain $(a'b'c'd') \Leftrightarrow 2(\alpha a + \beta - (\alpha b + \beta))^{-1} = (\alpha a + \beta - (\alpha c + \beta))^{-1} + (\alpha a + \beta - (\alpha d + \beta))^{-1} \Leftrightarrow 2(a-b)^{-1}\alpha^{-1} = ((a-c)^{-1} - (a-d)^{-1})\,\alpha^{-1} \Leftrightarrow (abcd)$. Further we infer $(abc\infty) \Leftrightarrow c' = \frac{1}{2}(a' + b') \Leftrightarrow \alpha c + \beta = \frac{1}{2}(\alpha a + \beta + \alpha b + \beta) \Leftrightarrow \alpha c + \beta = \frac{1}{2}\alpha(a+b) + \beta \Leftrightarrow c = \frac{1}{2}(a+b) \Leftrightarrow (abc\infty)$.

Similarly for (6,2).

Note. Given a homomorphism (antihomomorphism) between two alternative division rings we put $b = 1$ in (2) and obtain $a' = a'1'$ so that $1'$ is the unit. If we put $b = a^{-1}$ (for $a \neq 0$) then $(ab)' = 1' = a'(a^{-1})'$ and consequently $(a^{-1})' = (a')^{-1}$.

Similarly for antihomomorphism.

Lemma 4. *Semihomomorphism between alternative division rings with a characteristic dictinct from 2 preserve harmonic quadruplets.*

In the case $(abcd)$ the proof follows from theorem 2, equation (7) from (1) and from the preceding note.

In the case $(abc\infty)$ the proof follows from theorem 2, from (5,2) and (1).

§ 4. Proof of the main theorem.

Given lines $p \subset \pi$, $p' \subset \pi'$, where π, π' are projective planes in which the Theorem of Complete Quadrilateral holds. We investigate the point-mapping $\bar{\varrho}$ of p into p' preserving the harmonic quadruplets. We choose points $U, U' = U\bar{\varrho}$ and introduce on p (p') coordinates from some alternative division ring D (D') with a characteristic distinct from 2 so that U (U') is ideal point. We denote zero and unit from D (D') with the symbols 0, 1 ($0'$, $1'$).

By (6,1) it is possible to choose the mapping ϱ_1 of p on itself so that $x_1\varrho_1 = 0$, $x_2\varrho_1 = 1$ for given $x_1 \neq x_2$. For if $x_1 \neq 0 \neq x_2 \neq x_1$ we choose $\beta = (x_1^{-1} - x_2^{-1})^{-1}$, $\alpha = -\beta x_1$ and if $0 = x_1 \neq x_2$ we choose $\beta = 0$, $\alpha = -x_2$. In the mapping $\varrho = \varrho_1^{-1}\bar{\varrho}$ the zeros, units and ideal elements ∞, ∞' correspond.

Suppose $(abc0)$. Then by (5,1) $-(a-c)^{-1}a = (b-c)^{-1}b$. Given nonzero distinct elements $a, b, a+b \neq 0$ we compute c. We infer $a^{-1}(a-c) = b^{-1}(b-c) \Rightarrow -1 - a^{-1}c = 1 - b^{-1}c \Rightarrow$

$$c = 2(a^{-1} - b^{-1})^{-1}. \qquad (8,1)$$

Further we obtain by means of $(a^{-1} + b^{-1})^{-1} = (a^{-1}(a+b)b^{-1})^{-1} = a(a+b)^{-1}b$

$$c = 2a(a+b)^{-1}b \text{ for } a \neq 0 \neq b \neq -a. \qquad (8,2)$$

By hypothesis $(abc0) \Rightarrow (a\varrho\, b\varrho\, c\varrho\, 0')$ is satisfied so that

$$c\varrho = 2(a\varrho)(a\varrho - b\varrho)^{-1}b\varrho. \qquad (8,3)$$

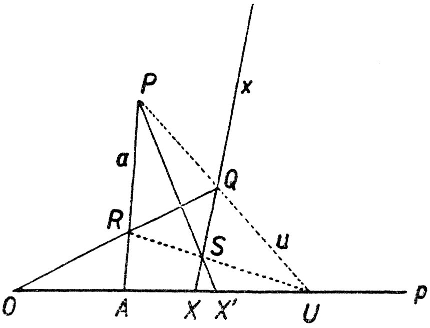

Fig. 2.

The case $c = \frac{1}{2}(a+b)$ is not possible in (8,1) because the relations $(abc\infty)$, $(abc0)$ are different.

Suppose $(abc\infty)$. From (5,2) and from $(abc\infty) \Rightarrow (a'b'c'\infty')$ follows $c\varrho = \frac{1}{2}(a\varrho + b\varrho)$.

From $(x\, \infty\, 2x\, 0)$ follows $(x\varrho\, \infty'\, 2(x\varrho)\, 0')$ by (5,2) and finally $(2x)\varrho = 2(x\varrho)$. Consequently from $\left(\frac{a}{2} + \frac{b}{2}\right)\varrho = \frac{a\varrho}{2} + \frac{b\varrho}{2}$ we deduce the identity

$$(x+y)\,\varrho = x\varrho + y\varrho. \qquad (1')$$

(The cases $x = 0$ or $y = 0$ are trivial.)

We put in (1') $x = 2v$, $y = 2w$ and obtain $(2(v+w))\,\varrho = (2v)\varrho + (2w)\varrho$.

Further we put in (8,3) $b = 1 - a$ so that we obtain $(2a(1-a))\,\varrho = -2(a\varrho)(1 - a\varrho)$. From this it follows $(a^2)\varrho = (a\varrho)^2$ by the preceding equation where we set $v = a$, $w = -a^2$.

Thus in the whole we have $(xy + yx)\,\varrho = ((x+y)^2 - x^2 - y^2)\,\varrho = (x\varrho + y\varrho)^2 - (x\varrho)^2 - (v\varrho)^2 = (x\varrho)(y\varrho) + (y\varrho)(x\varrho)$. Therefore ϱ is a Jordan homomorphism and by theorem 3 ϱ is a semihomomorphism. Thus $\bar{\varrho}$ has the type $x\bar{\varrho} = (\alpha\varrho)(x\varrho) + \beta\varrho$.

Lemma 5. *The mapping* (6,1) *of a line on itself is a projectivity.*

Proof (cf. [6], p. 766; [4], p. 278—9). We choose U as an ideal point on line p and using the known Hilbert configuration we introduce coordinates from an alternative division ring D with a characteristic distinct from 2.

In fig. 2 let us fix O, U, A, a, u, P, Q, R. We put $x = QX$, $X' = p \cap P(x \cap \cap RU)$. Point P mediates a perspectivity $X \to S$ between p, RU, point P mediates a perspectivity $S \to X'$ between RU, p. By composition of both preceding perspectivities we obtain the projectivity $X \to X'$. The algebraic meaning is known: $x' = x + a$, where small letters denote elements from D corresponding to the points of p.

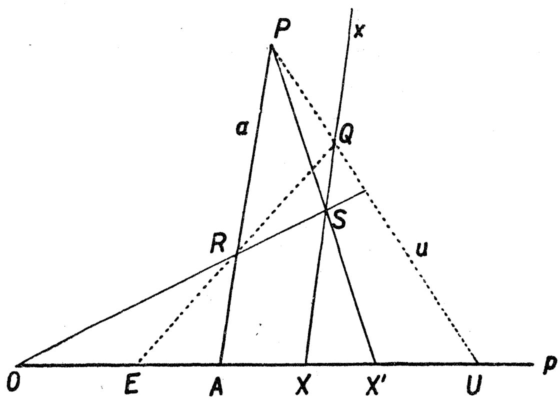

Fig. 3.

Now we choose in configuration of fig. 3 O, E, U, A, P, Q fixed; then R is also fixed. Let $A \neq 0$. We put $x = QX, X' = p \cap P(x \cap RS)$. Then Q mediates a perspectivity $X \to S$ between p, OR and P mediates a perspectivity $S \to X'$ between OR, p so that $X \to X'$ is the projectivity. The algebraic meaning is known: $x' = ax$.

Thus the mapping $x' = = \alpha x + \beta$ for $\alpha, \beta \in D$, $\alpha \neq 0$ is a projectivity.

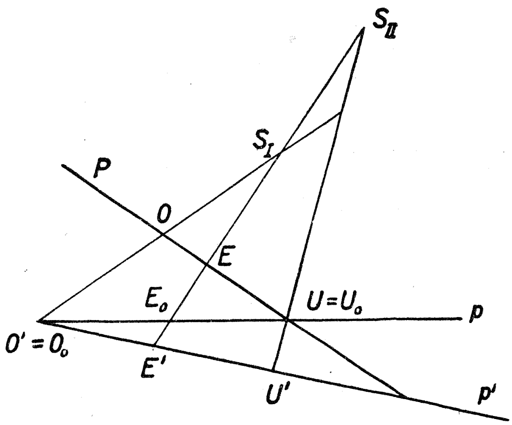

Fig. 4.

Note. If (by hypothesis of begin of this chapter) $\pi = \pi'$, then the auxiliary projectivity may be constructed which maps O on O', E on E' and U on U'. For $p = p'$ have given the proof at the beginning of this chapter. Let $p \neq p'$; we shall investigate only the case in which every point of O, E, U, O', E', U' is distinct from $p \cap p'$. We introduce the signification of fig. 4, thus we choose $p_0 = O'U$, $S_I = EE' \cap OO'$, $O' = O_0$, $U = U_0$, $E_0 = EE' \cap p_0$. Thus S_I mediates a perspectivity between p, p_0, S_{II} mediates a perspectivity between p_0, p' and the resulting projectivity is the composition of these perspectivities.

Similarly for the cases where some of the six points is equal to $p \cap p'$.

As a whole we have proved that *the Staudt projectivity between lines* $p, p' \subset \pi$

may be composed from a projectivity between p, p' and some semiautomorphism on p'.

At the end of this paper we shall prove this lemma:

Lemma 6. a) *Let the plane π be Pascalian. If the semiautomorphism is a projectivity it reduces to the identity mapping.*

b) *Let π be non Pascalian. Then there exist non-identical semiautomorphisms which are projectivities.*

Proof. ad a) By [4], theorem I, p. 275, projectivity is uniquely determined by three pairs of corresponding points. The elements 0,1 and the symbol ∞ are self-corresponding in any semiautomorphism on the line p. The only projectivity with these self-corresponding elements is the identity.

ad b) On the line p there exist at least two projectivities $\sigma_I \neq \sigma_{II}$ with self-corresponding symbols 0, 1, ∞. By lemma 1 and by the main theorem there exist the expressions $\sigma_i = \varrho_1 \varrho_i$ ($i = $ I, II) where ϱ_1 is an identity and ϱ_i are semiautomorphisms. The proof follows from these expressions.

BIBLIOGRAPHY

[1] *G. Ancochea:* Le théorème de von Staudt en géométries quaterniènnes, Journal für Math. *184* (1941), pp. 193—198.

[2] *G. Ancochea:* On semiautomorphisms of division algebras, Ann. of Math. *48* (1947), pp. 147—153.

[3] *V. Havel:* Harmonical quadruplet in Moufang plane, Czech. Mat. Journ.

[4] *W. V. D. Hodge-D. Pedoe*: Methods of algebraic geometry, Cambridge 1947.

[5] *Л. К. Хуа:* О семигомоморфизмах колец и их приложениях к проективной геометрии, Успехи мат. наук VIII, 3(55) 1953, pp. 143-7.

[6] *R. Moufang:* Die Schnittpunktsätze des projektiven speziellen Fünfecknetzes in ihrer Abhängigkeit voneinander (Das A-Netz), Math. Ann. *106* (1932), pp. 755—795.

[7] *R. Moufang:* Alternativkörper und der Satz vom vollständigen Vierseit, Abh. Math. Sm. Hamb. *9* (1933), pp. 207—222.

[8] *Л. А. Скорняков,* Проективные плоскости, Успехи мат. наук (вып. 6) 1951, стр. 112—154.

Резюме.

О ТЕОРЕМЕ ШТАУДТА НА ПЛОСКОСТИ МУФАНГ.

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(Поступило в редакцию 27/III 1954 г.)

Ло-Кен-Хуа доказал для плоскостей Дезарга теорему Штаудта о точечном отображении двух прямых на себя, при котором не нарушается гармоническое отношение точек. Самый общий тип проективной плоскости,

в котором еще можно говорить о гармоническом отношении, есть плоскость π с универсальной теоремой о полном тетраэдре. В настоящей статье справедливость теоремы Штаудта расширена на такие плоскости:

Пусть π, π' — две такие плоскости с аффинными координатами Муфанг (из альтернатовных тел T, T' характеристики $\neq 2$). Пусть, дальше, ϱ есть отображение координатной оси $p \subset \pi$ в координатную ось $p' \subset \pi'$ при котором сохраняются гармонические отношения и при котором несобственные точки прямых p, p' соответствуют друг другу. Это отображение можно получить из аффинного соответствия на прямой p (т. е. проективного отношения с неподвижной несобственной точки) и из семигомоморфного соответствия прямой p и p' (т. е. из соотношения, в котором соответствующие друг другу координаты связаны или соотношением гомоморфизма или анти-гомоморфизма). Теорема имеет аффинный характер, так как на место несобственных точек прямых p, p' при постановке координат была выбрана пара соответствующих друг другу точек. Но это не умаляет общности теоремы.