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ON THE STRUCTURE OF SIMPLE SEMIGROUPS WITHOUT ZERO.

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Dedicated to Prof. Dr. J. Hronec on the occasion of his 70th birthday.

This paper deals with the structure of simple semigroups without zero. A semigroup is called simple if it does not contain non-trivial two-sided ideals. It is shown especially: a simple semigroup, having at least one minimal left ideal, is the class sum of isomorphic groups if and only if it contains at least one idempotent. The paper contains further a study of the structure of minimal left and right ideals and of the so called Suschkewitsch (Сушкевич) kernel of general (i.e. not necessarily simple) semigroups.

By a semigroup we mean a non-vacuous set S of elements $a, b, c, \ldots$ closed under an associative univalent operation: $(ab)\,c = a(bc)$.

In such a system we introduce in usual way the notion of ideals. A non-vacuous subset L of S is called a left ideal if the relation $SL \subseteqq L$ holds, i. e. if for every $s \in S$, $l \in L$, $sl \in L$ holds. A right ideal is a subset $R \subseteqq S$ satisfying the relation $RS \subseteqq R$. A two-sided ideal is a subset, which is both a left and right ideal of S.

The intersection (if it is non-vacuous) and the sum of two (left, right, two-sided) ideals is a (left, right, two-sided) ideal.

An element $z \in S$ with the property $az = za = z$ for every $a \in S$ is called a "zero element". A semigroup contains at most one zero element.

A left (right, two-sided) ideal of the semigroup S is called a minimal ideal of S if it does not contain any proper subset, which is itself a left (right, two-sided) ideal of S.

Two minimal left (right) ideals have no element in common. A semigroup has at most one minimal two-sided ideal. The zero element (if it exists) is clearly a minimal two-sided ideal.

According to Rees [1] a semigroup is called simple if it contains no other two-sided ideals than S itself, except possibly the zero ideal consisting of the single element z (if S has a zero element).

The purpose of this paper is to study the structure of some types of simple semigroups such defined.

Simple semigroups without zero are of greatest importance in studying general (i. e. non necessarily simple) semigroups. If namely the intersection of all two-sided ideals of a semigroup S without a zero element is non-vacuous, then it is a simple semigroup without zero. This intersection is the so called "Suschkewitsch kernel" of S. The study of the kernel is identical with the study of simple semigroups without zero.

Clifford [3] introduced in the study of semigroups two very general conditions, which we shall use in the following formulation.

Condition A. *S is a semigroup without zero having at least one minimal left ideal.*

Condition B. *S is a semigroup without zero having at least one minimal left and at least one minimal right ideal.*

We shall prove: A simple semigroup satisfying Condition ***B*** is a class sum of disjoint isomorphic groups. This result — though not stated explicitly in Clifford [3] — can be deduced from his results [1], [2], [3]. We give first an independent proof of this theorem. The proof based on our Theorems 2,3 and 2,4 below seems to be simpler, since we need not use the notion of "primitive idempotents" and "completely simple semigroups" first introduced by Rees [1].

The form of our proof enables us to show clearly the role of the existence of a minimal right ideal. Then it is easy to find conditions equivalent to Condition ***B***. If S satisfies Condition ***A***, we show that the existence of at least one minimal right ideal is used only in proving the existence of at least one idempotent. On the other side we show that the existence of at least one idempotent (in a simple semigroup satisfying Condition ***A***) implies the existence of at least one minimal right ideal. Hence, in a simple semigroup satisfying Condition ***A***, the existence of at least one idempotent and of at least one minimal right ideal are equivalent.

In addition to these results we get in course of our investigation other important properties of minimal left (right) ideals.

In a forthcoming paper — which I intend to publish in this journal — we shall widely generalise the results of this paper to semigroups having a zero element and — more generally — to semigroups having a Suschkewitsch kernel.

Remark. In what follows we use the following notations. The symbol $A \subset B$ (in contrary to $A \subseteqq B$) means always that A is a proper subset of B. If the validity of the assertion α implies the validity of the assertion β, we shall write $\alpha \Rightarrow \beta$. Other notations have the usual meaning.

1.

In this section we do not suppose first that the semigroup S is simple.

Theorem I,1. (Known, see e. g. CLIFFORD [3], p. 522.) *Let L be a minimal left ideal of the semigroup S. Let c be any element of S. Then Lc is also a minimal left ideal of S.*

Proof. The set Lc is clearly a left ideal of S. Suppose — contrary to our statement — that L^* is a left ideal of S and $L^* \subset Lc$. Let L_1 be the set of all elements $a \in L$ with $ac \in L^*$. The set L_1 is a left ideal of S, since for every $s \in S$ we have $sac \in L^*$, so that also $sa \in L_1$ holds. Since further L is a minimal left ideal of S, we have $L_1 = L$, $Lc = L^*$. Hence Lc does not contain any proper subideal. This is a contradiction to the supposition.

Theorem I,2. (Known, see again CLIFFORD [3], p. 523.) *Let S be a semigroup satisfying Condition* **A**. *Then the sum of all the minimal left ideals of S is a two - sided ideal of S.*

Proof. Let M be the sum of all the minimal left ideals of S, $M = \sum_{\alpha} L_\alpha$. M is clearly a left ideal. We show that M is also a right ideal. Let s be any element $\in S$. It is $Ms = \sum_{\alpha} L_\alpha s$. Every summand $L_\alpha s$ is a minimal left ideal of S. Hence it is yet contained in M (since M was the sum of all the minimal left ideals). The relation $Ms \subseteqq M$ holds for every element $s \in S$, i. e. M is a two-sided ideal of the semigroup S.

Theorem I,3. (See analogously SCHWARZ [1], p. 31.) *A semigroup without zero*[1]) *is the class sum of its minimal left ideals if and only if the following condition holds: every relation of the form*

$$a = xb, \ a, \ b, \ x \in S$$

implies a relation

$$\bar{x}a = b$$

with some $\bar{x} \in S$.

Proof. 1. The condition is necessary. Let S be the sum of its minimal left ideals. Let be $a \in S$. The minimal left ideal containing a is necessarily (a, Sa). Let be $b \neq a$. The element b belongs to the minimal left ideal (b, Sb). Two minimal left ideals are either identical or have no element in common. In the first case $(a, Sa) = (b, Sb)$. Hence if $a \in Sb$ holds, i. e. if there exits an $x \in S$ with $a = xb$, then there holds also $b \in Sa$ i. e. there exists an element $\bar{x}$ satisfying the relation $b = \bar{x}a$. This proves our assertion.

2. The condition is sufficient. We show: if S satisfies the condition given in our theorem, then every element $a \in S$ is contained in some

[1]) For semigroups having a zero element this problem is trivial, since there exists only one minimal left ideal (i. e. the zero ideal) and S reduces to z alone.

minimal left ideal of S. Consider the element a and the ideal $L_a = (a, Sa)$. Suppose — in the way of an indirect proof — that L_a is not a minimal left ideal of S, i. e. L_a contains a subideal L with $L \subset L_a$. We prove that this is impossible.

Let be $b \in L$, $b \neq a$. The ideal $L_b = (b, Sb)$ is a left ideal of S satisfying the relation $L_b \subseteqq L \subset L_a$. Since $b \in L_a$, there holds $b = xa$, $x \in S$. According to the supposition this relation implies a relation $a = \bar{x}b$ with $\bar{x} \in S$, i. e. $a \in Sb$. Therefore

$$a \in L_b = (b, Sb).$$

Multiplying on the left by S, we get

$$\begin{aligned} Sa &\subseteqq (Sb, S^2b) = (Sb), \\ (a, Sa) &\subseteqq (b, Sb), \\ L_a &\subseteqq L_b. \end{aligned}$$

Hence $L_a = L_b$. This is a contradiction to the supposition. The ideal (a, Sa) is a minimal left ideal. Every element $a \in S$ belongs to some minimal left ideal. This proves our theorem.

The right dual theorem, which can be proved analogously, is the following:

Theorem I,4. *A semigroup without zero is a sum of its minimal right ideals if and only if every relation of the form*

$$a = by, \quad a, b, y \in S$$

implies a relation

$$a\bar{y} = b$$

with some $\bar{y} \in S$.

Corollary I,3. *If a semigroup without zero is the class sum of its minimal left ideals, then to every $a \in S$ there exists an element $e \in S$ such that $a = ea$.*

Proof. Since $L_a = (a, Sa)$ is a minimal left ideal, there holds $L_a = Sa$. In fact, the relation $Sa \subset (a, Sa)$ would be a contradiction to the assumption of minimality of the ideal L_a. In other words: for every $a \in S$ there holds $a \in Sa$, i. e. for every $a \in S$ there exists an element e such that $a = ea$.

Analogously:

Corollary I,4. *If a semigroup without zero is the sum of its minimal right ideals, then to every $a \in S$ there exists an element $f \in S$ such that $a = af$ holds.*

2.

In this section we prove first that the conditions of Theorems 1,3 and 1,4 are satisfied only for simple semigroups.

Theorem 2,I. *Let S be a semigroup satisfying Condition* **A**. *Then S is the sum of its minimal left ideals if and only if S is a simple semigroup.*

Proof. 1. Let S be simple. According to the supposition it contains at least one minimal left ideal L. According to Theorem 1,2 the sum of all the minimal left ideals of S is a two-sided ideal M. Hence $M = S$, since $M \subset S$ would be contrary to the simplicity of S.

2. Conversely. Let S be the sum of its minimal left ideals: $S = \sum_\alpha L_\alpha$. Suppose that S has a two-sided subideal M' different from S, i. e.

$$M'S \subseteqq M' \subset S. \tag{1}$$

Since $M'L_\alpha$ is a left ideal of S contained in L_α, hence — with respect to the minimality — equal to L_α, we have

$$M'S = M'\sum_\alpha L_\alpha = \sum_\alpha (M'L_\alpha) = \sum_\alpha L_\alpha = S,$$

contrary to the relation (1). This proves Theorem 2,1.

In the following we treat only simple semigroups.

Theorem 1, 1, Theorem 1,3 and Corollary 1,3 imply:

Theorem 2,2. *Let S be a simple semigroup satisfying Condition* **A**. *Then to every $a \in S$ there exists an $e \in S$ with $a = ea$. Moreover, the following implication holds*:

$$a = xb \Rightarrow \bar{x}a = b \quad (a,\, b,\, x,\, \bar{x} \in S).$$

Using the right dual theorem, we get:

Theorem 2,3. *Let S be a simple semigroup satisfying Condition* **B**. *Then to every $a \in S$ there exist two elements e, $f \in S$ such that $a = ea$, $a = af$. Moreover, the following implications hold*:

$$a = xb \Rightarrow \bar{x}a = b, \tag{2}$$

$$a = by \Rightarrow a\bar{y} = b, \tag{3}$$

$(a,\, b,\, x,\, \bar{x},\, y,\, \bar{y} \in S)$.

The following theorem (which we shall prove very easily) is from the general point of view the most important theorem of this paper.

Theorem 2,4. *Let the suppositions of Theorem 2,3 hold. Then the elements e, f are idempotents.*

Proof. 1. We prove first that e is an idempotent. It follows from the equation $a = ea$, with respect to (3), the existence of an element $\bar{a}$ with $a\bar{a} = e$. Now we can write successively

$$e = a\bar{a} = (ea)\,\bar{a} = e(a\bar{a}) = e\,.\,e = e^2.$$

Hence e is an idempotent.

2. The proof of idempotency of f is analogous. It follows from the equation $a = af$, according to the relation (2), the existence of an element $\bar{a} \in S$ with $\bar{a}a = f$. Hence

$$f = \bar{a}a = \bar{a}(af) = (\bar{a}a)\,f = f^2,$$

which completes the proof.

Remark. Note the following reciprocity. The existence of a minimal left ideal implies the existence of the element e. Its idempotency is proved by means of the relation (3), which itself is a consequence of the existence of minimal right ideals.

Dually: the existence of a minimal right ideal implies the existence of the element f. Its idempotency is proved by means of the relation (2), which itself is a consequence of the existence of minimal left ideals.

Condition ***A*** alone (without further suppositions) is not strong enough to assure the existence of idempotents.

3.

In this section we are studying first the structure of minimal left ideals of a simple semigroup S. The methods of investigation are in some points similar to those used yet earlier in other connections by Suschkewitsch (А. К. Сушкевич) [1], Clifford [1] and Schwarz [1].

It must be noted in advance that in Theorems 3,1—3,7 we use only Condition ***A*** and that consequence of Condition ***B*** that every minimal left ideal has at least one idempotent. Condition ***B*** is not used otherwise explicitly.

We show first that Condition ***B*** really implies that every minimal left ideal L contains at least one idempotent. In fact, let be $a \in L$. According to Theorem 1,1 La is a minimal left ideal of S. Since $La \subseteq L \,.\, L \subseteq L$, we have $La = L$. Hence to the element $a \in L$ there exists an element $e \in L$ such that $ea = a$ holds. According to Theorem 2,4 e is an idempotent.

Theorem 3,1. *Let the simple semigroup S satisfy Condition* ***B***. *Then every minimal left ideal L of S can be written in the form $L = Se$, where e is an idempotent. Analogously: every minimal right ideal R is of the form $R = fS$, where f is an idempotent.*

Proof. Let L be any minimal left ideal. As it was shown above, L contains an idempotent e. Consider the left ideal Se. The following relation holds: $Se \subseteq SL \subseteq L$. Hence (with respect to the minimality of L) $Se = L$, which proves our theorem.

The dual part follows analogously.

Theorem 3,2. *Let S be a simple semigroup satisfying Condition* ***B***. *Then every minimal left ideal L of S is a semigroup having the following properties:*

a) L has at least one right identity.

b) Every idempotent of L is a right identity for every element $\in L$.

c) In L the right cancellation law holds.

Proof. a) Since $L = Se$ (with e idempotent), every $a \in L$ is of the form $a = ue$ $(u \in S)$. Hence $ae = (ue)\,e = ue^2 = ue = a$.

b) There follows from the proof of Theorem 3,1 that for every idempotent $e^* \in L$ the relation $L = Se^*$ holds. This formula together with the statement a) implies the statement b).

c) Note first: since for every $a \in L$, $La = L$, there exists to every $b \in L$ such an element $x \in L$ that $xa = b$ holds. In other words: the equation $xa = b$ has a solution $x \in L$ for every couple $a, b \in L$. Let now be

$$ac = bc \quad (a, b, c \in L). \tag{4}$$

Find an element $\bar{c} \in L$ satisfying the equation $\bar{c}c = e$ (e idempotent $\in L$). Then

$$(c\bar{c})^2 = c(\bar{c}c)\,\bar{c} = ce\bar{c} = c\bar{c}.$$

Hence $c\bar{c} = e^*$ is also an idempotent $\in L$. Multiplying the relation (4) with $\bar{c}$ on the right, we get

$$ac\bar{c} = bc\bar{c},$$
$$ae^* = be^*,$$

and (with respect to the fact proved in b)) $a = b$. This completes the proof of our theorem.

Theorem 3,3. *Let S be a simple semigroup satisfying Condition* **B**. *Let L be a minimal left ideal of S. Let e_α be any idempotent $\in L$. Then $e_\alpha L$ is a group.*

Proof. Denote $\mathfrak{g}_\alpha = e_\alpha L$. Since $\mathfrak{g}_\alpha^2 = e_\alpha L \,.\, e_\alpha L = e_\alpha(Le_\alpha)\,L = e_\alpha L \,.\, L = e_\alpha L = \mathfrak{g}_\alpha$, the set $\mathfrak{g}_\alpha$ is a semigroup.

1. We prove that e_α is the only idempotent $\in \mathfrak{g}_\alpha$. In fact: let $e_\alpha \neq e_\beta$ be two idempotents $\in \mathfrak{g}_\alpha$. Then there would be $e_\beta^2 = e_\beta$ and, since further $e_\beta \in e_\alpha L$, there would be also $e_\alpha e_\beta = e_\beta$. Hence $e_\alpha e_\beta = e_\beta^2$. According to Theorem 3,2 c) this equation implies $e_\alpha = e_\beta$, contrary to the supposition.

2. We prove next that for every $a \in \mathfrak{g}_\alpha$ the equations

$$ya = e_\alpha, \tag{5}$$
$$ax = e_\alpha \tag{6}$$

have solutions with $x, y \in \mathfrak{g}_\alpha$.

a) For every $a \in L$ $La = L$ holds. Multiplying on the left by e_α, we get $e_\alpha L = e_\alpha La$, $e_\alpha \in \mathfrak{g}_\alpha a$. Hence for every $a \in \mathfrak{g}_\alpha$ there exists an $y \in \mathfrak{g}_\alpha$ with $e_\alpha = ya$.

b) We prove the solvability of the equation (6) in the following way. It follows from a) that there exists an $\bar{a} \in \mathfrak{g}_\alpha$ satisfying the equation $\bar{a}a = e_\alpha$. Our statement will be proved if we show that $\bar{a}$ satisfies also the equation $a\bar{a} = e_\alpha$. We have first $a\bar{a} \in \mathfrak{g}_\alpha$, since it is $a \in \mathfrak{g}_\alpha$, $\bar{a} \in \mathfrak{g}_\alpha$ and $\mathfrak{g}_\alpha$ is a semigroup. Secondly, there holds

$$(a\bar{a})^2 = a(\bar{a}a)\,\bar{a} = ae_\alpha\bar{a} = a\bar{a}.$$

Hence $a\bar{a}$ is an idempotent. But since $\mathfrak{g}_\alpha$ contains just one idempotent, we have $a\bar{a} = e_\alpha$, which proves the validity of our statement.

3. We prove at last the uniqueness of the solutions of (5) and (6). This follows immediately for the equation (5), since — according to Theorem 3,2 c) — $y_1 a = y_2 a$ implies $y_1 = y_2$. For the equation (6) we prove it in the following manner. Let be $ax_1 = ax_2 = e_\alpha (x_1, x_2 \in \mathfrak{g}_\alpha)$. Find an element $y \in \mathfrak{g}_\alpha$ with $ya = e_\alpha$. (This is possible with respect to (5).) Multiplying by y on the left, we get $yax_1 = yax_2$, $e_\alpha x_1 = e_\alpha x_2$. Since e_α is clearly a left unity for every element $\in \mathfrak{g}_\alpha$, we have $x_1 = x_2$.

The statements 1, 2, 3 imply that $\mathfrak{g}_\alpha$ is a group with e_α as unity element. Theorem 3,3 is completely proved.

Theorem 3,4. *Let the suppositions of Theorem 3,3 be satisfied. Let e_α, e_β be two different idempotents $\in L$. Then the groups $\mathfrak{g}_\alpha = e_\alpha L$ and $\mathfrak{g}_\beta = e_\beta L$ have no element in common.*

Proof. Suppose, contrary to our assertion, that the intersection $\mathfrak{g}_\alpha \cap \mathfrak{g}_\beta$ is non-vacuous. Let a be an element with $a \in \mathfrak{g}_\alpha$, $a \in \mathfrak{g}_\beta$. It can be written in two forms $a = e_\alpha a = e_\beta a$. According to Theorem 3,2 c) there would be, after cancellation by a on the right, $e_\alpha = e_\beta$. This contradicts our supposition.

Theorem 3,5. *Let the suppositions of Theorem 3,3 be satisfied. Then every element $a \in L$ is contained in some group, i. e. L is a sum of disjoint groups.*

Proof. It is sufficient to prove that every $a \in L$ can be written in the form $a = ea$, where e is an idempotent.[2]) The existence of such an element e follows from the relation $La = L$. Its idempotency follows from the relation $a = ea = e^2 a$, i. e. $ea = e^2 a$, which (with respect to Theorem 3,2 c)) implies $e = e^2$.

Remark. The groups of Theorem 3,5 are uniquely determined. Let be — contrary to this assertion — $L = \sum_\alpha \mathfrak{g}_\alpha$, $L = \sum_\alpha \bar{\mathfrak{g}}_\alpha$ two different decompositions of L into a sum of groups. Then there exists at least one element a belonging to two different groups $\mathfrak{g}_\beta$, $\bar{\mathfrak{g}}_\gamma$. Hence we can write $a = e_\beta a$, $a = \bar{e}_\gamma a$ (e_β, $\bar{e}_\gamma$ unity elements of the groups $\mathfrak{g}_\beta$, $\bar{\mathfrak{g}}_\gamma$). But $e_\beta a = \bar{e}_\gamma a$ implies $e_\beta = \bar{e}_\gamma$, hence $\mathfrak{g}_\beta = e_\beta L = \bar{e}_\gamma L = \bar{\mathfrak{g}}_\gamma$, contrary to the supposition.

Theorem 3,6. *The groups of Theorem 3,4 (or Theorem 3,5) are all isomorphic together.*

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[2]) This is simply the statement of Theorem 2,3. But in this section 3 (according to the agreement made at the beginning of the section) we use only the fact that every minimal left ideal L has at least one idempotent (without using otherwise Condition B explicitly).

Proof. 1. Let e_α, e_β be two idempotents $\epsilon\, L$. Let X be the set of all elements $\epsilon\, L$ such that $e_\alpha \,.\, X$ equals a single fixed element $x \,\epsilon\, \mathfrak{g}_\alpha$. Then it is

$$e_\beta X = (e_\beta e_\alpha)\, X = e_\beta(e_\alpha X) = e_\beta x,$$

i. e. $e_\beta X$ is again a single element $\epsilon\, \mathfrak{g}_\beta$.[3]) (In fact, it equals the element $e_\beta x$.) Conversely, let X' be the set of all elements $\epsilon\, L$, for which the relation $e_\beta X' = e_\beta x$ holds. Clearly $X' \supseteqq X$. On the other side

$$e_\alpha X' = (e_\alpha e_\beta)\, X' = e_\alpha(e_\beta X') = e_\alpha(e_\beta x) = (e_\alpha e_\beta)\, x = e_\alpha x = x$$

(the last relation being satisfied, since it is $x \,\epsilon\, \mathfrak{g}_\alpha$). Therefore $X' \subseteqq X$, whence $X' = X$.

As a consequence of the state just proved, we can decompose L in a sum of disjoint sets $X, Y, Z, \ldots$ having the following property: if we multiply any of these sets by any idempotent $\epsilon\, L$, we get always a single element $\epsilon\, L$. (For different idempotents and different sets we get — in general — different elements.)

2. Let now be x any element $\epsilon\, \mathfrak{g}_\alpha$. Let be $x = e_\alpha \xi$, $\xi \,\epsilon\, L$. We show that the correspondence

$$x \to y = e_\beta \xi \tag{7}$$

is an isomorphic mapping of the group $\mathfrak{g}_\alpha$ to the group $\mathfrak{g}_\beta$. The element $y = e_\beta \xi$ belongs to $\mathfrak{g}_\beta$. According to 1. above the element y (result of the multiplication of e_β and ξ) does not depend on the choice of the element $\xi \,\epsilon\, L$.[4])

Let $x_1 = e_\alpha \xi_1$, $x_2 = e_\alpha \xi_2$ be two elements $\epsilon\, \mathfrak{g}_\alpha$. The elements corresponding to them in the mapping (7) are $y_1 = e_\beta \xi_1$, $y_2 = e_\beta \xi_2$ ($y_1, y_2 \,\epsilon\, \mathfrak{g}_\beta$). To the product $x_1 x_2 = e_\alpha \xi_1 \,.\, e_\alpha \xi_2 = e_\alpha \xi_1 \xi_2$ correspond in the same mapping the element $e_\beta \xi_1 \xi_2$. But this element equals just $y_1 y_2$, since $e_\beta \xi_1 \xi_2 = e_\beta \xi_1 \,.\, e_\beta \xi_2 = y_1 y_2$. Hence the mapping (7) is a homomorphism of the group $\mathfrak{g}_\alpha$ on the group $\mathfrak{g}_\beta$. Since further $\mathfrak{g}_\beta = e_\beta L$, every element $\epsilon\, \mathfrak{g}_\beta$ is image of some element $x \,\epsilon\, \mathfrak{g}_\alpha$. To show that the correspondence (7) is an isomorphism, it is sufficient to prove that $x_1 \neq x_2$ implies $y_1 \neq y_2$. In fact, let be $e_\alpha \xi_1 \neq e_\alpha \xi_2$ but — contrary to this statement — $e_\beta \xi_1 = e_\beta \xi_2$. Multiplying the last relation on the left by e_α, we get $e_\alpha e_\beta \xi_1 = e_\alpha e_\beta \xi_2$, hence $e_\alpha \xi_1 = e_\alpha \xi_2$, which contradicts our supposition. Theorem 3,6 is completely proved.

Theorems 3,3—3,6 give together:

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[3]) We use here and in the following the fact that e_α, e_β are right unity elements in L.

[4]) Since for the set X, for which $e_\alpha X = x$ holds, the product $e_\beta X$ equals also a single element $\epsilon\, L$ (in our case the element y).

Theorem 3,7. *Let S be a simple semigroup satisfying Condition* **B**. *Then every minimal left ideal L of S is a sum of isomorphic groups.*

We shall call these groups "group-components" of L.

Analogous result holds naturally for minimal right ideals.

Since a simple semigroup satisfying Condition **B** is the class sum of its minimal left (right) ideals, there follows from Theorem 3,7 immediately that S is a sum of disjoint groups. We can sharpen this result by proving the following:

Theorem 3,8. *Let S be a simple semigroup satisfying Condition* **B**. *Then S is the class sum of disjoint isomorphic groups.*

Proof. Let be $S = \sum_{\xi} L^{(\xi)}$ ($\xi = \alpha, \beta, \gamma, \ldots$), where $L^{(\xi)}$ runs through all minimal left ideals of S. We know that $L^{(\xi)} = \sum_{\eta} \mathfrak{g}_{\eta}{}^{(\xi)}$, where $\mathfrak{g}_{\eta}{}^{(\xi)}$ are isomorphic groups. Since the group-components $\mathfrak{g}_{\eta}{}^{(\xi)}$ of every $L^{(\xi)}$ are isomorphic together, it is sufficient to show that in every $L^{(\xi)}$ ($\xi = \alpha, \beta, \gamma, \ldots$) there exists at least one group-component, which is isomorphic with the group-components of one fixed minimal left ideal, say the ideal $L^{(\alpha)}$.

Let R be a minimal right ideal. If the element $c \in \mathfrak{g}_{\eta}{}^{(\xi)}$ belongs to the ideal R, then all elements $\in \mathfrak{g}_{\eta}^{(\xi)}$ belong also to R. As a matter of fact, the relation $c \in R$ implies $cS \subseteqq RS \subseteqq R$, hence $c(\mathfrak{g}_{\eta}^{(\xi)} + \ldots) \subseteqq R$ and therefore $\mathfrak{g}_{\eta}{}^{(\xi)} \subseteqq R$.

Let us write further

$$R = R \cap S = R \cap \sum_{\xi} L^{(\xi)} = \sum_{\xi} (R \cap L^{(\xi)}) = (R \cap L^{(\alpha)}) + \\ + (R \cap L^{(\beta)}) + \ldots \tag{8}$$

Since $RL^{(\xi)} \subseteqq R \cap L^{(\xi)}$, every summand on the right hand side is non-vacuous. According to the remark just made, $R \cap L^{(\xi)}$ is a group or (perhaps) a sum of groups $\mathfrak{g}_{\eta}{}^{(\xi)}$, where $\mathfrak{g}_{\eta}{}^{(\xi)}$ runs through some of the group-components of the ideal $L^{(\xi)}$.*) (Especially $R \cap L^{(\alpha)}$ is a sum of some group-components $\mathfrak{g}_{\eta}^{(\alpha)}$ of the ideal $L^{(\alpha)}$.) Hence R is a sum of groups, this sum containing group-components of every $L^{(\xi)}$ ($\xi = \alpha, \beta, \gamma, \ldots$).

But on the other side we know that R is a sum of isomorphic groups. Therefore (since this decomposition is uniquely determined) all groups on the right-hand side of the relation (8) are isomorphic with $\mathfrak{g}_{\eta}^{(\alpha)}$ (group-component of $L^{(\alpha)}$). We proved: in every ideal $L^{(\xi)}$ there exists a group-component isomorphic with the group $\mathfrak{g}_{\eta}{}^{(\alpha)}$. This completes the proof of Theorem 3,8.

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*) It can be proved that $R \cap L^{(\xi)}$ is a single group.

4.

In proving Theorems 3,1—3,8 we used the existence of a minimal right ideal just two times:

a) first, to assure the existence of idempotents in every minimal left ideal of S,

b) secondly (in Theorem 3,8), to prove that the group-components of all ideals of S are isomorphic together.

This fact anticipates that Theorem 3,8 will be perhaps true if only Condition **A** with some supplementary suppositions will be satisfied. The weakest possible supposition is: S has at least one idempotent. We show that this supposition is really sufficient. The following theorem holds:

Theorem 4,1. *Let S be a simple semigroup satisfying Condition* **A**. *The necessary and sufficient condition that S should be a sum of isomorphic groups is: S has at least one idempotent.*

Proof. 1. The condition is necessary, since a group must contain an idempotent.

2. We prove that the condition is sufficient. Let S contain the idempotent e. Since Condition **A** holds, there exists a minimal left ideal L containing e. For this ideal there holds $L = Le$; moreover, L is a sum of (isomorphic) groups. To prove our theorem it is sufficient to show: S has also at least one minimal right ideal. For then the suppositions of Theorem 3,8 are satisfied.

We prove that $R = eS$ is a minimal right ideal of the semigroup S. This follows indirectly. Suppose that $R = eS$ contains a proper subideal R', $R' \subset R$. Since $R'L \subseteq R' \cap L$, the intersection $L \cap R'$ is non-vacuous. Let a be any element $\in L \cap R'$. Clearly $aS \subseteq R' \subset eS$. The element a (belonging to L) belongs also to some group $\mathfrak{g}_\alpha$, group-component of the ideal L. With a belong to R' also all elements aS and the more so all elements of the group $\mathfrak{g}_\alpha$. Therefore $\mathfrak{g}_\alpha \subseteq L \cap R'$. Let e_α be the unity element of this group. We have $\mathfrak{g}_\alpha \subseteq L \cap R' \subseteq R' \subset eS$. Since $a \in eS$, we have $ea = a$. Since further a belongs to $\mathfrak{g}_\alpha$, there holds also $e_\alpha a = a$. The elements e, e_α, a are contained in L and satisfy the equation $ea = e_\alpha a$. But in L the right cancellation law holds; therefore $e = e_\alpha$.

Find now in $\mathfrak{g}_\alpha$ an element $\bar{a}$ with $a\bar{a} = e$. Then it holds

$$R = eS = a\bar{a}S \subseteq aS \subseteq R'.$$

The result $R = eS \subseteq R'$ is a contradiction to the assumption $R' \subset R$. This proves Theorem 4,1.

We obtained a very interesting result. Let S be a simple semigroup satisfying Condition **A**. Then the suppositions

a) S contains at least one idempotent,

b) S contains at least one minimal right ideal, are equivalent. In fact: it was proved in Theorem 2,4 that b) implies a); conversely, we proved in Theorem 4,1 that a) implies b).

5.

We prove now one result concerning general (i. e. not necessarily simple) semigroups, which is an immediate consequence of our theorems and is itself of greatest interest.

Theorem 5,1. *Let S be a semigroup without zero having at least one minimal left ideal. If one of the minimal left ideals has at least one idempotent, then every minimal left ideal has idempotents. Moreover, S contains minimal right ideals, each of which has idempotents. The sum of all the minimal left ideals is identical with the sum of all the minimal right ideals. At last, every minimal left and right ideal is generated by idempotents.*[5])

Proof. A semigroup without zero having a minimal left ideal has a Suschkewitsch kernel K. (See e. g. CLIFFORD [3].) From Theorems 1,2 and 2,1 follows that the kernel is the sum of all the minimal left ideals and that K is a simple semigroup. One proves easily that every minimal left ideal of S is also a minimal left ideal of K. According to the supposition K has an idempotent. Hence K has at least one minimal right ideal R of K. One shows easily again that R (and every minimal right ideal of K) is also a minimal right ideal of S. Hence: K is a sum of isomorphic groups. Every minimal left and right ideal of S contains idempotents and is generated by them.

Corollary 5,1. *Let S be a semigroup without zero having at least one minimal left ideal. If one of the minimal left ideals contains at least one element of finite order, then every minimal left and right ideal of S is generated by idempotents.*

Proof. Let a be an element of finite order, which is contained in the minimal left ideal L. All powers of $a : a, a^2, a^3, \ldots$ belong to L. But it is known (and elementary to prove[6])) that for some integer $s > 0$ a^s is an idempotent. Hence the suppositions of Theorem 5,1 are satisfied and Corollary 5,1 holds.

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[5]) We say that the ideal L is generated by the idempotent e, if $L = Se$ holds.

[6]) See e. g. DUBREIL [1], p. 38.

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