Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 5 (1955), No. 1, 76–82

Persistent URL: http://dml.cz/dmlcz/100132

Чехословацкий математический журнал, т. 5 (80) 1955

HARMONICAL QUADRUPLET IN MOUFANG PLANE

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(Received March 15, 1954.)

Suppose that in projective plane the Theorem of Complete Quadrilateral is valid. If the coordinates from an alternative division ring with characteristic $p \neq 2$ are used it is possible to generalize some results for harmonic conjugacy in Desarguesian planes. The author investigates these generalized algebraic expressions of harmonic conjugacy of point-quadruplets.

§ 1. Geometrical considerations.

The usual axioms for a projective plane as a point-set are (cf. e. g. [3], § 2):

PG1 For any two points $X \neq Y$ one and only one subset (line) XY exists so that $X \in XY$, $Y \in XY$.

PG2′ Any two distinct lines have one and only one point in common.

PG3 There exist three points not on the same line.

PG4 There are at least three points on every line.

With the symbol $|A, B, C, \ldots|$ we denote the case, where points $A, B, C, \ldots$ are distinct and on the same line. Now we introduce the Theorem of Complete Quadrilateral ([6], § 1, p. 761):

$$|A, B, C|, |A, P_1, P_2|, AB \neq P_1P_2 \Rightarrow$$
$$D = (P_1B \cap A(P_1C \cap P_2B))\, P_2 \cap AB \tag{1}$$

is independent of the choice of P_1, P_2,

$$C \neq D\,. \tag{1'}$$

Note. If we construct D in this manner, we shall say that we have used (Q) for A, B, C, P_1, P_2 (in this order). For these four points A, B, C, D we take the symbol $(ABCD)$.

Let Theorem of Complete Quadrilateral be universally valid in given projective plane ([3], § 6). First we deduce some results in the geometrical way.

Lemma 1. $(ABCD) \Rightarrow (ABDC)$, $(ABCD) \Rightarrow (BACD)$.

Proof. We apply (Q) to the points A, B, C, P_1, P_2 to obtain D. Now we put $P_1' = P_2$, $P_2' = P_1$ and use (Q) for A, B, D, P_1', P_2'. This implies $(ABDC)$ so that the first case is proved. In the same manner, using (Q) for A, B, C, P_1, P_2 we obtain D. If we put $P_1' = P_1C \cap P_2B$, $P_2' = P_2$ and use (Q) for A, B, C P_1', P_2' we infer $(BACD)$ proving the second case.

Lemma 2a. $(ABCD)$, S non ϵ AB, $|S, A, A'|$, $|S, B, B'|$ $|S, C, C'|$, $|D, A', B', C'| \Rightarrow (A'B'C'D)$.

Proof. Assume the hypothesis of the lemma. Then we use (Q) for A', B', D, A, S, obtaining $C' \epsilon CS$, thus $(A'B'DC')$. Using lemma 1, we get $(A'B'C'D)$.

Lemma 2b. $(ABCD)$, $(A'B'C'D)$, $AD \neq A'D \Rightarrow (S = AA' \cap BB',\ S' = AB' \cap BA', C, C')$.

Proof. If given assumptions are satisfied we use (Q) for A, B, D, A', S. Hence (C, S, S'). Similarly we apply (Q) to the points A', B', D, A, S, proving (C', S, S'). This completes the proof.

Lemma 2c. If 1, 2, 3, 4 *are given points no three of which are collinear, then* $(12 \cap (13 \cap 24)(14 \cap 23),\ 34 \cap (13 \cap 24)(14 \cap 23),\ 13 \cap 24,\ 14 \cap 23)$.

Proof. We use (Q) for $1, 4, 14 \cap 23, 2, 12 \cap 34$ and obtain $(1, 4, 14 \cap 23, 14 \cap (12 \cap 34)(13 \cap 24))$. It follows by lemma 2a that by projection from the point $12 \cap 34$ this quadruplet get into the wanted quadruplet.

Theorem 1. $(ABCD) \Rightarrow (CDAB)$.

Proof. First we use (Q) for points A, B, C, P_1, P_2. We obtain D. Further the relation $(D, X' = P_1C \cap P_2D, P_2, X = P_2D \cap P_1B)$ is valid (by lemma 2c used for points $A, B, P_1, Q_1 = P_1C \cap P_2B$). The relation $|D, Y, Q_2|$ (where $Y = P_2B \cap Q_2D$, $Q_2 = P_2C \cap Q_1A$) holds by Theorem of Complete Quadrilateral. In similar way we obtain from the quadrilateral $A, B, P_1C \cap Q_2B, Q_1$ the relation $(D, Y' = DQ_2 \cap P_1C, Q, Y)$. It follows by projection from point C (lemma 2a) $(DY'YQ_2) \Rightarrow (BQ_1YP_2)$. Finally we have the relation $|X, Y, C|$ by lemma 2b, used for quadruples (BQ_1YP_2), $(DX'XP_2)$. Now we use (Q) for D, C, A, X, P_2 so that we obtain $(DCAB)$ and by lemma 1 also $(CDAB)$.

§ 2. Algebraic considerations.

We begin with the definition of the alternative division ring.

An alternative division ring is a non-void set with two binary operations $+, .$ *so that all elements form a commutative group with respect to addition, the elements distinct from zero form an alternative loop with respect to multiplication, in which the unit exist; further both distributive laws for multiplication over addition hold.*

We note, that in an alternative loop the equation $xy = z$ is uniquely solvable for any two given elements from x, y, z and the weak assotiative law for multiplication holds (this means, that the equations $x^2y = x(xy)$, $x(yx) = (xy)x$,

$xy^2 = (xy)y$ are valid). The characteristic of an alternative division ring has the usual sense.

Theorem of Complete Quadrilateral holds universally in projective plane if and only if it is possible to introduce for corresponding affine plane coordinates from an alternative division ring with characteristic $p \neq 2$.

Ruth Moufang introduces the following affine plane (cf. [5], § 1): Points are ordered pairs of elements from an alternative division ring A with characteristic $p \neq 2$. For two points we define $(x_1, y_1) = (x_2, y_2) \Leftrightarrow x_1 = x_2,\ y_1 = y_2$. Lines are sets of points the coordinates of which fulfil equations $\alpha x + y - \beta = 0$, $x - \beta = 0$ respectively (vor $\alpha, \beta \in A$).

Marshall Mall uses the right-multiplication ([3], theorem 6,4). He replaces the Theorem of Complete Quadrilateral by his Theorem L, which is more general (without condition (1')). In Hall plane characteristic 2 is not excluded.

Suppose that Moufang plane is given. We choose points $A = (a, 0)$, $B = (b, 0)$, $C = (c, 0)$, $P_1 = (a, 1)$ for various $a, b, c \in A$, where $2c \neq a + b$ and P_2 is ideal point of y-axis. The sense of the case $2c = a + b$ will be mentioned later.

We use formulas $\alpha = (b_2 - b_1)(a_1 - a_2)^{-1}$, $\beta = b_1 + \alpha a_1$, valid for the line $\alpha x + y - \beta = 0$ containing points $(a_1, b_1), (a_2, b_2), a_1 \neq a_2$. If $b_1 = 0$, then $y = b_2(a_1 - a_2)^{-1}(a_1 - x)$ is valid.

By easy computation we infer $P_1C \ldots y = (c - a)^{-1}(c - x)$, $(P_1C \cap P_2B) . A \ldots y = ((c - a)^{-1}(c - b)(a - b)^{-1})(a - x)$, $P_1B \ldots y = (b - a)^{-1}(b - x)$. Since $2c \neq a + b$, $(P_1C \cap P_2B)\, A$ and P_1B are not parallel. We obtain

$$(c - a)^{-1}(c - b)(a - b)^{-1} = (b - a)^{-1}(b - d)(a - d)^{-1}\,, \tag{2}$$

where $D = (d, 0)$ satisfies (1).

Auxiliary lemma $(x^{-1}(x - y))\, y^{-1} = x^{-1}((x - y)\, y^{-1}) = (y^{-1}(x - y))\, x^{-1} = y^{-1}((x - y)\, x^{-1}) = y^{-1} - x^{-1}$.

The proof follows immediately after multiplication.

Therefore if we put on the right side of (2) $x = b - a$, $y = d - a$ we infer by auxiliary lemma $(a - d)^{-1}(b - d)(a - b)^{-1} = - (a - c)^{-1}(b - c)(a - b)^{-1}$ and finally (by right-multiplication with $(a - b)$)

$$- (a - d)^{-1}(b - d) = (a - c)^{-1}(b - c)\,. \tag{3}$$

It follows that for inverse elements $- (b - d)^{-1}(a - d) = (b - c)^{-1}(a - c)$.

From (2) we deduce by auxiliary lemma $(a - b)^{-1}(b - d)(d - a)^{-1} = (a - b)^{-1}(b - c)(a - c)^{-1}$ and by left-multiplication with $(a - b)$ and by a simple arrangement $-(d - b)(d - a)^{-1} = (c - b)(c - a)^{-1}$. Consequently we have for inverse elements $-(d - a)(d - b)^{-1} = (c - a)(c - b)^{-1}$.

Let (3) be valid for various $a, b, c, d \in A$. The inequality $2c \neq a + b$ follows because $2c = a + b \Rightarrow (a - d)^{-1}(b - d) = 1 \Rightarrow a = b$. Hence the converse computation is possible so that points P_1, P_2 exist with this property: if we apply (Q) to the points A, B, C, P_1, P_2 we obtain D. This completes the proof of the following theorem:

Theorem 2. $(ABCD) \Leftrightarrow (3)$ *for distinct points* $A = (a, 0)$, $B = (b, 0)$, $C = (c, 0)$, $D = (d, 0)$.

From (2) it follows by auxiliary lemma $-((c - a)^{-1} - (b - a)^{-1}) = (a - b)^{-1} - (a - d)^{-1}$ and therefore

$$2(a - b)^{-1} = (a - c)^{-1} + (a - d)^{-1} . \tag{4}$$

The converse is also true. We have proved the following lemma:

Lemma 3. $(3) \Leftrightarrow (4)$ *for various* $a, b, c, d \in A$.

Theorem 1 yields for various $a, b, c, d \in A$ this corrolary: $-(a - d)^{-1} . (b - d) = (a - c)^{-1}(b - c) \Leftrightarrow -(a - d)^{-1}(a - c) = (b - d)^{-1}(b - c)$. Now we deduce this corrolary from (3).

We use

$$(b - a)^{-1}(c - b)(a - c)^{-1} = (b - a)^{-1}(b - d)(a - d)^{-1}$$

for the equation of $A(P_1C \cap P_2B)$ and we infer

$$D(P_2C \cap A(P_1C \cap P_2)) \dots y = (((b - - a)^{-1}(c - b))(d - c)^{-1})(d - x),$$

$$C(P_1B \cap A(P_1C \cap P_2B)) \dots y = (((b - a)^{-1} (b - d))(c - d)^{-1})(c - x).$$

If follows for the first line (by $(b - a)^{-1} = \frac{1}{2}((b - c)^{-1} + (b - d)^{-1})$ and by auxiliary lemma)

$$y = \tfrac{1}{2}((c - d)^{-1} - (d - b)^{-1} (c - b)(d - c)^{-1})(d - x) = \tfrac{1}{2}((c - d)^{-1} + (d - b)^{-1} - (d - c)^{-1}(d - x) = \tfrac{1}{2}(2(c - d)^{-1} + (d - b)^{-1})(d - x) = (c - d)^{-1}(d - x) + \tfrac{1}{2}(d - b)^{-1}(d - x).$$

Similarly for the second line:

$$y = \tfrac{1}{2}((b - c)^{-1}(b - d)(c - d)^{-1} + (c - d)^{-1}) (c - x) = \tfrac{1}{2}(- (c - b)^{-1}(b - d)(c - d)^{-1} + (c - d)^{-1})(c - x) = \tfrac{1}{2}((c - d)^{-1} - (c - b)^{-1} + (c - d)^{-1})(c - x) = (c - d)^{-1}(c - x) - \tfrac{1}{2}(c - b)^{-1}(c - x).$$

Both lines yield the same y for $x = b$. Further we have

$$1 = 1 \Rightarrow (c - d)^{-1}(c - d + b - b) = 1 \Rightarrow (c - d)^{-1}(d - b) + \tfrac{1}{2} = (c - d)^{-1}(c - b) - \tfrac{1}{2},$$

which completes the proof. Thus we may use (Q) for the points $C, D, A, P_2C \cap A(P_1C \cap P_2B), P_2$ and obtain $(CDAB)$.

Let $(ABCD)$ be valid. From (4) and from $-2(a - b)^{-1} = (b - c)^{-1} + (b - d)^{-1}$ (which is valid by $(BACD)$) we get by addition

$$(a - c)^{-1} + (b - c)^{-1} + (a - d)^{-1} + (b - d)^{-1} = 0.$$

Lemma 4. *If for every triad of various nonzero elements $\alpha, \beta, \gamma \in A$ no two of which are inverse, various $a, b, c, d \in A$ exist so that $\alpha = a - c$, $\beta = a - d$, $\gamma = b - c$ and* (3) *holds, then A is commutative field.*

Proof. It follows from (3) and from theorem 1, that the equations $-\alpha^{-1}\beta = \gamma^1\delta$, $-\alpha^{-1}\gamma = \beta^{-1}\delta$ (where $\delta = \beta + \gamma - \alpha$) are satisfied. We obtain therefore

$$\gamma(\alpha^{-1}\beta) = \beta(\alpha^{-1}\gamma)\,. \tag{5}$$

If two of elements α, β, γ are equal or inverse, then (5) holds also (since A is an alternative division ring, [7], § 1). The case $\gamma = 0$ or $\beta = 0$ is trivial. Thus (5) holds for every $\alpha \neq 0, \beta, \gamma$. If we choose $\alpha = 1$, we obtain the commutativity. From commutativity the associativity follows by ([10], § 3 and final note) and A is a field. In belonging plane the Theorem of Pappus holds.

Ruth Moufang had showed the validity of theorem 1 if the given plane is linearly ordered and the Theorem of Complete Quadrilateral holds universally.

By the results [2] (§ 5, theorem C), the plane is then Desarguesian. We have seen in § 1 thus the assumptions of order are superfluous. The assumption of affine plane is useful for algebraic considerations.

Lemma 5. *Let $A = (a, 0)$, $B = (b, 0)$, $C = (c, 0)$, $D = (d, 0)$ are various points and I ideal point of x-axis. Then $(ABCI) \Leftrightarrow 2c = a + b$, $(IBCD) \Leftrightarrow 2b = c + d$.*

Proof. 1. Let $(ABID)$ be valid. If $P_1 = (a, 1)$ and P_2 is ideal point of y-axis, then $P_1C \ldots y - 1 = 0$, $P_1CP_2B = (b, 1)$, $A(P_1C \cap P_2B) \ldots y + (a - b)^{-1}x - (a - b)^{-1}a = 0$, $P_1B \ldots y + (b - a)^{-1}x - (b - a)^{-1}b = 0$, $2d = a + b$. Conversely from $2d = a + b$ $(ABID)$ follows.

2. Let $(ABCX)$ be valid, where $c = \frac{1}{2}(a + b)$. Then $P_1C \ldots y + \frac{1}{2}(b - a)^{-1}x - \frac{1}{2}(b - a)^{-1} - \frac{1}{2}(a + b) = 0$, $P_1C \cap P_2B = (b, -1)$, $(P_1C \cap P_2B)A \ldots y + (b - a)^{-1}x - (b - a)^{-1}a = 0$, $P_1B \ldots y + (b - a)^{-1}x - (b - a)^{-1}b = 0$. Consequently $(P_1C \cap P_2B)\, A \cap P_1B$ is the ideal point of P_1B and X as a meet of ideal line and x-axis is the ideal point of x-axis. Conversely from $2c = a + b$ the relation $(ABCI)$ follows. Further we verify easily $(ABCI) \Rightarrow (BACI)$, $(ABCI) \Rightarrow (ABIC)$.

3. Let $(AICD)$ be valid. Then $P_1C \ldots y + (c - a)^{-1}x - (c - a)^{-1}c = 0$, $P_1C \cap P_2B$ is ideal point of P_1C, $(P_1C \cap P_2B)\, A \ldots y + (c - a)^{-1}x - (c - a)^{-1}a = 0$, $P_1B \ldots y - 1 = 0$, $d = (c - a)(-1 + (c - a)^{-1}a) = 2a - c$. Conversely from $d = 2a - c$ the relation $(AICD)$ follows. We verify again $(AICD) \Rightarrow (IACD)$.

Note. The generalised Theorem of Complete Quadrilateral (without assertion (1')) is valid universally in given projective plane if and only if the corresponding affine plane may be coordinatized from an alternative division ring with general characteristic p.

Some preceding considerations are valid also for characteristic $p = 2$. If $p = 2$ then $2c \neq a + b \Leftrightarrow 0 \neq a + b \Leftrightarrow a \neq b$. Thus for various $a, b, c \in A$ the hypothesis $2c \neq a + b$ is satisfied.

Lemma I. *If $p = 2$, then the generalized harmonic quadruplet $(ABCD)$ with distinct A, B, C satisfies $C = D$.*

Proof. $(4) \Rightarrow 0 = (a - c)^{-1} + (a - d)^{-1} \Rightarrow -(a - c) = a - d \Rightarrow 2a = 0 = c + d \Rightarrow c = d$. Further the proof of lemma 5 may be modified also for $p = 2$. In case 1 lines P_1B, $(P_1C \cap P_2B)\, A$ are parallel and $D = I$. In case 3 we obtain similarly $C = D$.

Lemma II. *Let p be general and* (3) *holds for some $a, b, c, d \in A$, $a \neq c = d \neq b$. Then $p = 2$.*

Proof. $(3) \Rightarrow -(a - c)^{-1}(b - c) = (a - c)^{-1}(b - c) \Rightarrow 2(a - c)^{-1}(b - c) = 0$. By hypothesis $(a - c)^{-1}(b - c) \neq 0$, thus finally $p = 2$.

Still we make this remark: If in the generalized sense $(ABII)$ or $(AIBB)$ holds for same triad A, B, I (A, B are various points distinct from the ideal point I) then $p = 2$.

Therefore we have proved this known geometric result ([9]; for Desarguesian planes also [4], p. 229): If the Generalized Theorem of Complete Quadrilateral is valid universally in given plane and $(ABCC)$ holds for some triad $|A, B, C|$ then: $(V_1V_2V_3V_4)$, $|V_1, V_2, V_3| \Rightarrow V_3 = V_4$.

If we use the notation of Generalized Theorem of Complete Quadrilateral, then $C = D \Leftrightarrow |C, P_2, P_1B \cap A(P_1C \cap P_2B)|$.

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Резюме.

ГАРМОНИЧЕСКАЯ СОПРЯЖЕННОСТЬ НА ПЛОСКОСТИ МУФАНГА

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(Поступило в редакцию 15/III 1954 г.)

В статье исследуется проективная плоскость, в которой универсально справедлива теорема о полном тетраэдре. Пусть $(ABCD)$ обозначает гармоническое отношение точек A, B, C, D. В § 1 доказано геометрическим методом, что отношения $(ABCD)$, $(CDAB)$ эквивалентны.

В § 2 использована аффинная плоскость с координатами из альтернативного тела T характеристики $\neq 2$ (плоскость Муфанга).

Если прямая p образует ось x этой плоскости с несобственной точкой N, тогда для различных точек $A = (a, 0)$, $B = (b, 0)$, $C = (c, 0)$, $D = (d, 0)$, $a, b, c, d \in$ T, справедливы со отношения:

$$(ABCD) \Leftrightarrow -(a-c)^{-1}(a-d) = (b-c)^{-1}(b-d) \Leftrightarrow 2(a-b)^{-1} = \\ = (a-c)^{-1} + (a-d)^{-1} \Rightarrow (a-c)^{-1} + (a-d)^{-1} + (b-c)^{-1} + \\ + (b-d)^{-1} = 0\,, \quad (ABCN) \Leftrightarrow c = \tfrac{1}{2}(a+b)\,.$$

Некоторые результаты § 2 остаются в силе также для характеристики 2. На основании этих результатов простым путем доказан результат Г. Пикэрта: Если на данной плоскости справедлива теорема о полном тетраэдре с линейно зависимыми диагональными точками для одного тетраэдра, то она справедлива универсально.