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Simple polynomials

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# SIMPLE POLYNOMIALS.

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We say that (1.1) is a simple polynomial in the closed circle $|z| \leqq 1$, if there is a $r > 1$ such that $P(z_1) \neq P(z_2)$ for $z_1 \neq z_2$, $|z_1| < r$, $|z_2| < r$. The necessary and sufficient condition for (1.1) to be simple is that the system of equations (2.5) has no solution $(u, x)$ such that $-2 \leqq u \leqq 2$. Eliminating $x$ from (2.5), we get the condition in an algebraic form: the resultant of (2.5) does not vanish in the interval $-2 \leqq u \leqq 2$. Detailed discussion of $P(z) = z + a_2z^2 + a_3z^3$.

**I. The fundamental theorem.** If a polynomial

$$w = P(z) = z + a_2z^2 + a_3z^3 + \ldots + a_nz^n, \ |a_n| > 0 \qquad (1.1)$$

sets up a one-to-one corespondence between the domain $|z| < r > 1$ and a certain domain $W$ of the $w$ plane, we will say that (1.1) is simple within and on the closed circle $|z| \leqq 1$. In this definition the condition $r > 1$ is essential.

Our aim is to deduce the necessary and sufficient conditions for the coefficients of such polynomials. If $P(z)$ is such a polynomial, then the increment ratio

$$\Delta(z_1, z_2) = \frac{P(z_1) - P(z_2)}{z_1 - z_2} = 1 + a_2(z_1 + z_2) + \ldots + \\ + a_n(z_1^{n-1} + z_1^{n-2}z_2 + \ldots + z_2^{n-1}) \qquad (1.2)$$

is different from zero for every two different or equal values $z_1, z_2$ within or on the closed circle $|z| \leqq 1$ and this is the only condition for the simplicity of $P(z)$. But it is not necessary to study all such couples of numbers $z_1, z_2$ because if the condition (1.2) $\Delta(z_1, z_2) \neq 0$ is satisfied for every two different or equal values $z_1, z_2$ on the circle $|z| = 1$, then and only then the polynomial $P(z)$ is simple within and on the circle $|z| \leqq 1$. It is obvious that this condition is necessary for the simplicity on the circle. But on the other hand if this condition is satisfied on the circle $|z| = 1$, then the boundary of the closed domain $W$ is a closed contour $C$

$$w_1 = P(e^{i\varphi}), \ 0 \leqq \varphi \leqq 2\pi$$

where every value $w_1$ is attained only once in the interval $0 \leqq \varphi < 2\pi$, which means that $C$ is a simple closed contour. It is quite easy to show that this property of the contour $C$ is sufficient for the simplicity of $P(z)$ within and on the circle $|z| \leqq 1$.[1]

Now suppose that $P(z)$ is not simple in the closed circle $|z| \leqq 1$. Then there must exist at least two different or equal numbers $z_1, z_2$ on the circle $|z| = 1$ for which the ratio (1.2) vanishes

$$\left.\begin{aligned} &\Delta(z_1, z_2) = 1 + \sum_2^n a_k(z_1^{k-1} + z_1^{k-2}z_2 + \ldots + z_2^{k-1}) = 0 \\ &\text{and therefore the conjugate complex number} \\ &\overline{\Delta}(\bar{z}_1, \bar{z}_2) = 1 + \sum_2^n \bar{a}_k\left(\frac{1}{z_1^{k-1}} + \frac{1}{z_1^{k-2}z_2} + \ldots + \frac{1}{z_2^{k-1}}\right) = 0 \end{aligned}\right\} \tag{1.3}$$

vanishes too. The two algebraic equations (1.3) for $z_1, z_2$ or, what is the same, the two equations

$$\left.\begin{aligned} &1 + \sum_2^n a_k(z_1^{k-1} + z_1^{k-2}z_2 + \ldots + z_2^{k-1}) = 0 \\ &z_1^{n-1} \cdot z_2^{n-1} + \sum_2^n \bar{a}_k z_1^{n-k} z_2^{n-k}(z_1^{k-1} + z_1^{k-2}z_2 + \ldots + z_2^{k-1}) = 0 \end{aligned}\right\} \tag{1.3$'$}$$

must therefore be satisfied by at least one couple of numbers $z_1, z_2$ on the circle $|z| = 1$. If such a couple does not exist, then and only then the polynomial $P(z)$ is simple within and on the circle $|z| \leqq 1$.

The first of the equations (1.3) is identical with the associated equation of J. DIEUDONNÉ.[2] We shall therefore denominate the system (1.3) as the system of two associated equations to the polynomial $P(z)$ (1.1).

Therefore our fundamental theorem runs as follows: The polynomial (1.1) is simple within and on the circle $|z| \leqq 1$ if and only if there exists no couple of numbers $(z_1, z_2)$ satisfying the associated system (1.3) and such that $|z_1| = |z_2| = 1$.

A further aim of this investigation is to simplify this theorem.

**II. Some properties of the associated system.** This system is symmetrical in $z_1$ and $z_2$. Thus if $(z_1, z_2)$ is a solution, then $(z_2, z_1)$ is a solution too. But it is quite easy to show that also $\left(\frac{1}{\bar{z}_1}, \frac{1}{\bar{z}_2}\right)$ and $\left(\frac{1}{\bar{z}_2}, \frac{1}{\bar{z}_1}\right)$ are

[1] The proof of this theorem due to G. DARBOUX (Leçons sur la théorie générale des surfaces, T. l. p. 173) can be found also in every textbook e. g. E. T. COPSON: An introduction to the theory of functions, p. 184.

[2] J. DIEUDONNÉ: Polynomes et fonctions bornées. Annales de l'école norm. sup. (3) **48**, 1931.

solutions of the system. If $(z_1, z_2)$ is a solution of (1.3), then also the conjugate numbers of the left sides of (1.3) are zero

$$1 + \sum_{2}^{n} \bar{a}_k(\bar{z}_1^{k-1} + \bar{z}_1^{k-2} . \bar{z}_2 + \ldots + \bar{z}_2^{k-1}) = 0$$

$$1 + \sum_{2}^{n} a_k\left(\frac{1}{\bar{z}_1^{k-1}} + \frac{1}{\bar{z}_1^{k-2} . \bar{z}_2} + \ldots + \frac{1}{\bar{z}_2^{k-1}}\right) = 0.$$

This proves the asertion that $\left(\frac{1}{\bar{z}_1}, \frac{1}{\bar{z}_2}\right)$ is a solution of (1.3). Knowing this we can simplify the fundamental theorem. If $P(z)$ is simple then it is obvious that the system cannot have a solution $(z_1, z_2)$ such that $|z_1| \leqq 1, |z_2| \leqq 1$ neither can it have a solution with $|z_1| \geqq 1, |z_2| \geqq 1$. But the same reasoning shows that the possibility $|z_1| = 1, |z_2| \neq 1$ is excluded too. Therefore the fundamental theorem can be stated as follows:

The polynomial (1.1) is simple within and on $|z| \leqq 1$ if and only if all solutions $(z_1, z_2)$ of the system (1.3) have the property

$$|z_1| < 1, \ |z_2| > 1 \ \text{or} \ |z_1| > 1, \ |z_2| < 1. \tag{2.1}$$

The method of resolving the assoc. system (1.3) is a purely algebraical problem and the symmetry of the equations suggests that the substitution $z_1 = x . y$, $z_2 = \frac{x}{y}$ can simplify the equations. If we use the substitution then both numbers $z_1$ and $z_2$ have a modulus equal to unity if and only if $|x| = 1$ and $|y| = 1$. The assoc. system appears now in the form

$$\begin{aligned} 1 + \sum_{2}^{n} a_k x^{k-1}\left(y^{k-1} + y^{k-3} + \ldots + \frac{1}{y^{k-3}} + \frac{1}{y^{k-1}}\right) &= 0 \\ x^{n-1} + \sum_{2}^{n} \bar{a}_k x^{n-k}\left(y^{k-1} + y^{k-3} + \ldots + \frac{1}{y^{k-3}} + \frac{1}{y^{k-1}}\right) &= 0 \end{aligned} \tag{2.2}$$

A further substitution

$$y + \frac{1}{y} = u \tag{2.3}$$

is obvious. The functions in the brackets are

$$y + \frac{1}{y} = u = P_1(u), \ y^2 + 1 + \frac{1}{y^2} = u^2 - 1 = P_2(u),$$

$$\begin{aligned} & y^{k-1} + y^{k-3} + \ldots + \frac{1}{y^{k-3}} + \frac{1}{y^{k-1}} = \\ & = \sum_{r=0}^{r \leq \frac{k-1}{2}} (-1)^r \binom{k-r-1}{r} u^{k-2r-1} = P_{k-1}(u) \end{aligned} \tag{2.4}$$

Instead of (2.2) we have now

$$1 + \sum_{2}^{n} a_k x^{k-1} P_{k-1}(u) = 0, \ \ x^{n-1} + \sum_{2}^{n} \bar{a}_k x^{n-k} P_{k-1}(u) = 0 \qquad (2.5)$$

If the polynomial (1.1) is simple, then there exists no solution $(u, x)$ of the system (2,5) such that $u$ is a real number of the interval $-2 \leq u \leq 2$. This condition is necessary and sufficient for the simplicity of (1.1). The necessity of this condition is obvious, because if such a couple $(u, x)$ where $-2 \leq u \leq 2$ exists, then, as consequence of (2.3), the number $y$ has a modulus equal to unity and $z_1 = x \cdot y$, $z_2 = \frac{x}{y}$. That means $|z_1| = |z_2| = |x|$ which contradicts (2.1); therefore the polynomial (1.1) is not simple. The sufficiency of the condition follows from the fact that if $u$ is always either a real number $|u| > 2$ or a complex number then $y$ has a modulus different from unity and therefore $\left|\frac{z_1}{z_2}\right| = |y^2| \neq 1$. The meaning of this is that the assoc. system (1.3) has no solution $(z_1, z_2)$ with $|z_1| = |z_2| = 1$ and the polynomial (1.1) is simple.

The decision about the simplicity of the polynomial depends therefore only on the properties of the numbers $u$ in all possible couples $(u, x)$ and does not depend on the properties of $x$. The number of solutions $(u, x)$ of the equations (2.5) can be finite or infinite. The last case takes place if and only if the left sides of (2.5) have a nonconstant common divisor. In this case the common divisor is a polynomial in $u$ and $x$.

**III. The associated resultant.** If the equations (2.5) have not a common divisor, then there exists only a finite number of solutions $(u, x)$ of the system. The numbers $u$ in these couples are roots of the resultant. To form the resultant we eliminate, using the method of Euler or Bézout, the number $x$ from the equations. The resultant is a polynomial of a degree not exceeding $N = 2(n - 1)^2$ in $u$:

$$R(u) = A_0 u^N + A_1 u^{N-1} + \ldots + A_{N-1} u + A_N = 0 \qquad (3.1)$$

where $A_k$ are polynomials of the coefficients

$$a_2, \bar{a}_2, a_3, \bar{a}_3, \ldots, a_n, \bar{a}_n$$

and all $A_k$ are real numbers. For special values of the coefficients $a_k$ the degree of $R(u)$ can be less then $2(n - 1)^2$. If e. g. $a_k = 0$ for $k = 2, 3, \ldots, n - 1$ and $|a_n| > 0$, then the equations (2.5) are

$$1 + a_n x^{n-1} P_{n-1}(u) = 0, \ \ x^{n-1} + \bar{a}_n P_{n-1}(u) = 0$$

and the resultant is $\quad R(u) = 1 - |a_n|^2 P_{n-1}^2(u)$

the degree of which is $N = 2(n - 1)$ only.

In the case when no common divisor exists, the necessary and sufficient condition for the simplicity of (1.1) is therefore:

The resultant (3.1) has no root in the interval $-2 \leqq u \leqq 2$. Now suppose that the polynomials (2.5), which are non-identical, have a greatest common divisor $Q(x, u)$, where $Q(x, u)$ is a polynomial in $x$ and $u$. We shall prove that in this case (1.1) is not simple. Is the polynomial $Q$ of degree $n_1 \geqq 1$ in $u$ and of degree $n_2 \geqq 1$ in $x$, then we can always find a couple $(x_1, u_1)$ where $-2 < u_1 < 2$ and $x_1 \neq 0$ so that $Q(x_1, u_1) = 0$. But then the number $y$ defined by $u_1 = y + \dfrac{1}{y}$ has modulus $|y| = 1$ and therefore, as we have proved by (2.5), the polynomial (1.1) is not simple. The same reasoning holds if $n_1 = 0$, $n_2 \geqq 1$. In the remaining case $n_1 \geqq 1$, $n_2 = 0$ the common divisor is a polynomial $Q(u)$ of $u$ only. Is now $u_1$ some root of $Q(u) = 0$ then the first equation (2.5) should be satisfied by $u_1$ and every $x$ e. g. $x = 0$, which is impossible. Therefore the divisor $Q$ cannot be a polynomial in $u$ only. But it is a well known theorem of algebra that the divisor $Q(x, u)$ exists if and only if in the resultant (3.1) all coefficients $A_k$ are zero. Thus the final form of the fundamental theorem is as follows:

*The polynomial (1.1) $P(z)$ is simple within and on the circle $|z| \leqq 1$ if and only if the associated resultant (3.1) $R(u)$ does not vanish identically and has no real root lying in the interval $-2 \leqq u \leqq 2$.*

It is very probable that the resultant $R(u)$ can never vanish identically. But I cannot prove this hypothesis.

As the coefficients $A_k$ of the resultant are real numbers, the determination of the number of roots in the interval $-2 \leqq u \leqq 2$ is a finite and purely algebraical question. We can e. g. use the theorem of Sturm or some equivalent method. But all these methods are very far from being satisfactory to our purpose. I do not know a general theorem of algebra which enables us to ascertain the *nonexistence* of roots in a given interval.

**IV. Some applications.** For the polynomial $P(z) = z + a_2 z^2$ the assoc. system (2.5) is

$$1 + a_2 x u = 0, \quad x + \bar{a}_2 u = 0$$

and the ass. resultant (3.1)

$$1 - a_2 \bar{a}_2 u^2 = 0.$$

The roots of this equation are $u_{1,2} = \pm \dfrac{1}{|a_2|}$ and therefore the polynomial is simple if $|a_2| < \frac{1}{2}$. Is $|a_2| = \frac{1}{2}$ then the polynomial is simple for

$|z| < 1$ and on the limiting contour $w = P(e^{i\varphi})$ the derivative $P'(z)$ vanishes for $z = -\frac{1}{2a_2}$. This result is trivial and follows directly from the condition (1.2)

$$\Delta(z_1, z_2) = 1 + a_2(z_1 + z_2) \neq 0$$

for every

$$|z_1| \leqq 1, \ |z_2| \leqq 1.$$

In a polynomial of third degree

$$P(z) = z + a_2 z^2 + a_3 z^3, \tag{4.1}$$

we can always suppose that $a_3$ is real and positive, because if (4.1) is simple then also $e^{-i\varphi}P(ze^{i\varphi})$ is simple for every $\varphi$. The ass. resultant (3.1) is

$$R(u) = (1 - a_3^2(u^2 - 1)^2)^2 - u^2(a_2 - a_3\bar{a}_2(u^2 - 1)) \,.$$
$$. \,(\bar{a}_2 - a_3 a_2(u^2 - 1)) = 0 \tag{4.2}$$

The condition for simplicity is therefore that this equation has not a single root $-2 \leqq u \leqq 2$. Is specially $a_2$ a real number the resultant is

$$(1 - a_3(u^2 - 1))^2\{(1 + a_3(u^2 - 1))^2 - a_2^2 u^2\} = 0 \tag{4.3}$$

The roots of the first bracket are given by $u^2 = 1 + \frac{1}{a_3}$. It is always $u^2 > 1$ because $a_3 > 0$. The first necessary condition is therefore

$$1 + \frac{1}{a_3} > 4, \ a_3 < \tfrac{1}{3}. \tag{4.4}$$

The second bracket of (4.3) represents two equations

$$a_3 u^2 \pm a_2 u + (1 - a_3) = 0 \tag{4.5}$$

The roots are complex if $a_2^2 < 4a_3(1 - a_3)$ and are real if $a_2^2 \geqq \geqq 4a_3(1 - a_3)$. In this case the four roots of (4.5) are

$$u = \frac{1}{2a_3}\{\pm a_2 \pm \sqrt{a_2^2 - 4a_3(1 - a_3)}\}.$$

The smallest modulus of these roots must be greater than 2:

$$|a_2| - \sqrt{a_2^2 - 4a_3(1 - a_3)} > 4a_3$$

or

$$0 \leqq \sqrt{a_2^2 - 4a_3(1 - a_3)} < |a_2| - 4a_3.$$

That is

$$4a_3 < |a_2| < \frac{1 + 3a_3}{2}, \ a_3 < \frac{1}{5}.$$

The n. a. s. conditions for the simplicity of $P(z)$ are therefore

$$\left.\begin{array}{l}\text{If } 0 < a_3 < \frac{1}{5} \text{ then } |a_2| < \frac{1}{2}(1+3a_3) < \frac{4}{5} \\ \text{If } \frac{1}{5} < a_3 < \frac{1}{3} \text{ then } a_2^2 < 4a_3(1-a_3) < \frac{8}{9}\end{array}\right\} \tag{4.6}$$

If $a_3 = \frac{1}{5}$ both bounds for $|a_2|$ in (4.6) are the same $|a_2| < \frac{4}{5}$.

The space of admissible coefficients $(a_3, a_2)$ is therefore limited by two straight lines

$$a_2 = \pm \tfrac{1}{2}(1 + 3a_3) \text{ in the interval } 0 \leqq a_3 \leqq \tfrac{1}{5},$$

by two arcs of the ellipse

$$a_2 = \pm \sqrt{4a_3(1-a_3)} \text{ if } \tfrac{1}{5} \leqq a_3 < \tfrac{1}{3}$$

and by the ordinate $a_3 = \frac{1}{3}$.

The ellipse touches the straight lines in the points $a_3 = \frac{1}{5}$, $a_2 = \pm \frac{4}{5}$. If the point $(a_3, a_2)$ is situated on the boundary of the space then for $r < 1$ the point $(a_3r^2, a_2r)$ belongs to the space and therefore $P(z)$ is simple within $|z| < 1$ but looses the simplicity on the circle $|z| = 1$. The radius of conformal mapping is therefore exactly equal to unity. The reason for the loss of simplicity on the circle $|z| = 1$ is different according to the case considered. In the case $0 < a_3 \leqq \frac{1}{5}$, $|a_2| = \frac{1}{2}(1+3a_3)$ one root of the equation $P'(z) = 0$ is $z = -1$. Both roots of $P'(z) = 0$ are situated on $|z| = 1$ if $a_3 = \frac{1}{3}$, $|a_2| \leqq \dfrac{2\sqrt{2}}{3}$. But if $\frac{1}{5} < a_3 < \frac{1}{3}$, $|a_2| = \sqrt{4a_3(1-a_3)}$ then the contour $w = P(e^{i\varphi})$ has a point of contact, but both roots of $P'(z)$ are $|z_{1,2}| > 1$.

This result enables us to calculate the radius $\varrho$ of conf. mapping for every polynomial

$$H(z) = z + A_2z^2 + A_3z^3, \tag{4.7}$$

where $A_3 > 0$ and $A_2$ is real. Is $\varrho$ this radius, then the polynomial

$$\frac{1}{\varrho}H(\varrho z) = z + A_2\varrho z^2 + A_3\varrho^2z^3$$

must have the radius of conf. mapping equal to unity. An easy calculus shows that

$$\left.\begin{array}{lll}\text{if } A_2^2 \geqq \frac{16}{5}A_3 & \text{then } \varrho = 1 : \{|A_2| + \sqrt{A_2^2 - 3A_3}\} \\ \text{if } \frac{16}{5}A_3 \geqq A_2^2 > \frac{8}{3}A_3 & \text{then } \varrho = \sqrt{4A_3 - A_2^2} : 2A_3 \\ \text{if } \frac{8}{3}A_3 \geqq A_2^2 \geqq 0 & \text{then } \varrho = 1 : \sqrt{3A_3}\end{array}\right\} \tag{4.8}$$

After this digression we return to the discussion of the resultant (4.2) if $a_2$ is a complex number

$$a_2 = \varrho e^{i\varphi},\ \varrho \geqq 0,\ \varphi \neq 0, \pi,\ \cos 2\varphi = \tau. \tag{4.9}$$

The resultant is now

$$R(u) = (1 - a_3^2(u^2 - 1)^2)^2 -$$
$$- \varrho^2 u^2 (1 - 2\tau a_3 (u^2 - 1) + a_3^2 (u^2 - 1)^2) = 0 \qquad (4.10)$$

Is $a_3 \geqq \frac{1}{3}$ then for $u^2 = 0$ the left side of (4.10) is zero or positive and for $u^2 = 1 + \frac{1}{a_3}$ the left side is negative and therefore the resultant has a root $0 \leqq u^2 \leqq 4$ and the polynomial is not simple. The first necessary condition is therefore

$$a_3 < \tfrac{1}{3} \qquad (4.11)$$

Now the left side of (4.10) must remain positive in the interval $0 \leqq \leqq u^2 \leqq 4$. Therefore the nec. a. s. conditions for the simplicity are

$$a_3 < \tfrac{1}{3}, \quad \varrho^2 < \inf_{0 \leq u^2 \leq 4} \frac{(1 - a_3^2(u^2 - 1)^2)^2}{u^2(1 - 2a_3\tau(u^2 - 1) + a_3^2(u^2 - 1)^2)}, \qquad (4.12)$$

where the symbol $\inf\limits_{a \leq x \leq b} f(x)$ denotes the lower bound of $f(x)$ for all $a \leqq \leqq x \leqq b$. Is the lower bound attained for $u^2 = 4$, then the n. a. s. conditions are

$$a_3 < \tfrac{1}{3}, \ \varrho^2 < \frac{(1 - 9a_3^2)^2}{4(1 - 6a_3\tau + 9a_3^2)}. \qquad (4.13)$$

This occurs e. g. always when $\tau = \cos 2\varphi \leqq 0$. A nearer analysis, which I do not intend to reproduce here, shows that the conditions (4.13) are n. a. s. not only for negative $\tau$ but also for all

$$-1 \leqq \tau \leqq \sqrt{\tfrac{15}{7}} \cdot \tfrac{43}{63} = 0.999136 \ldots \qquad (4.14)$$

The space of admissible values of $\varrho$ is limited by the axis $\varrho = 0$ and the curve

$$\varrho = \frac{1 - 9{a_3}^2}{2\sqrt{1 - 6a_3\tau + 9a_3^2}}. \qquad (4.15)$$

But if $\tau$ is greater than the bound (4.14) then the second condition (4.13) is only necessary, not sufficient. The space of admissible values of $\varrho$ is then limited by two different curves similarly as for real $a_2$. In the interval $0 < a_3 \leqq \frac{1}{5}$, the limiting curve has always the form (4.15) but the second limiting curve is very complicated and the equation of this curve can be calculated by elimination of $u$ from the two conditions $R(u) = 0$, $R'(u) = 0$.

The maximal values $|a_2|$, $a_3$ are attained by the polynomial $z + \frac{2\sqrt{2}}{3} z^2 + \frac{1}{3} z^3$, which is simple for $|z| < 1$ only.

For polynomials of higher degree than three the discussion of the resultant is not an easy task because not only the resultant is of high degree but the explicit calculation of the functions of STURM surpasses the possibilities of a normal man.

But two necessary conditions for the last two coefficients $a_{n-1}$ and $a_n$ can be deduced from the ass. equation of J. DIEUDONNÉ

$$1 + a_2 x \frac{\sin 2\varphi}{\sin\varphi} + a_3 x^2 \frac{\sin 3\varphi}{\sin\varphi} + \ldots + a_n x^{n-1} \frac{\sin n\varphi}{\sin\varphi} = 0.$$

If the polynomial (1.1) is simple then for every $\varphi$ the roots $x$ of this equation must all have modulus greater than unity. Therefore if we put $\varphi = 0$ the equation

$$P'(x) = 1 + 2a_2 x + 3a_3 x^2 + \ldots + n a_n x^{n-1} = 0 \tag{4.16}$$

must also have only such roots, which means

$$|a_n| < \frac{1}{n}. \tag{4.17}$$

But if we put $\varphi = \frac{\pi}{n}$, the ass. eq. is

$$1 + a_2 x \frac{\sin\frac{2\pi}{n}}{\sin\frac{\pi}{n}} + \ldots + a_{n-1} x^{n-2} = 0$$

and therefore

$$|a_{n-1}| < 1. \tag{4.17 bis}$$

This bound for $a_{n-1}$ is not the best possible for all $n$, because in the case $n = 3$ the exact bound is $|a_2| < \frac{2\sqrt{2}}{3}$. On the other hand the bound (4.17) is the best possible because e. g. the polynomial $z + a_n z^n$ is simple for every $|a_n| < \frac{1}{n}$.

In the general case of a polynomial $P(z)$ (1.1) a lower bound for the radius of conformal mapping can be easily deduced. The radius is always greater or equal to the positive root of the equation

$$1 - 2|a_2|x - 3|a_3|x^2 - \ldots - n|a_n|x^{n-1} = 0 \tag{4.18}$$

The proof of this theorem is easy. From (1.2) follows that

$$|\Delta(z_1, z_2)| \geqq 1 - \{|a_2|(|z_1| + |z_2|) + |a_3|(|z_1|^2 + |z_1 \cdot z_2| + |z_2|^2) + \ldots$$
$$+ \ldots + |a_n|(|z_1|^{n-1} + \ldots + |z_2|^{n-1})\}.$$

Is now $|z_2| \leqq |z_1|$ then

$$|\Delta(z_1, z_2)| \geqq 1 - \sum_{k=2}^{n} k|a_k| \cdot |z_1|^{k-1}$$

and this proves the assertion above, because for

$$0 \leqq |z_2| \leqq |z_1| < x$$

the increment ratio $\varDelta(z_1, z_2)$ does not vanish.

The family of polynomials

$$P(z) = z + \sum_{2}^{n} a_k z^k, \tag{4.19}$$

where $n$ and all $|a_k|$ are given numbers has therefore the following property:

The smallest radius of conf. mapping belongs to the polynomial

$$P_1(z) = z - \sum_{2}^{n} |a_k| z^k$$

and this radius is equal to the positive root of the equation $P_1'(z) = 0$. A consequence of this theorem is that all polynomials (1.1) where

$$1 - 2|a_2| - 3|a_3| - \ldots - n|a_n| \geqq 0 \tag{4.20}$$

are simple within $|z| < 1$. This condition is n. a s. for the polynomials $P_1(z)$, but for other members of the family the condition (4.20) is only sufficient, not necessary.

It is obvious that the theorem (4.19) remains true also for the family of power series

$$f(z) = z + \sum_{2}^{\infty} a_k z^k, \tag{4.21}$$

where $|a_k|$ are given numbers. The only difference from the polynomials is that the equation (4.18) needs not always have a positive root within the circle of convergence of the series. In this case the radius of conf. mapping is, for all members of the family, equal to the radius of convergence. Thus e. g. the family of series (4.21), where $|a_k| = \frac{1}{k^3}$, has the radius of convergence $r = 1$ and all members of the family are simple within $|z| < 1$, because

$$1 - \left(\frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} + \ldots\right) = 2 - \frac{\pi^2}{6} > 0.$$

The method described in the first three paragraphs is by no means limited to the polynomials. The same method can be used in every case where the increment ratio $\varDelta(z_1, z_2)$ of the function in question can be explicitly calculated. Such functions are e. g. all rational functions

$$Q(z) = z \frac{1 + a_1 z + a_2 z^2 + \ldots + a_n z^n}{1 + b_1 z + b_2 z^2 + \ldots + b_k z^k},$$

where the denominator does not vanish in the circle $|z| \leqq 1$. The method of forming the associated system (1.3) and the ass. resultant (3.1) does not change. For a power series $z + \sum_{2}^{\infty} a_k z^k$ with a radius of convergence $r > 1$ the system of assoc. equations (1.3) remains unchanged and also the fundamental theorem in the form (2.1) holds true. But the forming of the assoc. resultant (3.1) is a problem of transfinite algebra. I will return to these questions by another opportunity.